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STRUCTURAL PROPERTIES OF WEAK COTYPE 2 SPACES

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ABSTRACT. Several characterizations of weak cotype 2 and weak Hilbert spaces are given in terms of basis constants and other structural invariants of Banach spaces. For finite-dimensional spaces, characterizations depending on subspaces of fixed proportional dimension are proved.

1. **Introduction.** Results of this paper concern weak cotype 2 spaces and weak Hilbert spaces. Both classes are important in the local theory of Banach spaces, by virtue of their connection to the existence of large Euclidean subspaces. Recall that spaces of weak cotype 2 have been introduced by V. D. Milman and G. Pisier in [M-P] as spaces such that every finite dimensional subspace contains a further subspace of a fixed proportional dimension which is well Euclidean. This class has numerous other characterizations, either by geometric invariants combined with linear structure, or by inequalities between various ideal norms of related operators and their s-numbers (see *e.g.*, [P.3] and references therein).

On the other hand, recent results of the authors ([M-T.1], [M-T.3]) relate certain structural invariants of proportional dimensional quotients of a finite-dimensional space to volumetric invariants of the space; thus, via the well developed theory, to the existence of Euclidean subspaces or quotients. For example, finite-dimensional results in [M-T.3] imply, although it is not explicitly stated in the paper, that if a Banach space has the property that its all subspaces have a basis with a uniform upper bound for the basis constant, then the space is of weak cotype 2. This property is obviously much too strong to characterize Banach spaces of weak cotype 2; for instance, spaces L_p (with $p \neq 2$) contain subspaces without approximation property, hence without basis. A natural question then arises whether a weaker condition involving structural invariants of the same type can in fact characterize spaces of weak cotype 2. To put it more precisely, whether it is possible to replace in the original weak cotype 2 definition, the property of being "well Euclidean" by (much weaker) properties of having some structural invariants "well bounded".

One of the results of the present paper (Theorem 4.1 and the remark after Theorem 4.2) shows that this is indeed possible. There exists $\delta_0 > 0$ such that a Banach space X is of weak cotype 2 if and only if every finite dimensional subspace E contains a a subspace $E_0 \subset E$ with $\dim E_0 \geq \delta_0 \dim E$ such that certain structural invariants (of E_0) have a uniform upper bound. The invariants considered here are the basis constants or the complexification constants (for Banach spaces over reals only) or the symmetry constants or in fact some other related parameters.

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In the theory of proportional-dimensional subspaces of finite-dimensional spaces it is sometimes of interest to deduce properties of a fixed n-dimensional space X from an information on its all αn-dimensional subspaces, with a fixed proportion $0 < \alpha < 1$. A fundamental example is well known and follows from the theory of type and cotype: if all αn-dimensional subspaces of X are C-Euclidean than X itself is $f(C)$-Euclidean (*cf.*, *e.g.*, [T]). A recent more difficult example can be found in [B] and [M-T.1]. In Section 5 we study n-dimensional spaces X such that the basis constant of an arbitrary αn-dimensional subspace $E \subset X$ satisfies $\mathrm{bc}(E) \leq C$. This leads to proportional dimensional versions of a result mentioned above which follows from [M-T.3]. In particular we show that the bound $\mathrm{bc}(E), \mathrm{bc}(F) \leq C$ for all αn-dimensional subspaces E and all αn-dimensional quotients F of X implies that X is a weak Hilbert space, with the constant bounded above by a function of C.

The arguments in the paper are based on two related ideas. The first one is a technique developed in [M-T.1] and [M-T.3] of finding rather strange finite dimensional subspaces in spaces which fail to have weak cotype 2. In the dual setting, which is more convenient to use, it can be described as follows. First, for a finite dimensional Banach space E, using deep facts from the local theory of Banach spaces, we find a quotient F which can be placed in a special position in $\mathbb{R}^N$; and next we use a random (probabilistic) argument in order to prove that majority of quotients of F enjoys relative lack of well bounded operators. The second idea comes from [M-T.2], where the authors have proved that a random proportional-dimensional quotient F of l_1^n cannot be embedded into a Banach space F_1 with a nice Schauder basis and $\dim F_1 \leq (1+\delta)\dim E$ for some small fixed δ.

The paper is organized as follows. In Section 2 we collect a background material related to geometry and local theory of Banach spaces. In Section 3 we discuss spaces with few well bounded operators and related volumetric lower estimates. The main results of the paper are proved in Sections 4 and 5. Section 6 contains a proof of a random result which generalizes the result from [M-T.2] to arbitrary finite-dimensional Banach spaces.

We shall consider only Banach spaces over reals. The complex case can be dealt with in an analogous manner. Our notation will follow [P.3] and [T]. We also refer the reader to [P.3] for more details on Banach spaces of weak cotype 2.

2. **Preliminaries.** We will use the following geometric definition of the weak cotype 2 spaces. A Banach space X is of weak cotype 2 whenever there exist $0 < \delta_0 < 1$ and $D_0 \geq 1$ such that every finite-dimensional subspace E of X contains a subspace $\tilde{E} \subset E$ with $\dim \tilde{E} = k \geq \delta_0 \dim E$ and with the Banach-Mazur distance satisfying $\mathrm{d}(\tilde{E}, l_2^k) \leq D_0$. This definition is equivalent to the one most commonly used at present; in fact, it is shown in [P.3], Theorem 10.2, that the weak cotype 2 constant of X satisfies

$$wC_2(X) \leq C\delta_0^{-1}D_0, \tag{2.1}$$

where C is a universal constant.

