RESTRICTION OF SIEGEL MODULAR FORMS TO MODULAR CURVES
Cris Poor and David S. Yuen

We study homomorphisms from the ring of Siegel modular forms of a given degree to the ring of elliptic modular forms for a congruence subgroup. These homomorphisms essentially arise from the restriction of Siegel modular forms to modular curves. These homomorphisms give rise to linear relations among the Fourier coefficients of a Siegel modular form. We use this technique to prove that \( \dim S^1_0 = 1 \).

1. INTRODUCTION

A Siegel modular cusp form of degree \( n \) has a Fourier series \( f(\Omega) = \sum_t a(t)e(\text{tr}(\Omega t)) \) where \( t \) runs over \( \mathcal{X}_n \), the set of positive definite semi-integral \( n \times n \) forms. If we restrict attention to cusp forms of even weight then the Fourier coefficients are class functions of \( t \). The vector space \( S^k_n \) of cusp forms of weight \( k \) is finite dimensional and so there exist finite subsets \( S \subset \text{classes} (\mathcal{X}_n) \) such that the projection map \( \text{FS}_S : S^k_n \rightarrow \mathbb{C}^S \) given by \( f \mapsto \prod_{[t] \in S} a(t) \) is injective. The following Theorem [13, p. 218] gives one such \( S \) that is readily computable from \( n \) and \( k \). Instead of ordering semi-integral forms \( t \) by their determinant \( \det(t) \) we order them by their dyadic trace \( w(t) \). Denote by \( P_n(F) \) the positive definite \( n \times n \) symmetric matrices with coefficients in \( F \subset \mathbb{R} \). The dyadic trace \( w : P_n(\mathbb{R}) \rightarrow \mathbb{R}^+ \) is a class function and only a finite number of classes from \( \mathcal{X}_n \) will have a dyadic trace below any fixed bound, see [13].

**THEOREM 1.1.** Let \( n, k \in \mathbb{Z}^+ \). Let

\[
S = \left\{ [t] : t \in \mathcal{X}_n \text{ and } w(t) \leq n \frac{2k}{\sqrt{3}4\pi} \right\}.
\]

The map \( \text{FS}_S : S^k_n \rightarrow \mathbb{C}^S \) is injective.

This Theorem allows one to deduce equality in \( S^k_n \) from equality on the Fourier coefficients for \( S \). There are two obvious avenues for improvement. First, as is evident from Table 1, the bound \( \dim S^k_n \leq \text{card}(S) \) is tractable but crude and we would like to trim down the set \( S \) to make \( \text{card}(S) \) closer to \( \dim S^k_n \). Second, the image \( \text{FS}_S(f) \) determines \( f \) and one would like to compute some Fourier coefficients outside of \( S \).
directly from the Fourier coefficients in $S$. This paper realises both improvements. We give a method for producing linear relations on the Fourier coefficients of the elements in $S^k_n$. Table 1 gives $\dim S^k_4$, $\text{card}(S)$ and examples of linear relations for even $k \leq 12$. These are the only even weights for which $\dim S^k_4$ is known and the result $\dim S^{10}_4 = 1$ is a new one.

Table 1.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\dim S^k_4$</th>
<th>$\text{card}(S)$</th>
<th>linear relations</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>1</td>
<td>$a\left(\frac{1}{2}D_4\right) = 0$</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>2</td>
<td>$a\left(\frac{1}{2}D_4\right) + a\left(\frac{1}{2}A_4\right) = 0$</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>10</td>
<td>see equations (3.3)</td>
</tr>
<tr>
<td>12</td>
<td>2</td>
<td>23</td>
<td>21 uncomputed relations</td>
</tr>
</tbody>
</table>

For $k \leq 4$ we have $S = \emptyset$ and so Theorem 1.1 by itself proves $S^k_4 = 0$, results due to Christian [2] and Eichler [4, 5]. For $k = 6$ we have $S = \{[D_4/2]\}$ and the method in this paper provides the linear relation $a(D_4/2) = 0$ so that we conclude $\dim S^6_4 = 0$. For $k = 8$ we have $S = \{[D_4/2],[A_4/2]\}$ and the method provides the linear relation $a(D_4/2) + a(A_4/2) = 0$ showing that $\dim S^8_4 \leq 1$. The Schottky form $J$ is in $S^8_4$ [9] so we have $\dim S^8_4 = 1$, see [14, 11, 3] for these results. For $k = 10$ the $S$ consists of the ten classes in Table 3 and the method provides the nine linearly independent relations given in equation 3.3. We know the cusp form $G_{10}$ is in $S^{10}_4$, see [13, p. 232], so that we have $\dim S^{10}_4 = 1$, a result that has been beyond the reach of other methods [12, 3]. By the work of Erokhin $\dim S^{12}_4 = 2$ is already known, see [6, 7, 11]. Linear relations among Fourier coefficients for semi-integral forms not solely in $S$ allow the computation of Fourier coefficients outside of $S$.

