J. Austral. Math. Soc. **19** (Series B), (1975), 200–214.

A FORMULA FOR SUMMING DIVERGENT SERIES

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(Received 8 April 1975)

(Revised 8 October 1975)

Abstract

A formula is given for assigning sums to divergent series where the ratio of adjacent terms varies slowly along the series. This formula consists of a weighted average of partial sums and is shown to be a general formula which can be easily calculated using a simple recurrence relation. It appears to be more powerful than a repeated Aitken or Shanks e_1 process as long as the transformed series remains divergent and it is also compared with the Padé approximants. It is demonstrated on a factorial series, on a nearly geometric divergent series and for the extrapolation of a velocity formula for small amplitudes of motion.

1. Introduction

In classical analysis an infinite series $\Sigma_0^\infty a_n$ is termed divergent if the sequence of its partial sums $\{S_N\}$, where

$$S_N = \sum_{n=0}^{N} a_n,$$

is divergent. As is well-known, there is no particular reason why $\mathrm{Lim}_{N\to\infty} S_N$, even when it is convergent, should be taken as the only possible definition of a sum for Σa_n, many other limits being available and appropriate in particular contexts. Generally speaking, however, if $\{S_N\}$ is convergent then all the commonly used sums of the series will coincide with its limit. In this case the sequence $\{S_N\}$ may not afford the best means of determining the sum of the series numerically and we may prefer an alternative method of summation which converges more rapidly.

It is also well-known that an alternating series that is conditionally, but not absolutely convergent can be made to converge in the classical sense to any real number whatever, merely by altering the order of its terms. Between these extremes occur the summation methods that form a sequence of linear combinations (weighted sums) of the terms without altering their order, for instance the Euler transformation given below.

Originally these methods were developed to hasten the convergence of estimates of the sums of slowly convergent series, but when applied to divergent alternating series they are often found to converge, either absolutely or at least asymptotically, to a determinable limit which can meaningfully be accepted as the sums of these series.

For example, given a weakly divergent series with partial sums

$$S_N = \sum_{n=0}^{N} (-1)^n u_n \qquad (u_n \geqq 0)$$

in which the ratio u_{n+1}/u_n of successive terms is less than 3, the Euler transformation

$$U_N = -\sum_{n=0}^{N} (-\tfrac{1}{2})^{n+1} \Delta^n u_0,$$

where $\Delta^n u_o$ is the nth order forward difference of u_0, provides a rapidly converging sequence whose limit coincides with S_∞ when this exists, and will be a self-consistent definition of a sum of the series when $\{S_N\}$ diverges. A more widely-applicable general method, due to Aitken [1], may be regarded as replacing the tail of a series, t_n, where

$$t_n = u_n - u_{n+1} + u_{n+2} - \cdots,$$

by a two-term estimate, T_n, where

$$T_n = u_n^2/(u_n + u_{n+1}).$$

If the original series is truncated, then each term except the last is split into two parts, giving

$$\begin{aligned} S_n &= u_0 - u_1 + u_2 - u_3 + \cdots + (-1)^N u_N \\ &= [T_0 + (u_0 - T_0)] - [T_1 + (u_1 - T_1)] + \cdots + \\ &\quad (-1)^{N-1}[T_{N-1} + (u_{N-1} - T_{N-1})] + (-1)^N u_N, \end{aligned}$$

and the second part of each term is combined with the first part of the next term to form a new series of terms,

$$\begin{aligned} T_0 + S^*_{N-2} = T_0 &+ (u_0 - T_0 - T_1) - (u_1 - T_1 - T_2) + \cdots \\ &+ (-1)^{N-2}(u_{N-2} - T_{N-2} - T_{N-1}). \end{aligned}$$

The parts of terms at the end of the truncated series, omitted by this

https://doi.org/10.1017/S0334270000001089 Published online by Cambridge University Press

transformation, combine naturally with the other terms already deleted by truncation.

If the series is a convergent geometric progression (G.P.), then $T_n = t_n$, and hence $u_n - T_n - T_{n+1} = 0$ for all n. Hence the Aitken transformation reduces the series to T_0 in one operation. If we wish, we may also use analytic continuation to define T_0 as the geometric sum for a divergent G.P., provided only that $u_{n+1}/u_n \neq 1$.