Recall that a Banach space X is of weak type 2 whenever X^* is of weak cotype 2 and X is K-convex. In particular, the weak type 2 constant satisfies

$$wT_2(X) \leq K(X) wC_2(X^*). \tag{2.2}$$

A Banach space X is a weak Hilbert space if X is of weak type 2 and of weak cotype 2. We have no use of the technical definition of the weak Hilbert constant, let us just recall that this constant is controlled from above by $wT_2(X)$ and $wC_2(X)$. Finally, the following inequality is an easy consequence of the result of Pisier from [P.2], Corollary 9, which in turn are related to Pisier's deep K-convexity theorem. If X is a weak Hilbert space, then the K-convexity constant of X satisfies, for any $0 < \theta \leq 1$,

$$K(X) \leq C(\theta)\big(wT_2(X) wC_2(X)\big)^{\theta}. \tag{2.3}$$

Let $(E, \|\cdot\|)$ be an n-dimensional Banach space. For the Banach-Mazur distance $\mathrm{d}(E, L_2^n)$ from E to the Euclidean space l_2^n we will use a shorter notation of d_E. Fix a Euclidean norm $\|\cdot\|_2$ on E and identify E with $\mathbb{R}^n$ in such a way that $\|\cdot\|_2$ becomes the natural l_2-norm on $\mathbb{R}^n$. Let us recall that the volume ratio of E, $\mathrm{vr}(E)$, is defined by

$$\mathrm{vr}(E) = \big(\mathrm{vol}\, B_E / \mathrm{vol}\, \mathcal{E}_{\max}\big)^{1/n},$$

where $\mathcal{E}_{\max} \subset B_E$ is the ellipsoid of maximal volume contained in B_E.

More generally, for any ellipsoid $\mathcal{E} \subset B_E$, let $|\cdot|_2$ be the associated Euclidean norm, and let $\rho = (\mathrm{vol}\, B_E / \mathrm{vol}\, \mathcal{E})^{1/n}$. Szarek's volume ratio result (*cf.*, *e.g.*, [P.3] Theorem 6.1) says that for any $1 \leq k < n$, there exists a subspace $H \subset E$ with $\dim H = k$ such that

$$c\rho^{-n/(n-k)}|x|_2 \leq \|x\| \leq |x|_2 \quad \text{for } x \in H, \tag{2.4}$$

where $c > 0$ is a universal constant. It should be mentioned that some other volumetric invariants allow estimates with much better asymptotic dependence on $\lambda = k/n$, as $\lambda \longrightarrow 1$ (*cf.*, *e.g.*, [P.3]). However application of these more delicate methods would complicate proofs without making essential improvements to final inequalities.

For $k \leq n$ set

$$V_k(E) = V_k(B_E) = \sup\Big\{\big(\mathrm{vol}\, P_F(B_E) / \mathrm{vol}\, P_F(B_2^n)\big)^{1/k} \mid F \subset E, \dim F = k\Big\}.$$

This invariant is related to the notion of volume numbers of operators (*cf.* [P.3] Chapter 9). In particular, $V_k(E) \leq V_l(E)$ for $1 \leq l \leq k \leq n$.

The relevance of this invariant to the problem of finding Euclidean quotients of E is described by the following standard lemma. Its proof is based on Santalò inequality and the volume ratio method (2.4) used in the dual space E^*. We leave further details to the reader.

LEMMA 2.1. *Let $0 < \beta < 1$. Let $E = (\mathbb{R}^n, \|\cdot\|)$ be a Banach space such that $B_E \subset B_2^n$ and let $a > 0$ satisfies $V_{\beta n}(E) \geq a$. For every $\sigma > 0$ there is a quotient G of E such that $\dim G \geq \beta\sigma(1+\sigma)^{-1} n$ and $\mathrm{d}_G \leq Ca^{-(1+\sigma)}$, where $C \geq 1$ is a universal constant.*

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The following fact is an easy consequence of [P.3], Lemma 8.8. Let $\mathcal{E}$ be an ellipsoid on E with $\mathcal{E} \subset \mathcal{B}_E$ and let F be a quotient of E with $\dim F = \lambda n$ and with the quotient map $Q: E \longrightarrow F$. Then

$$(2.5) \qquad \big(\operatorname{vol} Q(B_E)/\operatorname{vol} Q(\mathcal{E})\big)^{1/\lambda n} \leq a(\lambda)\big(\operatorname{vol} B_E/\operatorname{vol} \mathcal{E}\big)^{1/\lambda n},$$

where $a(\lambda) \geq 1$ depends on λ only. In particular, $\operatorname{vr}(F) \leq a(\lambda)\operatorname{vr}(E)^{1/\lambda}$.

The next lemma shows that given a finite-dimensional Banach space one can dramatically improve geometric properties of its unit ball by passing to quotients of proportional dimensions. The argument is based on several deep results in the local theory of Banach spaces ([Mi], [B-S], *cf.* also [P.3]). The lemma implicitly uses the ellipsoid of minimal volume containing the unit ball B_E of a given n-dimensional space E; in fact it is concerned with the following property of a Euclidean norm $|\cdot|_2$ on E: there exists $c > 0$ such that every rank k orthogonal projection P in $(E, |\cdot|_2)$ satisfies

$$(2.6) \qquad \|P: E \longrightarrow (E, |\cdot|_2)\| \geq c(k/n)^{1/2}.$$

The norm $|||\cdot|||$ associated to the minimal volume ellipsoid satisfies (2.6) with $c = 1$ (*cf.*, *e.g.*, [T] Proposition 3.2.10).

Note that if E is a finite-dimensional Banach space and $|\cdot|_2$ is an Euclidean norm on E then each quotient map $Q: E \longrightarrow F$ induces on F in a natural way an Euclidean norm $|\cdot|_{2,F}$. Namely, we set $B_{2,F} = Q(B_{2,E})$, where $B_{2,F}$ and $B_{2,F}$ stand for Euclidean balls in F and E respectively.