The method of producing linear relations on Fourier coefficients from $S^k_n$ relies on the homomorphisms $\phi^*_s : S^k_n \rightarrow S^{\Gamma_0(\ell)}_1(\Gamma_0(\ell))$ which exist for any $s \in P_n(Z)$ and any $\ell \in Z^+$ with $\ell s^{-1}$ integral. We write elements of $\Gamma_1 = \text{Sp}_1(Z)$ as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and define the subgroup $\Gamma_0(\ell)$ by $\ell | c$ and the subgroup $\Delta_1$ by $c = 0$. We define $\phi^*_s(\tau) = s\tau$ so that for $f \in M^k_n$ we have $(\phi^*_s f)(\tau) = f(s\tau)$. There are three important points about these homomorphisms:

1. The image ring $M_1(\Gamma_0(\ell))$ is amenable to computation.
2. The Fourier coefficients of $\phi^*_s f$ at each cusp are linear combinations of the Fourier coefficients of $f$, see Proposition 2.3.
3. There are lots of $n \times n$ integral forms $s$. 

https://doi.org/10.1017/S00049727000020281 Published online by Cambridge University Press
The first point allows us to work out the linear relations among the Fourier coefficients at all cusps of elements in $S^1_{1k}(\Gamma_0(\ell))$. The second point induces linear relations on the Fourier coefficients of elements in $S^k_n$ from the linear relations on $S^1_{1k}(\Gamma_0(\ell))$. The third point allows us to continue producing linear relations if more are desired.

We illustrate the technique in weights 6 and 8 when the number of Fourier coefficients remains small. Let $f \in S^k_4$ have the Fourier expansion $f(\Omega) = \sum a(t)e(\langle \Omega, t \rangle)$ where $\langle \Omega, t \rangle = \text{tr}(\Omega t)$. Let $D_4$ represent the $4 \times 4$ form of this root lattice ($D_4 = 2B_0$ from Table 3). We compute the Fourier expansion of $\phi_{D_4}^* f$ in powers of $q = e(\tau)$. For any $s \in \mathcal{P}_n(Q)$ we expand $\phi_{D_4}^* f$ into a Fourier series as

$$
(\phi_{D_4}^* f)(\tau) = \sum_{j \in \mathbb{Q}^+} \left( \sum_{t : \langle s, t \rangle = j} a(t) \right) q^j.
$$

For simplicity we shall henceforth assume that $k$ is even. If we introduce the notation $\mathcal{V}(j, s, t) = \text{card}\{v \in \mathcal{X}_n : [v] = [t], \langle v, s \rangle = j\}$ then we can write

$$
(\phi_{D_4}^* f)(\tau) = \sum_{j \in \mathbb{Q}^+} \left( \sum_{[t]} \mathcal{V}(j, s, t) a(t) \right) q^j.
$$

Table 2 is a table of the representation numbers $\mathcal{V}(j, D_4, t)$ for $j \leq 7$, omitted entries are zero. See Table 3 for the list of $B_0, B_1, \ldots, B_9$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>$B_0$</th>
<th>$B_1$</th>
<th>$B_2$</th>
<th>$B_3$</th>
<th>$B_4$</th>
<th>$B_5$</th>
<th>$B_6$</th>
<th>$B_7$</th>
<th>$B_8$</th>
<th>$B_9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>16</td>
<td>48</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>144</td>
<td>288</td>
<td>216</td>
<td>48</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>384</td>
<td>1488</td>
<td>864</td>
<td>288</td>
<td>144</td>
<td>432</td>
<td>240</td>
<td>288</td>
<td>48</td>
<td>16</td>
</tr>
</tbody>
</table>

Thus we have the following expansion:

$$
(\phi_{D_4}^* f)(\tau) = a(B_0)q^4 + (16a(B_0) + 48a(B_1))q^5
$$
$$
+ (144a(B_0) + 288a(B_1) + 216a(B_2) + 48a(B_3))q^6
$$
$$
+ (384a(B_0) + 1488a(B_1) + 864a(B_2) + 288a(B_3) + 144a(B_4) + 432a(B_5))q^7
$$
$$
+ 240a(B_6) + 288a(B_7) + 48a(B_8) + 16a(B_9))q^8 + \cdots.
$$

The function $\phi_{D_4}^* f \in S^1_{4k}(\Gamma_0(2))$ is invariant under the Fricke operator because $D_4^{-1}$ is equivalent to $D_4/2$, see Proposition 2.2. The ring $M_1(\Gamma_0(2))$ is generated by $E_{2,2}^- \in M_1^2(\Gamma_0(2))$ and $E_{4,2}^- \in M_1^4(\Gamma_0(2))$ and the ring of cusp forms is principally generated.
by $C_{8,2}^+ \in S_8^0(\Gamma_0(2))$. The $\pm$ superscript indicates an eigenvalue of $\pm1$ under the Fricke operator. In general we define $E_{k,d}^\pm(\tau) = (E_k(\tau) \pm d^{k/2}E_k(d\tau))/(1 \pm d^{k/2})$ where the $E_k(\tau) = 1 - (2k/B_k)\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$ are the Eisenstein series and the $B_k$ are given by $t/(e^t - 1) = \sum_{k=0}^{\infty}B_k t^k/k!$. We have $E_{k,d}^\pm \in M_k^0(\Gamma_0(d))$ except in the case of $E_{2,d}^\pm$. The Fourier expansions of these generators are given by

$$E_{2,2}^-(\tau) = 1 + 24\sum_{n=1}^\infty(\sigma_1(n) - 2\sigma_1(n/2))q^n = 1 + 24q + 24q^2 + 96q^3 + 24q^4 + 144q^5 + \ldots$$

$$E_{4,2}^-(\tau) = 1 - 80\sum_{n=1}^\infty(\sigma_3(n) - 4\sigma_3(n/2))q^n = 1 - 80q - 400q^2 - 2240q^3 - 2960q^4 - \ldots$$

$$C_{8,2}^+(\tau) = \frac{1}{256}(E_{2,2}^-(\tau)^4 - E_{4,2}^-(\tau)^2) = q - 8q^2 + 12q^3 + 64q^4 - 210q^5 - 96q^6 - \ldots$$