Now let us describe a series for which the ratio u_{n+1}/u_n tends to a bounded limit, R, when $n \to \infty$, as nearly geometric and a series for which the ratio varies slowly along the series, such as an asymptotic series, as locally nearly geometric. Then the terms, u_n^*, of the transformed series, where

$$u_n^* = u_n - T_n - T_{n+1},$$

will be smaller than the corresponding terms of the original series, not only for *nearly geometric* but also for *locally nearly geometric* series. An examination of Table 1 shows that for at least one asymptotic series the Aitken sum of the first few terms appears to converge to the same number as is obtained by other methods.

Other comments about Aitken's method are listed below.

(1) The transformation may be repeated on the transformed series, S_{N-2}^*, giving a new first term and a new series, S_{N-4}^{**}. The series is reduced by two terms at each transformation. This may be repeated until the series reduces to fewer than two terms or the process becomes unstable.

(2) We may wish to sum a series in which the ratio of successive terms changes rapidly. This rapid variation usually occurs at the start of a series and the ratio of successive terms settles down to a slow variation further along the series. This may result in an unevenness or instability near the start of the successively transformed series $S_{N-2}^*, S_{N-4}^{**}, \cdots$. This can easily be detected if the terms of the series are tabulated. The initial instability does not affect the sum of the series nor the values of the later terms of the transformed series.

(3) Aitken's transformation does not alter the order of terms so it can be used to speed the convergence of conditionally convergent series.

(4) We may alter the sum of a divergent series by a biased splitting and regrouping such as

$$\begin{aligned} S &= 1 - 1 + 1 - 1 + 1 - \cdots \\ &= (\tfrac{2}{3} + \tfrac{1}{3}) - (\tfrac{1}{3} + \tfrac{2}{3}) + (\tfrac{2}{3} + \tfrac{1}{3}) - \cdots \\ &= \tfrac{2}{3} + (\tfrac{1}{3} - \tfrac{1}{3}) - (\tfrac{2}{3} - \tfrac{2}{3}) + (\tfrac{1}{3} - \tfrac{1}{3}) - \cdots, \end{aligned}$$

where the positive and negative terms are split in different ways. However, both the Euler and the Aitken transformation are not sign-dependent so they do not have any obvious bias which might affect the sum of a series.

A second general method, due to S. Lubkin [3], is to use a three-term approximant, L_n, to the tail, t_n, of a series, where

$$L_n = \frac{u_n(1 + u_{n+1}/u_n)}{1 + 2u_{n+1}/u_n + u_{n+1}/u_{n-1}}.$$

This is used in the same way as T_n to transform a series and effectively speeds the convergence of a slowly convergent series of positive terms. However, on divergent alternating series, two applications of Lubkin's transformation, using up six terms of a series, do not reduce the size of the remaining terms as much as three applications of Aitken's method. Nevertheless, L_n can be written as a weighted mean of T_n and $u_{n-1} - T_{n-1}$. Hence Aitken and Lubkin sums of alternating series are the same.

A third general set of formulae, called e_n *processes* was derived by D. Shanks [5], assuming that each term of a series may be represented by a sum of n exponentials. The formulae were demonstrated on a number of convergent and divergent series. They are not sign-dependent, so appear to be unbiassed and e_1 is Aitken's two-term formula. While e_n is exact for a series whose terms are a combination of n or fewer exponentials, one of Shank's examples shows that for other series e_2 does not reduce the size of the transformed terms as much as a double use of e_1, both using up four terms. Shanks's formulae are complicated but P. Wynn [7] gives a recurrence relation for calculating the successive e_n's.

The author's [2] own weighted Euler sum is another formula which is relatively simple and uses the reciprocals of the terms for weights in the Euler sum. It is identical with Aitken's or Lubkin's formula when two or three terms are used. It appears to sum divergent alternating series to the same sum as Aitken's formula, but the rate of convergence is slower. On the other hand, it does not appear to suffer seriously from initial instability and is convenient when a reasonably large number of terms of the series is available.