LEMMA 2.2. *For every $0 < \lambda < 1$ there is $\rho = \rho(\lambda) \geq 1$ and for every $c > 0$ there is $\tilde{\kappa} = \tilde{\kappa}(\lambda, c) \geq 1$, such that the following conditions hold for any n-dimensional Banach space E:*

(i) There exists a Euclidean norm $|\cdot|_2$ on E, with the unit ball B_2, such that

$$(2.7) \qquad (2^{1/2}\mathrm{d}_E)^{-1}B_2 \subset B_E \subset B_2$$

and it satisfies (2.6) with $c = 2^{-1/2}$;

(ii) there exists a λn-dimensional quotient F of E satisfying $\operatorname{vr}(F) \leq \rho$*;*

(iii) if $|\cdot|_2$ is a Euclidean norm on E satisfying (2.6) for some $c > 0$, then there exists a λn-dimensional quotient $\tilde{F}$ of E and an orthonormal basis $\{x_i\}$ in $(\tilde{F}, |\cdot|_2)$ such that

$$(2.8) \qquad \max_i \|x_i\|_{\tilde{F}} \leq \tilde{\kappa}.$$

PROOF. The proof of (ii) and (iii) was already given in [M-T.1], Proposition 3.5. As for (i), the Euclidean norm described in this condition combines properties of the norm $|||\cdot|||$ associated with the ellipsoid of minimal volume and a norm which determines the Euclidean distance d_E. Indeed, let $|||\cdot|||'$ be a norm satisfying

$$|||x|||' \leq \|x\| \leq \mathrm{d}_E|||x|||' \quad \text{for } x \in E.$$

Then for $x \in E$ set

$$|x|_2 = 2^{-1/2}(|||x|||^2 + |||x|||'^2)^{1/2}.$$

Clearly, $|x|_2 \leq \|x\| \leq 2^{1/2}\mathrm{d}_E|x|_2$, for $x \in E$, hence (2.7) holds. To prove that $|\cdot|_2$ satisfies (2.6), observe that for every $x \in E$ and every rank k orthogonal projection P in $(E, |\cdot|_2)$ one has

$$\begin{aligned} |Px|_2 &= 2^{-1/2}(|||Px|||^2 + |||Px|||'^2)^{1/2} \\ &\geq 2^{-1/2}|||Px||| \geq 2^{-1/2}|||P_1Px||| = 2^{-1/2}|||P_1x|||, \end{aligned}$$

where P_1 is the orthogonal projection in $(E, |||\cdot|||)$ with $\ker P_1 = \ker P$ and next apply (2.6) in the space $(E, |||\cdot|||)$. ∎

If F is a finite-dimensional space and B_2 is a Euclidean ball on F, then, for a fixed orthonormal basis $\{x_i\}$ in F, by B_1 we shall denote *abs* $\mathrm{conv}\{x_i\}$.

COROLLARY 2.3. *For every* $0 < \lambda < 1$ *there is* $\rho = \rho(\lambda) \geq 1$ *and* $\kappa = \kappa(\lambda) \leq 1$, *such that an arbitrary n-dimensional Banach space E satisfies:*

(i) there exist a λn*-dimensional quotient F of E and a Euclidean ball* B_2 *on F such that* $\mathrm{vr}(F) \leq \rho$ *and*

$$\kappa B_1 \subset B_F \quad \textit{and} \quad (2^{1/2}\mathrm{d}_E)^{-1}B_2 \subset B_F \subset B_2.$$

(ii) there exist a $\lambda n/2$*-dimensional quotient F of E and a Euclidean ball* B_2 *on F such that for some* $(2^{1/2}\mathrm{d}_E)^{-1} \leq a \leq 1$ *we have*

$$\kappa B_1 \subset B_F, \quad aB_2 \subset B_F \subset B_2, \quad \big(\mathrm{vol}\, B_F/\mathrm{vol}\,(aB_2)\big)^{2/\lambda n} \leq \rho.$$

PROOF. Condition (i) follows directly from Lemma 2.2 by chosing a Euclidean norm on E satisfying (2.7) and then passing twice to quotient spaces satisfying the conclusions of Lemma 2.2(ii) and (iii). Notice that a Euclidean ball B_2 on E determines the natural Euclidean ball on every quotient F on E, and if B_2 satisfies (2.6) and $\dim F$ is proportional to $\dim E$ then the ball on F satisfies (2.6) as well, with the constant depending on the proportion λ. The estimate for volume ratio follows from Lemma 2.2 (ii) by using (2.5). Note, however, that the resulting function $\rho(\lambda)$ is a power, depending on λ, of the upper bound from Lemma 2.2(ii).

To get (ii), fix $0 < \lambda' < 1$ to be determined later. First pass to a $\lambda' n$-dimensional quotient F' of E satisfying Lemma 2.2(i) and (ii). Let $\mathcal{E}$ be the maximal volume ellipsoid on F'; there exists a quotient F'' of F' with $\dim F'' = \dim F'/2$ and the quotient map $Q: F' \longrightarrow F''$ such that on F'' we have $Q(\mathcal{E}) = aQ(B_2)$, for some a. Clearly, $a \leq 1$ and (i) implies that $a \geq (2^{1/2}\mathrm{d}_E)^{-1}$. Passing to a quotient F of F'' with $\dim F = \lambda' \dim F''$, and using Lemma 2.2(iii), we get all required inclusions; the bound for the ratio of volumes follows from (2.5). Given $1/2 < \lambda < 1$ choose λ' such that $\dim F = \lambda n$, then all constants involved will depend on λ. Again, the function $\rho(\lambda)$ is a power of the upper bound from Lemma 2.2(ii). ∎

Volumetric techniques for finding Euclidean sections provide Euclidean subspaces of small proportional dimensions. The next lemma describes a method from [M-P] of constructing Euclidean subspaces of large proportional dimensions in spaces saturated with small Euclidean ones. The proof of the statement below can be found in [M-T.1] Theorem 4.2.