The vanishing order of $\phi_{D_4}f$ at the cusp $[\mathcal{I}]$ is at least $4$ and because $\phi_{D_4}f$ is an eigenfunction of the Fricke operator the vanishing order at the cusp $[\mathcal{J}]$ is the same. Thus we have $(C_{6,2}^+)^4 | \phi_{D_4}f$ in $M_1(\Gamma_0(2))$. For $k = 6$ this means $\phi_{D_4}f = 0$ and so every coefficient in equation 1.3 gives a homogeneous linear relation; in particular we must have $a(B_0) = 0$ (or $a(D_4/2) = 0$) and hence by Theorem 1.1 we have $S_4^6 = 0$. For $k = 8$ there is a parameter $c \in \mathbb{C}$ such that

$$\phi_{D_4}^*f = c(C_{6,2}^+)^4 = c(q^4 - 32q^5 + 432q^6 - 2944q^7 + 7192q^8 + 39744q^9 - \ldots).$$

Elimination of the parameter $c$ provides the following 3 linear relations for any $f \in S_4^8$.

$$a(B_0) + a(B_1) = 0;$$

$$-24a(B_0) + 4a(B_1) + 18a(B_2) + 4a(B_3) + 4a(B_4) = 0;$$

$$208a(B_0) + 93a(B_1) + 54a(B_2) + 18a(B_3) + 9a(B_4) + 27a(B_5)$$

$$+ 15a(B_0) + 18a(B_7) + 3a(B_8) + a(B_9) = 0. \quad (1.4)$$

As mentioned, the first relation alone, $a(D_4/2) + a(A_4/2) = 0$ (note $B_1 = A_4/2$), implies that $\dim S_4^8 \leq 1$.

For $k = 10$ there are parameters $\alpha, \beta \in \mathbb{C}$ such that $\phi_{D_4}^*f = (C_{8,2}^+)^4(\alpha(E_{2,2}^-)^4 + \beta C_{8,2}^+).$ The element $(E_{2,2}^-)^2E_{4,2}^-$ cannot occur in this representation because it has eigenvalue $-1$ under the Fricke operator. Elimination of the parameters $\alpha$ and $\beta$ provides two linear relations:

$$224a(B_0) = 184a(B_1) + 18a(B_2) + 4a(B_3) + a(B_4);$$

$$21376a(B_1) = -16110a(B_2) - 3916a(B_3) - 1231a(B_4) - 1512a(B_5) - 840a(B_6) - 1008a(B_7) - 168a(B_8) - 56a(B_9). \quad (1.5)$$

https://doi.org/10.1017/S0004972700020281 Published online by Cambridge University Press
In conjunction with Theorem 1.1 these two relations imply \( \dim S^0_4 \leq 8 \) but it will require another homomorphism \( \phi_H^*: S^0_4 \rightarrow S^0_1(\Gamma_0(6)) \) and a more extensive computation to prove that \( \dim S^0_4 \leq 1 \).

2. PROPOSITIONS

We let \( \Gamma_n = \text{Sp}_n(\mathbb{Z}) \). We write elements of \( \text{Sp}_n(\mathbb{R}) \) as \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \). The group \( \text{Sp}_n(\mathbb{R}) \) acts on functions from the right via \( \left( f \left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) \right)(\Omega) = \det (C\Omega + D)^{-k} \left( (A\Omega + B)(C\Omega + D)^{-1} \right) \).

**PROPOSITION 2.1.** Let \( n, \ell \in \mathbb{Z}^+ \). Let \( s, \ell s^{-1} \in \mathcal{P}_n(\mathbb{Z}) \). The map \( \phi^*_s : M^k_n \rightarrow M^k_1(\Gamma_0(\ell)) \) is a graded ring homomorphism.

**PROOF:** For \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}_1(\mathbb{R}) \) we have

\[
\left( \phi^*_s | \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)(\tau) = (a\tau + b)(cs^{-1}\tau + dI)^{1-k} = (a\tau + b)(cs^{-1}\tau + dI)^{1-k}
\]

If we now assume that \( s \in \Gamma_0(\ell) \) then \( cs^{-1} \) is integral and so \( \begin{pmatrix} aI & bs \\ cs^{-1} & dI \end{pmatrix} \in \text{Sp}_n(\mathbb{Z}) \). Therefore we have \( \left( f \left| \begin{pmatrix} aI & bs \\ cs^{-1} & dI \end{pmatrix} \right) \right)(\tau s) = f(\tau s) = \phi^*_s(f)(\tau) \). It is straightforward to see that \( \phi^*_s f \) is holomorphic on \( \mathcal{H}_1 \) and that it is bounded on domains of type \( \{ \tau \in \mathcal{H}_1 : \text{Im} \tau > y_0 \} \). Thus we have \( \phi^*_s : M^k_n \rightarrow M^k_1(\Gamma_0(\ell)) \).

For \( \ell \in \mathbb{Z}^+ \) let \( W_\ell = \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix} / \sqrt{\ell} \) denote the Fricke involution.

**PROPOSITION 2.2.** Let \( n, \ell \in \mathbb{Z}^+ \). Let \( s, \ell s^{-1} \in \mathcal{P}_n(\mathbb{Z}) \). Let \( f \in M^k_n \). Assume that \( s \) is \( \text{GL}_n(\mathbb{Z}) \)-equivalent to \( \ell s^{-1} \). Then \( \phi^*_s f \in M^k_1(\Gamma_0(\ell)) \) is an eigenfunction of the Fricke operator \( W_\ell \). The eigenvalue is \( +1 \) unless \( s \) is improperly equivalent to \( \ell s^{-1} \) and \( k \) is odd in which case \( \phi^*_s f \) has eigenvalue \( -1 \) under \( W_\ell \).