A different method again for summing series is the use of Padé approximants [4, 6] which are compared in Section 3 (vi) with the new formula.

While there are better specialist formulae, Aitken's [1] formula seems to be the best unbiassed general formula for summing a general divergent alternating series when we have a limited number of terms available, and we assume only that the later terms of the transformed series alternate.

The formula which follows is a weighted sum which appears to be more powerful initially in speeding the convergence or reducing the size of terms than Aitken's repeated formula: it is a general formula; it is not sign-dependent so appears to be unbiassed; it may be calculated either directly or from a recurrence relation with comparable labour to that for Aitken's formula and while it may, like other formulae, suffer from initial instability, this can easily be observed and avoided when it occurs.

2. The formula

Let S_{mn} denote the sequence

$$S_{mn} = u_m - u_{m+1} + \cdots + (-1)^{n-m} u_n \qquad (n \geqq m).$$

Let R_r be the ratio of two successive terms, i.e. $R_r = u_{r+1}/u_r$. Then an estimate of $S_{0,\infty}$, using the terms u_m to u_n as weights, is

$$T_{mn} = P_{mn}/Q_{mn},$$

where

$$\begin{aligned} P_{mn} = S_{0n} &+ (R_m + R_{m+1} + \cdots + R_{n-1}) S_{0,n-1} \\ &+ (R_m^2 + R_m R_{m+1} + \cdots + R_{n-2}^2) S_{0,n-2} \\ &+ \cdots + R_m^{n-m} S_{0m} \end{aligned}$$

and

$$Q_{mn} = 1 + (R_m + R_{m+1} + \cdots + R_{n-1}) + (R_m^2 + R_m R_{m+1} + \cdots + R_{n-2}^2) + \cdots + R_m^{n-m}$$

Recurrence relations for generating P_{mn} and Q_{mn} are

$$\begin{aligned} P_{mm} &= S_{0m} \qquad Q_{mm} = 1 \\ P_{mn} &= P_{m+1,n} + R_m P_{m,n-1} \\ Q_{mn} &= Q_{m+1,n} + R_m Q_{m,n-1}. \end{aligned}$$

3. Properties of the formula

(i) The estimators, T_{mn}, are weighted means of the truncated sums $S_{0n}, S_{0,n-1}, \cdots, S_{0m}$ with weights

$$1, (R_m + R_{m+1} + \cdots + R_{n-1}), (R_m^2 + R_m R_{m+1} + \cdots + R_{n-2}^2), \cdots, R_m^{n-m}.$$

If this is compared with the author's [2] other weighted Euler sum, whose weights were

$$1, \binom{n-m}{1} R_{n-1}, \binom{n-m}{2} R_{n-1}^2, \cdots, R_{n-1}^{n-m},$$

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we see that the weighting function for divergent series uses the ratios of terms earlier in the series.

(ii) The first estimator of $S_{0\infty}$ is the trivial S_{0m} and the second estimator is the Aitken formula or Shanks's e_1. Thus $T_{mn} = S_{0m}$ and with a little rearranging the second estimator gives

$$T_{m,m+1} = \frac{S_{0m} + (-1)^{m+1}u_{m+1} + R_m S_{0m}}{1+R_m} = S_{0m} + (-1)^{m+1}\frac{u_m u_{m+1}}{u_m + u_{m+1}}$$

$$= S_{0,m-1} + (-1)^m \frac{u_m^2}{u_m + u_{m+1}},$$

which we recognise as Aitken's transformation.

(iii) If the ratios R_r are constant and equal to R, say, then P_{mn} and Q_{mn} are binomials in $(1+R)$ and

$$T_{mn} = S_{0,m-1} + (-1)^m u_m \frac{(1+R)^{n-m-1}}{(1+R)^{n-m}}.$$

Proof. If $R_r = R$ for all values of r, then $Q_{mn} = (1+R)^{n-m}$. Similarly,

$$P_{mn} = S_{0,m-1}Q_{mn}$$
$$+ (-1)^m u_m\{(1 - R_m + \cdots + (-1)^{n-m-1}R_m R_{m+1}\cdots R_{n-1}) + \cdots + R_m^{n-m}\}$$
$$= S_{0,m-1}Q_{mn} + (-1)^m u_m (1+R)^{n-m-1}.$$