LEMMA 2.4. *Let* $0 < \delta < \xi < 1$. *Let* Z *be an* N*-dimensional space such that every* ξN*-dimensional subspace* Z_1 *of* Z *contains a subspace* H *with* $\dim H \geq (\xi - \delta)N$ *such that* $\mathrm{d}_H \leq D$, *for some* $D \geq 1$. *Then for every* $0 < \eta < 1 - \delta$ *there exists a subspace* $\tilde{H}$ *of* Z *with* $\dim \tilde{H} \geq (1 - \delta - \eta)N$ *such that* $\mathrm{d}_{\tilde{H}} \leq cD$, *where* $c = c(\xi, \delta, \eta)$.

Let E be an n-dimensional Banach space and let $\|\cdot\|_2$ be a Euclidean norm on E. Recall that an operator $T\colon E \longrightarrow E$ is said to be (k, β)*-mixing* for $k, \beta \geq 0$, if and only if there is a subspace $F \subset E$ with $\dim F \geq k$ such that $|P_{F^\perp} Tx|_2 \geq \beta |x|_2$ for every $x \in F$, where $P_{F^\perp}$ denotes the the orthogonal projection onto $F^\perp$. If this is the case then we write $T \in \mathrm{Mix}_n(k, \beta)$. The fact whether a fixed operator T is in $\mathrm{Mix}_n(k, \beta)$ may depend on the choice of the Euclidean structure on E. Clearly, if $k \geq \ell$, then $\mathrm{Mix}_n(k, \beta) \subset \mathrm{Mix}_n(\ell, \beta)$. The following proposition is a folklore one (*cf.*, *e.g.*, [Sz.2] Lemma 3.4A, [Ma.2]).

PROPOSITION 2.5. *Let* E *be an* n*-dimensional Banach space. For an arbitrary Euclidean norm* $|\cdot|_2$ *on* E *one has*

(i) *for every projection* P *of rank* $k \leq n/2$, *we have* $2P \in \mathrm{Mix}_n(k, 1)$;

(ii) *if the basis constant* $\mathrm{bc}(E) \leq M$ *then for every* $k \in \mathbb{N}$ *there is an operator* $T\colon E \longrightarrow E$ *with* $\|T\| \leq 2M$ *and* $T \in \mathrm{Mix}_n(k, 1)$;

(iii) *let* $T\colon E \longrightarrow E$, $k \in \mathbb{N}$ *and* $\beta \geq 0$. *Then* $T \in \mathrm{Mix}_n(k, \beta)$ *if and only if* $T^* \in \mathrm{Mix}_n(k, \beta)$ *(with respect to the dual Euclidean norm* $|\cdot|_2^*$*).*

3. **Volumetric estimates.** Technical result which this paper is based upon yields the existence, for a given finite-dimensional Banach space, of a quotient space, say F, of proportional dimension which admits relatively few well bounded operators. Moreover the same property is satisfied in any further quotient F_0 of F and in any space $\tilde{F}$ which admits F as its quotient, provided that the dimension of the new spaces is close enough to the dimension of F.

In this section we shall work with an N-dimensional Banach space $E = (\mathbb{R}^N, \|\cdot\|)$, on which we always consider the (natural) Euclidean norm $\|\cdot\|_2$ and the associated Euclidean ball B_2^N. If F is a quotient of E, with the quotient map $Q\colon E \longrightarrow F$, then the natural Euclidean norm on F has the unit ball $Q(B_2^N)$. Unless otherwise stated, these natural Euclidean norms are used for all geometric invariants.

First we discuss quotients which admit lower estimates for norms of mixing operators. The main analytic estimate is stated in the following theorem; the proof involves a random construction and it is postponed until Section 6.

THEOREM 3.1. *For an arbitrary* $0 < \delta < 1$, $0 < \eta < 3/8$ *and* $0 < \varepsilon < 2^{-5}\eta$, *set* $\gamma = 2^{-5}\delta\varepsilon\eta$, *and for* $n \in \mathbb{N}$ *set* $N = (1 + \varepsilon)n$. *Let* $E = (\mathbb{R}^N, \|\cdot\|)$ *be an* N*-dimensional*

Banach space such that $B_E \subset B_2^N$. *Let* $\rho \geq 1$, $0 < \kappa \leq 1$ *and* $0 < a \leq 1$ *satisfy*

$$\text{(3.1)} \qquad \mathrm{vr}(E) \leq \rho, \quad \kappa B_1^N \subset B_E, \quad aB_2^N \subset B_E.$$

Then E *admits an* n*-dimensional quotient* F *such that for every operator* $T: F \longrightarrow F$ *with* $T \in \mathrm{Mix}_n(\eta n, 1)$, *and every quotient map* $Q: F \longrightarrow Q(F)$ *with* $\mathrm{rank}\, Q \geq (1-\gamma)n$, *one has*

$$\text{(3.2)} \qquad \|QT: F \longrightarrow Q(F)\| \geq cV_{\eta n/4}(E)^{-1}a^{\delta},$$

where $c = c(\varepsilon, \eta, \kappa, \rho) > 0$.

REMARK. Theorem 3.1 remains valid also for $\delta = 0$. In this case we have $\gamma = 0$ (note that the constant c *does not* depend on γ) and the theorem reduces to Theorem 2.2 from [M-T.3].

Formula 3.2 implies that the space F as well as its further quotients F_0 admit relatively few well bounded operators. Indeed, one can formally deduce from it well bounded operators on these spaces are small perturbations of a multiple of the identity operator.