**PROOF:** When \( s \) is equivalent to \( \ell s^{-1} \) we have \( UsU' = \ell s^{-1} \) for some \( U \in \text{GL}_n(\mathbb{Z}) \). We shall show that \( \left( \phi^*_s f \right)|_{W_\ell} = \det(U)^k \phi^*_s f \). The factor \( \det(U)^k \) is one
except in the case noted. We first check that $\phi_\tau \circ W_\ell = \begin{pmatrix} 0 & U^* \\ -U & 0 \end{pmatrix} \circ \phi_\tau$. For every $\tau \in \mathcal{H}_1$ we have

$$(\phi_\tau \circ W_\ell)(\tau) = \phi_\tau \left( -\frac{1}{\ell} \right) = -\frac{1}{\ell} s\tau^{-1} = -U^*s^{-1}U^{-1}\tau^{-1} = U^*(-U\tau)^{-1}$$

$$= \begin{pmatrix} 0 & U^* \\ -U & 0 \end{pmatrix} (s\tau) = \left( \begin{pmatrix} 0 & U^* \\ -U & 0 \end{pmatrix} \circ \phi_\tau \right)(\tau).$$

Noting that $\begin{pmatrix} 0 & U^* \\ -U & 0 \end{pmatrix} \in \Gamma_n$ we compute

$$[(\phi_\tau^* f) \mid W_\ell]_{nk}(\tau) = \left( \sqrt{\ell} \tau \right)^{-nk} (\phi_\tau^* f)(W_\ell(\tau))$$

$$= \left( \sqrt{\ell} \tau \right)^{-nk} (f \circ \phi_\tau \circ W_\ell)(\tau)$$

$$= \left( \sqrt{\ell} \tau \right)^{-nk} \left( f \circ \begin{pmatrix} 0 & U^* \\ -U & 0 \end{pmatrix} \circ \phi_\tau \right)(\tau)$$

$$= \left( \sqrt{\ell} \tau \right)^{-nk} \det(-U\tau)^k f(\phi_\tau(\tau))$$

$$= \left( -\sqrt{\ell} \tau \right)^{-nk} \det(U)^k \det(s)^k (\phi_\tau^* f)(\tau)$$

$$= \det(U)^k (\phi_\tau^* f)(\tau).$$

In the last line above we have used the fact that $\det(s)^2 = \ell^n$ and that when $nk$ is odd we must have $f$ identically zero. \hfill \Box

The next Proposition shows how to develop the Fourier expansion of $\phi_\tau^* f$ at any cusp.

**Proposition 2.3.** Let $n \in \mathbb{Z}^+$. Let $s \in \mathcal{P}_n(\mathbb{Q})$. Let $f \in S^k_n$ have the Fourier expansion $f(\Omega) = \sum_t a(t)e((\Omega, t))$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$. There exist $A, B \in \mathbb{Q}^{n \times n}$ such that $\begin{pmatrix} aI & bs \\ cs^{-1} & dl \end{pmatrix} \in \Gamma_n \begin{pmatrix} A & B \\ 0 & A^* \end{pmatrix}$ and for any such $A, B$ we have

$$(\phi_\tau^* f \mid nk)(\begin{pmatrix} a & b \\ c & d \end{pmatrix})(\tau) = \det(A)^k f(\tau A^* + BA')$$

$$= \det(A)^k \sum_{j \in \mathbb{Q}^+} \left( \sum_{t: (A^*t, A't) = j} a(t)e((t, BA')) \right) t^j$$

**Proof:** We now wish to study $\left( \phi_\tau^* f \right) \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}_1(\mathbb{Z})$. Then as in the proof of Proposition 2.1 we have

$$\left( \phi_\tau^* f \mid nk \right) \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)(\tau) = \left( f \right) \left( \begin{pmatrix} aI & bs \\ cs^{-1} & dl \end{pmatrix} \right)(\tau s).$$

https://doi.org/10.1017/50004972700020281 Published online by Cambridge University Press
Now, we can always decompose any matrix in $\text{Sp}_n(\mathbb{Q})$ as something in $\text{Sp}_n(\mathbb{Z})$ times something in $\text{Sp}_n(\mathbb{Q})$ with $C = 0$ [8, p. 125]. So let 
\[
\left( \begin{array}{cc}
 aI & bs \\
 cs^{-1} & dI
\end{array} \right) \in \Gamma_n \left( \begin{array}{cc}
 A & B \\
 0 & A^* \end{array} \right).
\]
Since $f$ is automorphic with respect to $\Gamma_n$, we have
\[
(f|_{\text{Sp}_n(\mathbb{Q})} \left( \begin{array}{cc}
 aI & bs \\
 cs^{-1} & dI
\end{array} \right))(\tau) = (f|_{\text{Sp}_n(\mathbb{Z})} \left( \begin{array}{cc}
 A & B \\
 0 & A^* \end{array} \right))(\tau_s) = (\text{det} \, A)^k f(\tau As A^* + BA'). \]

The Fourier expansion for $(\phi^* f|_{\text{Sp}_n(\mathbb{Q})} \left( \begin{array}{cc}
 aI & bs \\
 cs^{-1} & dI
\end{array} \right))(\tau)$ follows from the Fourier expansion for $f$ under the substitution $\Omega = \tau As A^* + BA'$. \[\square\]