Therefore

$$T_{mn} = S_{0,m-1} + (-1)^m u_m \frac{(1+R)^{n-m-1}}{(1+R)^{n-m}}.$$

(iv) T_{0n} is a general formula in the sense that if the R_r are small and slowly varying and if T_{0n} is expanded as a power series in the R_r starting from u_0, then the $(n+1)$ largest terms will coincide exactly with the first $(n+1)$ terms of $S_{0\infty}$ and the larger of the later terms form an approximate G.P. Similarly, this holds if the R_r are large and slowly varying and $(-1)^n(S_n - T_{0n})$ is expanded as an inverse power series starting backwards from u_n.

Proof. If the R_r are small then $S_{0\infty}$ may be written as

$$u_0(1 - R_0 + R_0R_1 - R_0R_1R_2 + \cdots)$$

and

$$T_{01} = \frac{u_0}{1+R_0} = u_0(1 - R_0 + R_0^2 - R_0^3 + \cdots).$$

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If, on the other hand, the R_r are large then

$$-S_1 = u_1\left(1 - \frac{1}{R_0}\right)$$

and

$$-S_1 + T_{01} = u_1\left(1 - \frac{1}{R_0} + \frac{1}{R_0^2} - \frac{1}{R_0^3} + \cdots\right).$$

Hence the two largest terms of our two-term approximant and the original series agree exactly whether the series is convergent or divergent. Also the other large terms will approximately agree if R_r varies slowly along the series.

If T_{0n}, as defined in the formula, is expanded as a power series, using R_0 to R_{n-1}, where R_0 to R_{n-1} are small and slowly varying, and if the first $(n+1)$ terms of T_{0n} are the same as in the original series and the remainder can be summed, then

$$T_{0n} = \frac{P_{0n}}{Q_{0n}} = u_0(1 - R_0 + \cdots(-1)^n R_0 \cdots R_{n-1} + (-1)^{n+1} R^{n+1}/(1+R)),$$

using an order of magnitude expression for the remainder in which R is comparable with the R_r. Also

$$T_{1,n+1} = \frac{P_{1,n+1}}{Q_{1,n+1}} = u_0(1 - R_0 + \cdots + (-1)^{n+1} R_0 \cdots R_n + (-1)^{n+2} R^{n+2}/(1+R)).$$

If R^* is any positive number, a weighted mean is given by

$$\frac{P_{1,n+1} + R^* P_{0n}}{Q_{1,n+1} + R^* Q_{0n}} = u_0\left[1 - R_0 + \cdots + (-1)^{n+1} R_0 \cdots R_n \right.$$
$$\left. + (-1)^{n+2} \frac{\dfrac{R^{n+2}}{1+R} Q_{1,n+1} - \dfrac{R^* R^{n+1}}{1+R} Q_{0n} + R_0 \cdots R_n R^* Q_{0n}}{Q_{1,n+1} + R^* Q_{0n}}\right].$$

Hence if R^* is comparable with the members of the slowly varying set, R_0 to R_n, and if $Q_{1,n+1} \doteqdot Q_{0n}$, then the first $(n+2)$ terms of the expansion agree exactly with the first $(n+2)$ terms of the original series and the larger of the later terms form an approximate G.P. Similarly, this holds for the expansion in reciprocal powers. Finally, T_{01} is a general formula. Hence, using the recurrence formulae, T_{02} and T_{03} and so on are all general formulae, in the sense that the $(n+1)$ largest terms of T_{0n} and the original series coincide exactly and the other larger terms form comparable G.Ps.

(v) The best choice of R^* for T_{mn} for a divergent, almost geometric, alternating series is R_m.

Justification: The error in a formula which fits $(n-m)$ terms of a series exactly is a fraction of the smallest term used, which is the first term for a divergent series, and the fraction will not change rapidly along the series if R_m varies slowly with m. Hence the ratio of errors in $T_{m,n-1}$ and $T_{m+1,n}$ is approximately $-R_m$, so we can substantially reduce the error by combining them in the ratio $R^*:1$, where R^* is close to R_m.