PROPOSITION 3.2. *With the same notation as used in Theorem 3.1 if* $E = (\mathbb{R}^N, \|\cdot\|)$ *is an* N*-dimensional Banach space such that* $B_E \subset B_2^N$ *and satisfying (3.1), then* E *admits an* n*-dimensional quotient* F *which satisfies the following two conditions:*

(i) for every quotient F_0 *of* F *with* $\dim F_0 = k \geq (1-\gamma)n$ *and every operator* $T: F_0 \longrightarrow F_0$, *with* $T \in \mathrm{Mix}_k(\eta_0 k, 1)$, *where* $\eta_0 = \eta/(1-\gamma)$, *we have*

$$\text{(3.3)} \qquad \|T\| \geq cV_{\eta n/4}(E)^{-1}a^{\delta},$$

(ii) every Banach space $\tilde{F}$ *with* $\dim \tilde{F} = l \leq (1+\gamma)n$ *such that* F *is a quotient of* $\tilde{F}$ *admits a Euclidean norm such that every operator* $\tilde{T}: \tilde{F} \longrightarrow \tilde{F}$, *which is* $(\tilde{\eta} l, 1)$*-mixing with respect to this norm, where* $\tilde{\eta} = (\eta + 2\gamma)$, *satisfies*

$$\text{(3.4)} \qquad \|\tilde{T}\| \geq cV_{\eta n/4}(E)^{-1}a^{\delta}.$$

Here $c = c(\varepsilon, \eta, \kappa, \rho) > 0$.

PROOF. Let F be the quotient of E satisfying Theorem 3.1. In particular, F admits the natural Euclidean norm inherited from E. Set

$$K = \inf_Q \inf\{\|Q\tilde{T}: F \longrightarrow Q(F)\| \mid \tilde{T}: F \longrightarrow F, \tilde{T} \in \mathrm{Mix}_n(\eta n, 1)\},$$

where the first infimum runs over all quotient maps $Q: F \longrightarrow Q(F)$ with $\mathrm{rank}\, Q \geq (1-\gamma)n$. To prove (i), pick a quotient $F_0 = F/G_0$, with $\dim F_0 = k \geq (1-\gamma)n$. Identify F_0 with the linear subspace $G_0^{\perp} \subset F$, under the norm whose unit ball is $Q_{G_0}(B_F)$; here Q_{G_0} is the orthogonal projection with $\ker Q_{G_0} = G_0$. Fix an arbitrary $(\eta' k, 1)$-mixing operator $T: F_0 \longrightarrow F_0$. Pick any $\tilde{T}: F \longrightarrow F$ such that $Q_{G_0}\tilde{T} = TQ_{G_0}$. Since for $x \in G_0^{\perp}$ we have $\tilde{T}x = Tx + z$, for some $z \in G_0$, then $\tilde{T} \in \mathrm{Mix}_n(\eta' k, 1)$. Since $\eta' k \geq \eta n$, then $\tilde{T} \in \mathrm{Mix}_n(\eta n, 1)$. Thus, by the definition of K we have

$$\|T: F_0 \longrightarrow F_0\| = \|Q_{G_0}TQ_{G_0}: F \longrightarrow F_0\| = \|Q_{G_0}\tilde{T}: F \longrightarrow F_0\| \geq K,$$

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and (i) by the estimate (3.2).

The proof of (ii) is very similar. Fix an l-dimensional space $\tilde{F}$ with $l \leq (1+\gamma)n$ such that $q_F:\tilde{F} \longrightarrow F$ is the quotient map. Consider an arbitrary Euclidean norm on $\tilde{F}$, with the unit ball $\tilde{B}_2$, such that $q_F(\tilde{B}_2)$ is the natural Euclidean ball on F. Let $\tilde{T}:\tilde{F} \longrightarrow \tilde{F}$ be $(\tilde{\eta}l, 1)$-mixing. If $T:F \longrightarrow F$ satisfies $Tq_F = q_F\tilde{T}$ then clearly, $\|\tilde{T}\| \geq \|q_F\tilde{T}\| = \|T\|$. Moreover, since $\dim\ker q_F \leq \gamma n$, then $T \in \mathrm{Mix}_n(\tilde{\eta}l - 2\gamma n, 1) \subset \mathrm{Mix}_n(\eta n, 1)$. Thus $\|T\| \geq K$ and the lower estimate for $\|\tilde{T}\|$ follows. ■

It is well known that if for a space F all mixing operators have large norms, then F itself and other related spaces have several structural invariants, such as basis constant or symmetry constant or complexification constant, also bounded below (*cf., e.g.*, [M-T.3] Section 6). We give an example of an estimate of this type.

COROLLARY 3.3. *For an arbitrary* $0 < \varepsilon < 2^{-13}$ *and* $0 < \delta < 1$, *set* $\gamma = 2^{-13}\varepsilon\delta$, *and for* $n \in \mathbb{N}$ *set* $m = (1+2\varepsilon)n$. *Let E be an m-dimensional Banach space and let* $\mathrm{d}_E = \mathrm{d}(E, \ell_2^m)$. *There exists an n-dimensional quotient F of E such that for any further quotient F_0 of F with* $\dim F_0 \geq (1-\gamma)n$, *and for any space $\tilde{F}$ with* $\dim\tilde{F} \leq (1+\gamma)n$, *for whose F is a quotient, denoting by F' either F_0 or $\tilde{F}$, we have*

$$\mathrm{bc}(F') \geq cV_{2^{-10}n}(E)^{-1}\mathrm{d}_E^{-\delta}. \tag{3.5}$$

Moreover, for any Banach space Z such that every n-dimensional subspace of Z is D-Euclidean we have

$$\mathrm{bc}(F' \oplus_2 Z) \geq cD^{-1}V_{2^{-10}n}(E)^{-1/2}\mathrm{d}_E^{-\delta/2}. \tag{3.6}$$

Here $c = c(\varepsilon, \delta) > 0$.

The choice of $\eta = 2^{-8}$ made below when applying Proposition 3.2 was done for convenience of past references. With appropriate modifications the same argument would work for an arbitrary $0 < \eta < 3/8$.

PROOF. By Corollary 2.3(i), there exists a $(1+\varepsilon)n$-dimensional quotient $\tilde{E}$ of E and a Euclidean ball on $\tilde{E}$, $\tilde{B}_2 \supset B_{\tilde{E}}$, such that (3.1) holds, with some $\rho = \rho(\varepsilon)$, $\kappa = \kappa(\varepsilon)$ and $a = (2^{1/2}\mathrm{d}_E)^{-1}$.