The above Proposition provides for the computation of the Fourier expansion of $\phi^* f|_{\sigma}$ in general. When $\ell$ is squarefree however the computation of the character $e(t, BA')$ may be finessed. We introduce a new notation: Notice that $A$ in Proposition 2.3 is determined up to $uA$ with $u \in \text{GL}_n(\mathbb{Z})$. Thus $As A'$ is determined up to equivalence class. We define
\[
s [ \left( \begin{array}{cc} 
 a & b \\
 c & d
\end{array} \right) ] = As A'
\]
with the understanding that this is well-defined only up to equivalence class. Since $f$ is automorphic with respect to $\left( \begin{array}{cc}
 u & 0 \\
 0 & u^*
\end{array} \right)$, we have $f(usu^* \tau) = f(\tau)$ and it makes sense to talk about $f\left( \left( s [ \left( \begin{array}{cc} 
 a & b \\
 c & d
\end{array} \right) ] \right) \tau \right)$ and $\phi^* \left( \left( \begin{array}{cc} 
 a & b \\
 c & d
\end{array} \right) \right) \tau$.

**PROPOSITION 2.4.** Let $s \in \mathcal{P}_n(\mathbb{Z})$. Let $\ell \in \mathbb{Z}^+$ such that $\ell s^{-1}$ is integral and primitive. Let $\left( \begin{array}{cc}
 a & b \\
 c & d
\end{array} \right) \in \Gamma_1$. Suppose $\text{gcd}(c, (\ell/c)) = 1$. Let $\overline{c} \in \mathbb{Z}$ such that $\overline{c} \equiv 1 \mod \ell/c$. For any $A$ with $\left( \begin{array}{cc} 
 aI & bs \\
 cs^{-1} & dI
\end{array} \right) \in \Gamma_n \left( \begin{array}{cc}
 A & B \\
 0 & A^* \end{array} \right)$ we have
\[
(\phi^* f|_{\text{Sp}_n(\mathbb{Q})} \left( \begin{array}{cc}
 a & b \\
 c & d
\end{array} \right))(\tau) = (\text{det} \, A)^k \phi^* \left( \left( \begin{array}{cc} 
 a & b \\
 c & d
\end{array} \right) \right) f(\tau + d\overline{c}).
\]

**PROOF:** We have $\left( \begin{array}{cc} 
 A & B \\
 0 & A^* \end{array} \right)\left( \begin{array}{cc} 
 aI & bs \\
 cs^{-1} & dI
\end{array} \right)^{-1} \in \text{Sp}_n(\mathbb{Z}).$ Thus we have
\[
\left( \begin{array}{cc} 
 A & B \\
 0 & A^* \end{array} \right) \left( \begin{array}{cc} 
 dl & -bs \\
 -cs^{-1} & aI
\end{array} \right) = \left( \begin{array}{cc} 
 dA - cBs^{-1} & -bAs + aB \\
 -cA^* s^{-1} & aA^* \end{array} \right) \in \text{Sp}_n(\mathbb{Z}).
\]
Note that each of the four blocks must be in $\mathbb{Z}^{n \times n}$. Multiplying $dA - cBs^{-1}$ by the integral $s$ implies $dAs - cB$ is integral. Both $As$ and $B$ are integral because we have

\[
As = a(dAs - cB) + c(-bAs + aB),
\]
\[
B = b(dAs - cB) + d(-bAs + aB).
\]

Since $cBs^{-1} = (c/\ell)B\ell s^{-1}$ and $\ell s^{-1} \in \mathbb{Z}^{n \times n}$, we have $cBs^{-1} \in (c/\ell)\mathbb{Z}^{n \times n}$. This combined with $dA - cBs^{-1} \in \mathbb{Z}^{n \times n}$ implies $dA \in (c/\ell)\mathbb{Z}^{n \times n}$. Also we have $A = (1/\ell)(As)\ell s^{-1} \in (1/\ell)\mathbb{Z}^{n \times n}$ and consequently $A = a(dA) - b(cA) \in (c/\ell)\mathbb{Z}^{n \times n}$. Since $As$ is integral, its transpose $sA'$ is also integral. Then multiplying $dA - cBs^{-1}$ by the integral $cBs'$ implies that $d\tilde{c}AsA'$ and $cBA'$ differ by an integer matrix. But $c \equiv 1 \mod (\ell/c)$ and $BA' \in (c/\ell)\mathbb{Z}^{n \times n}$ imply that $cBA'$ and $BA'$ differ by an integer matrix. Hence $d\tilde{c}AsA'$ and $BA'$ differ by an integer matrix. Finally, from Proposition 2.3 we have

\[
(d \det A)^{-k} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} (\tau) = f(\tau AsA' + BA')
\]
\[
= f(\tau AsA' + d\tilde{c}AsA')
\]
\[
= f(AsA'(\tau + d\tilde{c}))
\]
\[
= \phi^*(a, b) f(\tau + d\tilde{c}).
\]

3. THE SPACE $S_{4}^{10}$

We shall apply the technique of the Introduction to $S_{4}^{10}$. Theorem 1.1 says a form in $S_{4}^{10}$ is determined by its coefficients $a(t)$ with $w(t) \leq 3.5$. Table 3 gives the list of these 10 quadratic forms, see [10, 13]. For uniformity of notation we shall refer to these as $B_0, \ldots, B_9$. Here the number under $\ell$ for $B_i$ is the smallest positive integer such that $\ell(2B_i)^{-1}$ is integral.