We also wish not to waste terms of the series, so in constructing T_{mn} from $T_{m,n-1}$ and $T_{m+1,n}$ we will only construct R^* from u_m to u_n or R_m to R_{n-1}. In addition we wish to avoid the growth of rounding off errors in calculations, so we want a positive denominator. This means that we cannot use forms like $(2R_m - R_{m+1})$, which approximates R_{m-1} but introduces negative terms into the denominator, so our only choice for a divergent series is the single term $R^* = R_m$. This choice of weight also requires that $Q_{m,n-1} \doteqdot Q_{m+1,n}$. It can be seen in Tables 2 and 3 that this soon ceases to be true at the start of the series, but the repeated transformations keep feeding in information from later terms of the series, where the transformation is better, thus damping down initial instability.

On the other hand, if we had a convergent series, we should use the ratio of last terms, i.e. R_{n-1}, in combining $T_{m,n-1}$ and $T_{m+1,n}$ to form T_{mn}. Hence if, after several transformations of a weakly divergent series, our residuals formed a convergent series, then this process would become less powerful. The second numerical example exhibits a rapid initial convergence of this type, which peters out.

(vi) The formula in Section 2 is similar to the Padé approximants [4, 6]. It contains more terms and contains no negative terms in the denominator.

The Padé approximants for a series $F(z) = \sum_{r=0}^{\infty} a_r z^r$ are the ratios of two polynomials. The $[L, M]$ Padé approximant to $F(z)$ is the quotient of two polynomials $P_L(z)$ and $Q_M(z)$, of degree L and M respectively, i.e.

$$[L, M] \equiv \frac{P_L(z)}{Q_M(z)} \equiv \frac{p_0 + p_1 z + \cdots + p_L z^L}{1 + q_1 z + \cdots + q_M z^M},$$

chosen so that the power series expansion of $[L, M]$ for small z agrees exactly with the terms of $F(z)$ up to z^{L+M}.

Now, if $S_{0\infty}$ is a power series in z, with $u_r = a_r z^r$, then $R_r = a_{r+1} z / a_r$, so P_{mn} is a polynomial of degree n or $(n-1)$ and Q_{mn} is a polynomial of degree $(n-m)$ and we have deduced in Section 2(iv) that T_{mn} agrees exactly with the terms of $S_{0\infty}$ up to z^n.

Hence the first row of the Padé table $[n, 0]$ and the T_{nn} are the same truncated sums of the series up to z^n. The second row of the Padé table

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$[n-1, n]$ are the first Aitken or Shanks e_1 transformation of the series, using the last two terms for the transformation. Thereafter the number of terms in T_{mn} grows at twice the rate of the number of terms in $[L, M]$ when we match more terms of the original series.

Thus if we use 4 terms of the series

$$S_{0\infty} = u_0(1 - R_0 + R_0R_1 - R_0R_1R_2 + \cdots),$$

then

$$[1,2] = u_0\frac{(R_0 - R_1) + (-R_0^2 + 2R_0R_1 - R_1R_2)}{(R_0 - R_1) + R_1(R_0 - R_2) + R_0R_1(R_1 - R_2)},$$

and

$$T_{03} = u_0\frac{1 + (R_1 + R_2) + (R_0R_1 + R_1^2 - R_0R_2)}{1 + (R_0 + R_1 + R_2) + (R_0^2 + R_0R_1 + R_1^2) + R_0^3}.$$

The similarities between the formulae are that they are both quotients of two polynomials and fit four terms of the original series exactly and appear to converge with comparable rapidity. The differences are that the Padé approximants may be found from a relationship between five approximants, whereas the P_{mn} and Q_{mn} are obtained from simpler recurrence relations. Also, the denominators of the Padé approximants contain negative terms, so they must be calculated more accurately to avoid rounding-off errors.

4. Numerical examples

(i) The sum of alternating factorials

If we have an asymptotic series whose terms decrease at first, then as long as the terms are becoming smaller using a summation formula is no better than evaluating one more term of the series, since the truncation error is comparable with the first term omitted. Hence it only becomes profitable to use a summation formula on the terms which are not rapidly converging.