Fix $\eta = 2^{-8}$. Then operators on F' which are $(5 \cdot 2^{-9}\dim F', 1)$-mixing are also $(\eta'\dim F', 1)$-mixing, where $\eta' = \eta_0$ or $\eta' = \tilde{\eta}$, depending on the choice of F' being F_0 or $\tilde{F}$. Therefore, by Proposition 3.2, these operators have norms bounded below by $K = cV_{2^{-10}n}(E)^{-1}\mathrm{d}_E^{-\delta}$. Thus the conclusion follows from Theorem 2.1 in [M-T.3]. ■

REMARK. The quotient space F itself satisfies (3.5) and (3.6) as well, with $\delta = 0$. This was the content of Theorem 2.4 in [M-T.3], and it followed from the construction in [M-T.3], Theorem 2.2, which preceded the present construction.

It is of independent interest to study a relationship between various s-numbers of operators acting in spaces discussed in Theorem 3.1. The advantage of this approach lies in the fact that resulting estimates are valid for all operators, and not only for mixing ones.

Let us recall relevant definitions. Let X and Y be Banach spaces and let $T: X \longrightarrow Y$ be a bounded operator. Let k be a positive integer. The k-th Kolmogorov number $d_k(T)$ is defined by

$$d_k(T) = \inf_{Z \subset Y} \sup_{x \in B_X} \inf_{y \in Z} \|Tx - y\|,$$

where the infimum on Z runs over all subspaces Z of Y with $\dim Z < k$. The dual concept is that of Gelfand numbers which are defined by

$$c_k(T) = \inf\{\|T|_Z\| \mid Z \subset X, \operatorname{codim} Z < k\}.$$

We have $c_k(T) = d_k(T^*)$ for arbitrary X and Y and T.

As a consequence of Theorem 3.1 we get.

PROPOSITION 3.4. *For an arbitrary* $0 < \varepsilon < 2^{-10}$ *and* $0 < \delta < 1$ *set* $\gamma = 2^{-10}\delta\varepsilon$, *and for* $n \in \mathbb{N}$ *set* $N = 2(1+2\varepsilon)n$. *Let* $E = (\mathbb{R}^N, \|\cdot\|)$ *be an* N*-dimensional Banach space. Then* E *admits an* n*-dimensional quotient* F *such that for every operator* $T: F \longrightarrow F$ *we have*

$$d_{\gamma n}(T) \geq c V_{2^{-7}n}(E)^{-1} \mathrm{d}_E^{-\delta} \inf_{\lambda \in \mathbb{R}} c_{n/4}(T - \lambda\, \mathrm{Id}_F),$$

where $c = c(\varepsilon) > 0$.

PROOF. Let F_0 be a $(1+\varepsilon)n$-dimensional quotient of E satisfying condition (ii) of Corollary 2.3, for a Euclidean ball B_2 and some $0 < \kappa = \kappa(\varepsilon) \leq 1$, $\rho = \rho(\varepsilon)$ and $0 < a \leq 1$. Let F be a quotient of F_0 satisfying (3.2) of Theorem 3.1, with $\eta = 2^{-5}$. Let $Q: F_0 \longrightarrow F$ be the quotient map. Let be the Euclidean norm on F corresponding to $Q(B_2)$; in particular, $\|x\| \leq a^{-1}\|x\|_2$ for $x \in F$. Then (2.5) implies that

$$\left(\operatorname{vol} B_F / \operatorname{vol} Q(aB_2^N)\right)^{1/n} \leq C(\varepsilon)\rho^{1+\varepsilon}.$$

By the volume ratio argument, pick a $15n/16$-dimensional subspace H of F such that

$$Aa^{-1}\|x\|_2 \leq \|x\| \leq a^{-1}\|x\|_2 \quad \text{for } x \in H, \tag{3.7}$$

with $A = A(\varepsilon) = (C(\varepsilon)\rho^{1+\varepsilon})^{16}$.

Fix an arbitrary $T: F \longrightarrow F$ satisfying

$$\inf\{\|(T - \lambda\, \mathrm{Id})|_G\|_2 \mid \lambda \in \mathbb{R}, G \subset F, \dim G = 7n/8\} = 1.$$

Pick $G_0 \subset F$ with $\dim G_0 = 7n/8$ and λ_0 such that $\|(T - \lambda_0\, \mathrm{Id})|_{G_0}\|_2 \leq 2$. Set $G_1 = \{x \in H \cap G_0 \mid Tx \in H\}$. Then $\dim G_1 \geq 3n/4$. By (3.7) one has

$$\|(T - \lambda_0\, \mathrm{Id})x\| \leq 2A\|x\| \quad \text{for } x \in G_1.$$

Hence $\|(T - \lambda_0\, \mathrm{Id})|_{G_1}\| \leq 2A$, which means

$$\inf_{\lambda \in \mathbb{R}} c_{n/4}(T - \lambda\, \mathrm{Id}_F) \leq 2A. \tag{3.8}$$

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On the other hand, the normalization condition for T and Lemma 2.1 in [Ma.3] imply that $8\,T$ is $(2^{-5}n, 1)$-mixing. Denoting the right hand side of (3.2) by K, we get

$$\inf \|QT\colon F \longrightarrow Q(F)\| \geq K/8,$$

with the infimum taken over all quotient maps $Q\colon F \longrightarrow Q(F)$ with rank $Q \geq (1-\gamma)n$. By the definition of Kolmogorov numbers this means that $d_{\gamma n}(T) \geq K/8$. Combining this estimate with (3.8) we conclude the proof. ■

4. **Characterizations of infinite-dimensional spaces.** As mentioned in the introduction, if a Banach space X has the property that its all subspaces have a basis with a uniform upper bound for the basis constant, then X is of weak cotype 2. In fact, if there exists $M < \infty$ such that $\mathrm{bc}(E) \leq M$ for every subspace E of X, then $wC_2(X)$ admits an upper estimate by a function of M. Indeed, let F be an arbitrary finite-dimensional quotient of X^* and consider the Euclidean structure of F determined by the ellipsoid of minimal volume containing the unit ball B_F. Then the dual version of Theorem 2.4 in [M-T.3] implies that $V_{\beta n}(F) \geq c/M$, for some universal constants $\beta > 0$ and $c > 0$. Then the conclusion follows immediately from Lemma 2.1, by passing back to the space X.