We shall apply the technique to $H = 2B_4$ for which $6H^{-1}$ is integral. By Proposition 1.1 we have $\text{Im} \phi_H f \subset M_1(\Gamma_0(6))$ and our calculations will occur inside this ring. The ring $M_1(\Gamma_0(6))$ is generated by three forms $A$, $B$, $C$ of weight 2. There is one relation $C^2 = 9B^2 - 8A^2$. The ring of cusp forms is principally generated by a form of weight 4, $D = (A^2 - B^2)/4$. There are 4 cusps in $\Gamma_0(6) \backslash \Gamma_1/\Delta_1$, represented by $I$, $\sigma_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ and $J$ with respective widths 1, 3, 2 and 6. We now give the Fourier expansions of the generators at all four cusps. The definition of $E_{2,3}^-$ has already been given, similarly define

\[
E_{2,3}^- (\tau) = 1 + 12 \sum_{n=1}^{\infty} (\sigma_1(n) - 3\sigma_1(n/3))q^n = 1 + 12(q + 3q^2 + q^3 + 7q^4 + 6q^5 + \cdots).
\]
Restriction of Siegel modular forms

Define the following elements in $M_4(\Gamma_0(6))$:

\[
A(\tau) = (3/4)E_{2,2}(3\tau) + (1/4)E_{2,2}^{-}(\tau) = 1 + 6q + 6q^2 + 42q^3 + \cdots,
\]

\[
B(\tau) = (2/3)E_{2,3}(2\tau) + (1/3)E_{2,3}^{-}(\tau) = 1 + 4q + 20q^2 + 4q^3 + \cdots,
\]

\[
C(\tau) = (3/2)E_{2,2}(3\tau) - (1/2)E_{2,2}^{-}(\tau) = 1 - 12q - 12q^2 - 12q^3 + \cdots.
\]

Table 3. Semi-integral quaternary forms with dyadic trace $\leq 3.5$.

<table>
<thead>
<tr>
<th>Name</th>
<th>Form</th>
<th>Dyadic trace</th>
<th>16-Determinant</th>
<th>$\ell$</th>
</tr>
</thead>
</table>
| $B_0$ | $\frac{1}{2}$ \[
\begin{pmatrix}
1 & 2 & 1 & 1 \\
1 & 2 & 0 & 0 \\
1 & 0 & 0 & 2 \\
1 & 0 & 0 & 2
\end{pmatrix}
\] | 2 | 4 | 2 |
| $B_1$ | $\frac{1}{2}$ \[
\begin{pmatrix}
2 & 1 & 0 & 1 \\
0 & 0 & 2 & 1 \\
1 & 1 & 0 & 2 \\
0 & 0 & 1 & 2
\end{pmatrix}
\] | 2.5 | 5 | 5 |
| $B_2$ | $\frac{1}{2}$ \[
\begin{pmatrix}
2 & 1 & 0 & 0 \\
2 & 0 & 0 & 1 \\
1 & 1 & 0 & 2 \\
0 & 0 & 1 & 2
\end{pmatrix}
\] | 3 | 8 | 4 |
| $B_3$ | $\frac{1}{2}$ \[
\begin{pmatrix}
2 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 2
\end{pmatrix}
\] | 3 | 9 | 3 |
| $B_4$ | $\frac{1}{2}$ \[
\begin{pmatrix}
2 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 4 \\
0 & 0 & 1 & 2
\end{pmatrix}
\] | 3 | 12 | 6 |
| $B_5$ | $\frac{1}{2}$ \[
\begin{pmatrix}
2 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 1 & 2
\end{pmatrix}
\] | 3.5 | 12 | 6 |
| $B_6$ | $\frac{1}{2}$ \[
\begin{pmatrix}
2 & 1 & 1 & 0 \\
1 & 1 & 0 & 4 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 2
\end{pmatrix}
\] | 3.5 | 13 | 13 |
| $B_7$ | $\frac{1}{2}$ \[
\begin{pmatrix}
2 & 1 & 0 & 1 \\
2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 4
\end{pmatrix}
\] | 3.5 | 17 | 17 |
| $B_8$ | $\frac{1}{2}$ \[
\begin{pmatrix}
2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 4 \\
1 & 1 & 1 & 4
\end{pmatrix}
\] | 3.5 | 20 | 10 |
| $B_9$ | $\frac{1}{2}$ \[
\begin{pmatrix}
2 & 1 & 1 & 1 \\
1 & 2 & 0 & 1 \\
1 & 1 & 1 & 2 \\
1 & 1 & 2 & 4
\end{pmatrix}
\] | 3.5 | 25 | 5 |

The elliptic modular forms $A$, $B$, $C$ transform nicely as

\[
(A \mid J)(\tau) = -\frac{1}{6}A(\tau/6), \quad (A \mid \sigma_2)(\tau) = +\frac{1}{3}A((\tau - 1)/3),
\]

\[
(A \mid \sigma_3)(\tau) = -\frac{1}{2}A((\tau - 1)/2)
\]

\[
(B \mid J)(\tau) = -\frac{1}{6}B(\tau/6), \quad (B \mid \sigma_2)(\tau) = -\frac{1}{3}B((\tau - 1)/3)
\]

\[
(B \mid \sigma_3)(\tau) = +\frac{1}{2}B((\tau - 1)/2)
\]

\[
(C \mid J)(\tau) = +\frac{1}{6}C(\tau/6), \quad (C \mid \sigma_2)(\tau) = -\frac{1}{3}C((\tau - 1)/3)
\]

\[
(C \mid \sigma_3)(\tau) = -\frac{1}{2}C((\tau - 1)/2)
\]
We use Propositions 2.3 and 2.4 to work out the Fourier expansion of \( \phi_H^* f \mid \sigma \) for \( \sigma = I, \sigma_2, \sigma_3, J \). We implement the algorithms from [8, pp. 125, 322–328] to produce a factorisation
\[
\begin{pmatrix}
aI & bH \\
cH^{-1} & dI
\end{pmatrix} \in \Gamma_n \begin{pmatrix} A & B \\ 0 & A^* \end{pmatrix}
\]
and obtain \( \det(A) \) and \( H \Box \sigma = AH_\sigma A' \).