As an example we take

$$I = \int_1^\infty e^{1-x}\,dx/x = 0.596347361\cdots$$

$$= 0! - 1! + 2! - 3! + 4! - \cdots.$$

In Table 1 we use Aitken's method repeatedly, first calculating $T_n = u_n^2/(u_n + u_{n+1})$ and the transformed series $u'_n = u_n - T_n - T_{n+1}$ then repeating this process.

The numbers left over at the start of each transformation are marked with a star and later added.

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TABLE 1
Repeated Aitken sum of factorials

n	u_n	T_n	u'_n	T'_n	u''_n	T''_n	u'''_n	T'''_n
0	1	$\frac{1^*}{2}$						
1	-1	$-\frac{1}{3}$	$\frac{1}{2.3}$	$\frac{4^*}{2.3.8}$				
2	2	$\frac{2}{4}$	$\frac{-2}{3.4}$	$\frac{-2.5}{3.4.15}$	$\frac{2.16}{3.4.8.14}$	.0115377*		
3	-6	$-\frac{6}{5}$	$\frac{6}{4.5}$	$\frac{6.6}{4.5.22}$	$\frac{-6.26}{4.5.14.22}$	$-$.0093617	.0029101	.0013372*
4	24	$\frac{24}{6}$	$\frac{-24}{5.6}$	$\frac{-24.7}{5.6.32}$	$\frac{24.38}{5.6.22.32}$	.0125397	$-$.0034232	
5	-120	$-\frac{120}{7}$	$\frac{120}{6.7}$	$\frac{120.8}{6.7.44}$	$\frac{-120.52}{6.7.32.44}$			
6	720	$\frac{720}{8}$	$\frac{-720}{7.8}$					
7	-5040							

TABLE 2
T_{mn} for the sum of factorials

n	u_n	R_n	T_{nn}	$T_{n,n+1}$	$T_{n,n+2}$	$T_{n,n+3}$	$T_{n,n+4}$	$T_{n,n+5}$	$T_{n,n+6}$	$T_{n,n+7}$
0	1	1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{9}{15}$	$\frac{31}{52}$	$\frac{121}{203}$	$\frac{523}{877}$	$\frac{2469}{4140}$
1	-1	2	$\frac{0}{1}$	$\frac{2}{3}$	$\frac{6}{10}$	$\frac{22}{37}$	$\frac{90}{151}$	$\frac{402}{674}$	$\frac{1946}{3263}$	
2	2	3	$\frac{2}{1}$	$\frac{2}{4}$	$\frac{10}{17}$	$\frac{46}{77}$	$\frac{222}{372}$	$\frac{1142}{1915}$		
3	-6	4	$\frac{-4}{1}$	$\frac{4}{5}$	$\frac{16}{26}$	$\frac{84}{141}$	$\frac{476}{799}$			
4	24	5	$\frac{20}{1}$	$\frac{0}{6}$	$\frac{20}{37}$	$\frac{140}{235}$				
5	-120	6	$\frac{-100}{1}$	$\frac{20}{7}$	$\frac{40}{50}$					
6	720	7	$\frac{620}{1}$	$\frac{-80}{8}$						
7	-5040		$\frac{-4420}{1}$							

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Using 8 terms of the original series

$$I = .5000000 + .0833333 + .0115377 + .0013372 + \cdots,$$

the truncation error at this stage being .0001392. The new series ceases to be monotonic or becomes unstable after using 14 terms of the original series.

In Table 2 we list n, u_n, R_n and T_{nn} ($= S_{0n}/1$) in the first four columns and calculate P_{mn}/Q_{mn} using the recurrence relations. The fractions are all estimates of I which improve towards the top right corner of Table 2. To compare them with the results using Aitken's formula, we write the sequence T_{01}, T_{03}, T_{05} and T_{07} as a series

$$I = .5000000 + .1000000 - .0039409 + .0003177 - \cdots,$$

the truncation error at this stage being $-.0000294$. This series is more rapidly convergent than Aitken's and does not become unstable until 18 terms of the original series are used. Even if we had used weights from the other end of the series, namely $R^* = R_{n-1}$, giving the author's weighted Euler sum, we still get a convergent process. Using the same terms we obtain

$$I = .50000 + .08824 + .00684 + .00100 + \cdots .$$

(ii) A nearly geometric series

The following is a demonstration that T_{mn} is useful for divergent, alternating, almost geometric series in which R_n increases with n.