A similar general line of argument is used to prove characterizations of weak cotype 2 spaces in terms of the basis constant; we also obtain related characterizations in terms of an existence of uniformly bounded projections and mixing operators.

THEOREM 4.1. *There exists a constant $\gamma_0 > 0$ such that a Banach space X is of weak cotype 2 if and only if there exist a constant $M \geq 1$ such that every finite-dimensional subspace E of X contains a subspace $E_0 \subset E$ with $k = \dim E_0 \geq (1-\gamma_0)\dim E$ satisfying one of the following conditions:*

(i) $\mathrm{bc}(E_0) \leq M$,

(ii) there exists a projection $Q\colon E_0 \longrightarrow E_0$ of rank $k/8$ such that $\|Q\| \leq M$,

(iii) for every Euclidean norm on E_0 there exists an operator $T\colon E_0 \longrightarrow E_0$ which is $(k/8, 1)$-mixing with respect to this norm, such that $\|T\| \leq M$.

Moreover, if one of the conditions (i)–(iii) holds then $wC_2(X) \leq CM^4$, where C is a universal constant.

The counterpart of this result for superspaces is less satisfactory, as it gives implication in one direction only.

THEOREM 4.2. *There exists a constant $\gamma_0 > 0$ such that whenever X is a Banach space for which there exist a constant $M \geq 1$ such that for every finite-dimensional subspace E of X there is a Banach space $\tilde{E} \supset E$ with $k = \dim\tilde{E} \leq (1+\gamma_0)\dim E$ satisfying one of the following conditions*

(i) $\mathrm{bc}(\tilde{E}) \leq M$,

(ii) there exists a projection $Q\colon \tilde{E} \longrightarrow \tilde{E}$ of rank $k/8$ such that $\|Q\| \leq M$,

(iii) for every Euclidean norm on $\tilde{E}$ there exists an operator $T\colon \tilde{E} \longrightarrow \tilde{E}$ which is $(k/8, 1)$-mixing with respect to this norm, such that $\|T\| \leq M$,

then X is of weak cotype 2 and $wC_2(X) \leq CM^4$, where C is a universal constant.

REMARK. Beside conditions (i) and (ii) of the above theorem there is a number of other invariants whose uniform boundedness implies condition (iii). They are symmetry constant (*cf.* [Ma.1], [Ma.2]), complexification constant (for real Banach spaces) or the Banach-Mazur distance from the space to its complex conjugate (for complex Banach spaces) (*cf.* [Sz.2]). All these and other invariants could be used for versions of all theorems of this section (*cf.* also [M-T.3], Section 6).

REMARK. Note that there is an essential difference between the Euclidean case in the definition of weak cotype 2 and the results above. Namely, we do not know whether analogous characterizations are valid for an *arbitrary* proportion $\delta \in (0,1)$ (not necessary $\delta > 1-\gamma_0$). Recall that it is so in the Euclidean case: if for a Banach space X there exists $\delta_0 > 0$ such that every finite dimensional subspace E of X contains a C_0-Euclidean subspace $E_0 \subset E$ with $\dim E_0 \geq \delta_0 \dim E$, then an analogous condition holds for every $\delta \in (0,1)$, with the constant C depending on δ. It seems that the present difficulty is connected with a problem of Pełczyński in [Pe.1] whether every finite dimensional Banach space E can be embedded into a Banach space F with $\dim F \leq 2\dim E$ having a nice Schauder basis. For $\dim F$ close enough to $\dim E$, this question was answered in [M-T.2] in the negative.

PROOF OF THEOREM 4.1. Clearly, the weak cotype 2 assumption implies property (i), which implies (ii), which implies (iii). We shall prove that conversely, property (iii) implies that X is of weak cotype 2. Set $\gamma_0 = 2^{-20}$. Under our assumptions we have the following.

CLAIM. *For every finite-dimensional subspace $E \subset X$ there exists a subspace $H \subset E$ with $\dim H \geq 2^{-9}\dim E$ and $d_H \leq cM^2 d_E^{1/2}$, where c is a universal constant.*

PROOF OF THE CLAIM. Let $\varepsilon = 2^{-9}$, $\eta = 2^{-4}$, $\delta = 2^{-2}$, and $\gamma = \gamma_0$. Fix an arbitrary finite-dimensional $E \subset X$ and let $n \in \mathbb{N}$ be such that $\dim E = (1+2\varepsilon)n$.

Set $Z = E^*$. Let Z_1 be a quotient of Z with $\dim Z_1 = (1+\varepsilon)n$ satisfying condition (i) of Corollary 2.3. We also fix the Euclidean structure on Z_1 introduced in this condition. Let Z_2, with $\dim Z_2 = n$, be a quotient of Z_1 constructed in Proposition 3.2. Now we use (iii) in the dual form valid for every finite-dimensional quotient of X^* (*cf.* Proposition 2.5(iii)). It follows that there exists a Banach space Z_3, with $\dim Z_3 = k \leq (1+\gamma)n$, such that Z_2 is a quotient of Z_3 and such that Z_3 admits a $(k/8, 1)$-mixing operator S_0 with $\|S_0\| \leq M$. On the other hand, by the choice of Z_2, every $(\eta' k, 1)$-mixing operator T on Z_3 satisfies

$$\|T\| \geq cV_{2^{-6}n}^{-1}(Z_1)\, d_Z^{-\delta},$$

where $\eta' = \eta/(1-\gamma)$. Since $\eta' \leq 1/8$, the same estimate holds for S_0. Thus $V_{2^{-6}n}(Z_1) \geq c d_Z^{-\delta} M^{-1}$. Applying Lemma 2.1 with $\beta = 2^{-6}$ and $\sigma = 1$, we obtain a quotient Z_4 of Z_3 with $\dim Z_4 \geq 2^{-7}k \geq 2^{-9}\dim E$ and $d_{Z_4} \leq C d_Z^{2\delta} M^2$. Since $d_Z = d_E$, the proof of the Claim is concluded by setting $H = Z_4^*$.