We display \( H \Box \sigma_2, H \Box \sigma_3, H \Box J \) and mention that the associated \( |\det(A)| \) equals 3, 4, 12, respectively:

\[
H \Box \sigma_2 = \frac{1}{3} \begin{pmatrix} 4 & 2 & 1 & -1 \\ 2 & 4 & -1 & 1 \\ 1 & -1 & 4 & -1 \\ -1 & 1 & -1 & 4 \end{pmatrix};
\]
\[
H \Box \sigma_3 = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 3 \end{pmatrix};
\]
\[
H \Box J = \frac{1}{6} \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 5 & -1 & 2 \\ 1 & -1 & 5 & 2 \\ 1 & 2 & 2 & 5 \end{pmatrix}.
\]

Note that all three of the cases \( \sigma_2, \sigma_3, J \) satisfy the hypotheses of Proposition 2.4.

Note that for \( c = 2 \), we can take \( \overline{c} = -1 \) so that \( cc = 1 \mod 3 \); for \( c = 3 \), we can take \( \overline{c} = -1 \) so that \( cc = 1 \mod 2 \). Thus we have
\[
(\phi_H^* f \mid I)(\tau) = \phi_H^* f(\tau),
\]
\[
(\phi_H^* f \mid \sigma_2)(\tau) = 3^{-10} \phi_H^* \sigma_2 f(\tau - 1),
\]
\[
(\phi_H^* f \mid \sigma_3)(\tau) = 4^{-10} \phi_H^* \sigma_3 f(\tau - 1),
\]
\[
(\phi_H^* f \mid J)(\tau) = 12^{-10} \phi_H^* J f(\tau).
\]

Hence the Fourier expansions may be computed from the numbers \( \mathcal{V}(j, H \Box \sigma, t) \) given in Tables 4, 5, 6 and 7. Among the computations we perform, the computation of these representation numbers is by far the most expensive.

<table>
<thead>
<tr>
<th>j</th>
<th>B_0</th>
<th>B_1</th>
<th>B_2</th>
<th>B_3</th>
<th>B_4</th>
<th>B_5</th>
<th>B_6</th>
<th>B_7</th>
<th>B_8</th>
<th>B_9</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>12</td>
<td>36</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>96</td>
<td>168</td>
<td>114</td>
<td>24</td>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>196</td>
<td>760</td>
<td>384</td>
<td>108</td>
<td>60</td>
<td>168</td>
<td>108</td>
<td>96</td>
<td>12</td>
<td>4</td>
</tr>
</tbody>
</table>

https://doi.org/10.1017/S0004972700020281 Published online by Cambridge University Press
Table 5. $\mathcal{V}(j, H \square \sigma_2, t)$

<table>
<thead>
<tr>
<th>$j$</th>
<th>$B_0$</th>
<th>$B_1$</th>
<th>$B_2$</th>
<th>$B_3$</th>
<th>$B_4$</th>
<th>$B_5$</th>
<th>$B_6$</th>
<th>$B_7$</th>
<th>$B_8$</th>
<th>$B_9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10/3</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11/3</td>
<td>4</td>
<td>12</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12/3</td>
<td>24</td>
<td>24</td>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13/3</td>
<td>12</td>
<td>96</td>
<td>24</td>
<td>12</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14/3</td>
<td>78</td>
<td>192</td>
<td>120</td>
<td>24</td>
<td>6</td>
<td>12</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15/3</td>
<td>144</td>
<td>312</td>
<td>192</td>
<td>24</td>
<td>28</td>
<td>84</td>
<td>48</td>
<td>36</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 6. $\mathcal{V}(j, H \square \sigma_3, t)$

<table>
<thead>
<tr>
<th>$j$</th>
<th>$B_0$</th>
<th>$B_1$</th>
<th>$B_2$</th>
<th>$B_3$</th>
<th>$B_4$</th>
<th>$B_5$</th>
<th>$B_6$</th>
<th>$B_7$</th>
<th>$B_8$</th>
<th>$B_9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6/2</td>
<td>5</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7/2</td>
<td>24</td>
<td>72</td>
<td>12</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8/2</td>
<td>120</td>
<td>264</td>
<td>138</td>
<td>48</td>
<td>15</td>
<td>12</td>
<td>12</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 7. $\mathcal{V}(j, H \square J, t)$

<table>
<thead>
<tr>
<th>$j$</th>
<th>$B_0$</th>
<th>$B_1$</th>
<th>$B_2$</th>
<th>$B_3$</th>
<th>$B_4$</th>
<th>$B_5$</th>
<th>$B_6$</th>
<th>$B_7$</th>
<th>$B_8$</th>
<th>$B_9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10/6</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11/6</td>
<td></td>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12/6</td>
<td>24</td>
<td>12</td>
<td>6</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13/6</td>
<td>24</td>
<td>96</td>
<td>12</td>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

From these expansions we see that $\phi^*_H f$ vanishes to order at least 6 at every cusp so that there are parameters $\alpha_0, \ldots, \alpha_8$ and $\beta_0, \ldots, \beta_7 \in \mathbb{C}$ such that

$$\phi^*_H f = (D)^6 (\alpha_0 A^8 + \alpha_1 A^7 B + \cdots + \alpha_8 B^8 + C(\beta_0 A^7 + \beta_1 A^6 B + \cdots + \beta_7 B^7)).$$