Almost geometric, divergent series may be divided into two groups, one in which the ratio of successive terms rises to a constant limit greater than 1 and the other where the ratio of successive terms decreases to a constant limit greater than 1.

If the series is a binomial with exponent -1 or a G.P. such as $S = 1 - 2 + 4 - 8 + \cdots$, then all the formulae give the sum in one operation as 1/3 without remainder.

As an example of a series in which R_n increases we take the logarithmic series.

The function $ln(1+z)$ has a logarithmic singularity in the complex plane at $z = -1$ so we cannot expect to represent it as a rational function for all z. Even so we still obtain considerable convergence before the T_{mn} become unstable. As an example, we use

$$1.7918 \cdots = ln(1+5) = 5 - 5^2/2 + 5^3/3 - 5^4/4 + 5^5/5 - \cdots .$$

In Table 3, the values of n, u_n, R_n and S_{1n} or T_{nn} are listed in the first four columns and the recurrence relation is used to generate the higher order T_{mn} values.

TABLE 3

T_{mn} for a nearly geometric series

n	u_n	R_n	T_{nn}	$T_{n,n+1}$	$T_{n,n+2}$	$T_{n,n+3}$	$T_{n,n+4}$	$T_{n,n+5}$
0	5	$\frac{5}{2}$	$\frac{5}{1}$	$\frac{5}{3.5}$	$\frac{21.667}{13.083}$	$\frac{90.764}{51.903}$	$\frac{386.14}{216.55}$	$\frac{1688.3}{941.4}$
1	$\frac{-25}{2}$	$\frac{10}{3}$	$\frac{-7.5}{1}$	$\frac{9.1667}{4.3333}$	$\frac{36.597}{19.194}$	$\frac{159.23}{86.79}$	$\frac{722.99}{400.03}$	
2	$\frac{125}{3}$	$\frac{15}{4}$	$\frac{34.167}{1}$	$\frac{6.0417}{4.75}$	$\frac{37.240}{22.812}$	$\frac{192.22}{110.71}$		
3	$\frac{-625}{4}$	$\frac{20}{5}$	$\frac{-122.08}{1}$	$\frac{14.583}{5}$	$\frac{52.569}{25.167}$			
4	$\frac{3125}{5}$	$\frac{25}{6}$	$\frac{502.92}{1}$	$\frac{-5.7639}{5.1667}$				
5	$\frac{-15625}{6}$		$\frac{-2101.25}{1}$					

The successive sums for 2, 4 and 6 terms using Aitken's formula are 1.4286, 1.7368 and 1.7862, while for comparison T_{01}, T_{03} and T_{05} are 1.4286, 1.7487 and 1.7934. If further terms are calculated, Aitken's formula continues to provide a convergent sequence but the new formula provides a sequence that overshoots and then oscillates.

TABLE 4

Calculation of an inverse square root

n	u_n	R_n	T_{nn}	$T_{n,n+1}$	$T_{n,n+2}$	$T_{n,n+3}$	$T_{n,n+4}$	$T_{n,n+5}$
0	1	4	$\frac{1}{1}$	$\frac{1}{5}$	$\frac{7}{27}$	$\frac{47}{157.7}$	$\frac{316.7}{988}$	$\frac{2173}{6552}$
1	−4	6	$\frac{-3}{1}$	$\frac{3}{7}$	$\frac{19}{149/3}$	$\frac{128.7}{357.1}$	$\frac{906.0}{2601}$	
2	24	20/3	$\frac{21}{1}$	$\frac{1}{23/3}$	$\frac{44/3}{532/9}$	$\frac{134.0}{458.3}$		
3	−160	7	$\frac{-139}{1}$	$\frac{8}{8}$	$\frac{181/5}{321/5}$			
4	1120	36/5	$\frac{981}{1}$	$\frac{-99/5}{41/5}$				
5	−8064		$\frac{-7083}{1}$					

The corresponding Padé approximants using the same terms are [1, 2], [2, 3] and [3, 4]. These give 1.4286, 1.7213 and 1.7787, which are comparable but not quite as good as the repeated Aitken and the T_{mn} approximations.

As a second example of a series with increasing R_n we take the binomial series for the inverse square root of 9:

$$(1+8)^{-1/2} = 1 - 4 + 24 - 160 + 1120 - 8064 + \cdots = 1/3 = .333\cdots$$

The details for T_{mn} are given in Table 4 and the sums for 2, 4 and 6 terms to three decimal places are:

Repeated Aitken	.200	.299	.328
Padé $[n, n+1]$	.200	.294	.323
$T_{0,2n-1}$	.200	.298	.332.

As an example of a series with decreasing R_n we take a binomial with power $-3/2$:

$$(1+8)^{-3/2} = 1 - 12 + 120 - 1120 + 10080 - 88704 + \cdots = .\dot{0}3\dot{7}.$$

The new formula is inferior to the older formulae on this example. The sums using different formulae on 6 terms are:

Repeated Aitken	.0385
Padé [3, 4]	$.038\dot{3}$
T_{05}	.0551.

(iii) A velocity formula for an inextensible sheet oscillating sinusoidally in a viscous liquid

If σ is the angular frequency, k is the wave number and b is the amplitude, then the velocity, v, of the sheet relative to the fluid is given by

$$v = \frac{\sigma}{2k}\left[(bk)^2 - \frac{19}{16}(bk)^4 + \frac{41}{32}(bk)^6 - \frac{16913}{12288}(bk)^8 + \cdots\right].$$

Two transformations of this series are

$$T_{13} = \frac{\sigma}{2k}\,\frac{b^2k^2 + \frac{288875}{299136}b^4k^4 - \frac{10143611}{90937344}b^6k^6}{1 + \frac{644099}{299136}b^2k^2 + \frac{1681}{1444}b^4k^4}$$

and

https://doi.org/10.1017/S0334270000001089 Published online by Cambridge University Press

$$T_{03} = \frac{\sigma}{2k} \frac{b^2k^2 + \frac{644099}{299136} b^4k^4 + \frac{106369861}{90937344} b^6k^6}{1 + \frac{999323}{299136} b^2k^2 + \frac{356313}{92416} b^4k^4 + \frac{6859}{4096} b^6k^6}.$$

The original series appears to have a radius of convergence given by $bk \doteqdot .93$ and if we want a result, say, to 3% accuracy the truncated power series may be used for bk up to 0.6. When the series is transformed, T_{03} and T_{13} agree to better than 3% for amplitudes up to 1.1, which is beyond the original radius of convergence.

5. Conclusion

The choice of weight $R^* = R_m$ in Section 3 (v) was made to reduce the error for a divergent series. When a new sequence or series is constructed using the transformation of Section 2 and the original series is locally nearly geometric, the new terms are smaller than the original terms. If the later terms of the original series are more nearly geometric than the earlier terms, then the fractional reduction is greater further along the series, so the ratio of successive terms is also reduced.

If the weights, R_m, that we use in the transformation decrease along the original series, then in the second transformation we will use weights which are larger than ideal and so cause a slow convergence. On the other hand, if the ratios, R_m, increase along the series and each transformation reduces the length of the series by one term, then we consistently use the smaller weights available. Hence for a while at least the new formula remains efficient.

An examination of the numerical examples shows that for the asymptotic series the transformation may be repeated 9 times and gives a sum accurate to 9 decimal places before it starts to show an instability or loss of efficiency. In the two examples of geometric series with increasing ratio, the instability starts earlier but the transformation is efficient until it becomes unstable and the instability can easily be observed by tabulating a few adjacent terms in the table of T_{mn}. It then appears that the formula of Section 2 is useful for divergent series in which the ratio of the terms increases along the series.

The final example is a series which is truncated because of the difficulty of calculating later terms. A transformation of this series increases both its accuracy and range.

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