Without introducing any new parameters we also have equalities for any $\sigma \in \Gamma_1$:

$$\phi^*_H f \mid \sigma = (D \mid \sigma)^6 (\alpha_0 (A \mid \sigma)^8 + \cdots + \alpha_8 (B \mid \sigma)^8 + (C \mid \sigma)(\beta_0 (A \mid \sigma)^7 + \cdots + \beta_7 (B \mid \sigma)^7)).$$

For $\sigma = I, \sigma_2, \sigma_3, J$ the left side of equation 3.2 is computed from equation 3.1, equation 1.2 and Tables 4 through 7. The right side is computed from the expansions of the elliptic modular forms $A$, $B$ and $C$. At the cusp $[I]$ we equate the coefficients for
At the cusp \([\sigma_2]\) for \(j = 6/3, \ldots, 15/3\); at the cusp \([\sigma_3]\) for \(j = 6/2, \ldots, 8/2\) and at the cusp \([J]\) for \(j = 6/6, \ldots, 13/6\). Elimination of the 17 parameters \(\alpha_i, \beta_i\) from the \(4 + 10 + 3 + 8 = 25\) linear equations results in 8 linearly independent equations:

\[
\begin{align*}
    a(B_2) &= -86/21 a(B_0) - 188/21 a(B_1) \\
    a(B_3) &= 100/3 a(B_0) + 58/3 a(B_1) \\
    a(B_4) &= -300/7 a(B_0) + 24/7 a(B_1) \\
    a(B_5) &= -1892/21 a(B_0) + 568/21 a(B_1) \\
    a(B_6) &= 288/7 a(B_0) - 53/7 a(B_1) \\
    a(B_7) &= -1892/21 a(B_0) + 568/21 a(B_1) \\
    a(B_8) &= 288/7 a(B_0) - 53/7 a(B_1) \\
    a(B_9) &= -1892/21 a(B_0) + 568/21 a(B_1).
\end{align*}
\]

When we combine these 8 linear relations with the 2 linear relations in equation 1.5 obtained by considering \(\Phi^*_D\), we see that the rank is actually 9, so that we have a total of 9 linearly independent relations in \(a(B_0), \ldots, a(B_9)\):

\[
\begin{align*}
    a(B_1) &= 2a(B_0) \\
    a(B_2) &= -22a(B_0) \\
    a(B_3) &= 72a(B_0) \\
    a(B_4) &= -36a(B_0) \\
    a(B_5) &= -36a(B_0) \\
    a(B_6) &= 26a(B_0) \\
    a(B_7) &= -232a(B_0) \\
    a(B_8) &= 1200a(B_0) \\
    a(B_9) &= 2480a(B_0).
\end{align*}
\]

These relations and Theorem 1.1 imply that \(\dim S_{j}^{10} \leq 1\). Since we can come up with one nonzero cusp form \(G_{10}\) in \(S_{j}^{10}\) we have a theorem.

**Theorem 3.4.** We have \(\dim S_{j}^{10} = 1\) and \(S_{j}^{10} = \mathbb{C}G_{10}\).

4. **Final Comments**

The computations that have been performed for the form \(H\) are largely independent of the weight \(k\). Applied to the space \(S_{j}^{8}\) we may extend the Fourier expansion of the Schottky form \(J\) beyond that given in [1]. Table 8 gives the Fourier coefficients \(a(B_i)\) for \(J/2^{16}\) and \(G_{10}/2^{18}3^{4}5\).
Table 8. (Fourier Coefficients)

<table>
<thead>
<tr>
<th></th>
<th>$B_0$</th>
<th>$B_1$</th>
<th>$B_2$</th>
<th>$B_3$</th>
<th>$B_4$</th>
<th>$B_5$</th>
<th>$B_6$</th>
<th>$B_7$</th>
<th>$B_8$</th>
<th>$B_9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J/2^{16}$</td>
<td>1</td>
<td>-1</td>
<td>2</td>
<td>6</td>
<td>-12</td>
<td>-12</td>
<td>11</td>
<td>2</td>
<td>-72</td>
<td>116</td>
</tr>
<tr>
<td>$G_{10}/2^{18}3^{4}5$</td>
<td>1</td>
<td>2</td>
<td>-22</td>
<td>72</td>
<td>-36</td>
<td>-36</td>
<td>26</td>
<td>-232</td>
<td>1200</td>
<td>2480</td>
</tr>
</tbody>
</table>

Although the parameters $\alpha_i$ and $\beta_i$ were simply eliminated in Section 3, their values are also determined by this process. It may be of interest to present the images of $\phi_s^*f$ for $s = D_4, H$ and $f = J, G_{10}$.

\[
\phi_{D_4}^* J = 2^{16}(C_{8,2}^+)^4 \\
\phi_{D_4}^* G_{10} = 2^{18}3^45(C_{8,2}^+)^4 ((E_{2,2}^-)^4 + 48C_{8,2}^+) \\
\phi_H^* J = 2^{12}D^6(A + C)^4 \\
\phi_H^* G_{10} = 2^{14}3^35D^6(A + C)^4 (25A^4 - 8A^3B - 7A^3C - 8A^2BC - ABC^2 + 4AC^3 - BC^3 - C^4)
\]

It is interesting to note that the image of $G_{10}$ comes out to a multiple of the image of $J$ under both $\phi_{D_4}^*$ and $\phi_H^*$. As a final comment we note that linear relations among Fourier coefficients can be viewed as linear relations among Poincare series.

REFERENCES