Passing to the proof of the theorem, fix an arbitrary finite-dimensional subspace $X_0 \subset X$. Denote by D the smallest number such that every subspace X_1 of X_0 contains a subspace $G \subset X_1$ with $\dim G \geq \dim X_1/2$ and $d_G \leq D$. Clearly, D is finite.

We then know that for any m and any subspace X_1 of X_0 with $\dim X_1 = m$ there is a subspace $H \subset X_1$ with $\dim H \geq 2^{-10}m$ such that $d_H \leq cM^2D^{1/2}$. Indeed, first pick $E \subset X_1$ with $\dim E \geq m/2$ such that $d_E \leq D$, and then apply Claim to obtain H. Specifying $m = (1/2)\dim X_0$ we can use Lemma 2.4 with $\xi = 1/2$, $\delta = (1 - 2^{-10})\xi$ and $\eta = 2^{-10}\xi$, to get a subspace $\tilde{H} \subset X_0$ satisfying $\dim \tilde{H} \geq (1/2)\dim X_0$ and $d_{\tilde{H}} \leq c'M^2D^{1/2}$, where c' is a universal constant. By the definition of D, this implies $D \leq c'M^2D^{1/2}$, hence $D \leq c''M^4$. By (2.1), this completes the proof. ∎

Theorem 4.2 has almost identical proof, with condition (ii) of Proposition 3.2 replacing (i). We shall omit further details.

REMARK. Using Lemma 2.1 in a more delicate way and choosing δ sufficiently small one can get in the theorems above $wC_2(X) \leq C(\sigma)M^{1+\sigma}$, for every $\sigma > 0$.

We now pass to characterizations of weak Hilbert spaces.

THEOREM 4.3. *There exists a constant* $\gamma_0 > 0$ *such that a Banach space* X *is a weak Hilbert space if and only if there exists a constant* $M \geq 1$ *such that one of the following conditions is satisfied for every subspace* $Y \subset X$:

(i) every finite-dimensional subspace E *of* Y *contains a subspace* E_0 *with* $\dim E_0 \geq (1 - \gamma_0)\dim E$ *which admits a projection* $Q: E_0 \longrightarrow E_0$ *with* $\operatorname{rank} Q = \dim E_0/8$ *and* $\|Q\| \leq M$, *and every finite-dimensional quotient* F *of* Y *admits a quotient* F_0 *with* $\dim F_0 \geq (1 - \gamma_0)\dim F$ *which admits a projection* $R: F_0 \longrightarrow F_0$ *with* $\operatorname{rank} R = \dim F_0/8$ *and* $\|Q\| \leq M$,

(ii) for every finite-dimensional subspace E *of* Y *there is a Banach space* $\tilde{E}$ *containing* E *with* $\dim \tilde{E} \leq (1 + \gamma_0)\dim E$ *which admits a projection* $\tilde{Q}: \tilde{E} \longrightarrow \tilde{E}$ *with* $\operatorname{rank} \tilde{Q} = \dim \tilde{E}/8$ *and* $\|\tilde{Q}\| \leq M$, *and for every finite-dimensional quotient* F *of* Y *there is a Banach space* $\tilde{F}$ *which has* F *as a a quotient and* $\dim \tilde{F} \leq (1 + \gamma_0)\dim F$ *which admits a projection* $R: \tilde{F} \longrightarrow \tilde{F}$ *with* $\operatorname{rank} R = \dim \tilde{F}/8$ *and* $\|R\| \leq M$.

PROOF. Clearly, if X is a weak Hilbert space then both (i) and (ii) are satisfied. Conversely, assuming that one of conditions (i) or (ii) is satisfied, by Theorem 1 in [M-T.2], X does not contain l_1^n's uniformly. Hence, by Pisier's result, [P.1], (*cf.* also [P.3], Theorem 11.3), X is K-convex.

If (i) holds, by Theorem 4.1 we infer that both X and X^* are of weak cotype 2. Thus X is a weak Hilbert space. In case of (ii), we use Theorem 4.2. ∎

REMARK. If X has the approximation property, the above theorem remains valid if we restrict ourselves to the case $Y = X$ only. This can be easily seen by using for example [P.2], Theorem 2.

REMARK. In Theorem 4.3, the assumption on the existence of uniformly bounded rank $k/8$ projections can be replaced by the existence of uniformly bounded $(k/8, 1)$-mixing operators or by a uniform bound for basis constants.

5. **Subspaces and quotients of proportional dimension.** The main result of this section is concerned with subspaces and quotients of a fixed proportional dimension.

THEOREM 5.1. *Let $0 < \alpha < (1+2^{-8})^{-1}$ and let $M \geq 1$. Let G be an n-dimensional Banach space. If every αn-dimensional subspace E and every αn-dimensional quotient F of G have the basis constants* $\mathrm{bc}(E) \leq M$ *and* $\mathrm{bc}(F) \leq M$, *then G is a weak Hilbert space and the weak type 2 and the weak cotype 2 constants satisfy the estimates*

$$wT_2(G) \leq C(\alpha)M^{31\cdot 23/11} \quad \textit{and} \quad wC_2(G) \leq C(\alpha)M^{31\cdot 21/11}.$$

The proof of the theorem requires several steps. To begin with we consider only one-sided assumptions on G, that is, the assumptions on its subspaces. In such a situation, the following proposition establishes a weak cotype 2 property, provided that the space admits a nice direct sum decomposition.

PROPOSITION 5.2. *Let $0 < \alpha < 1$, $M \geq 1$ and let G be an n-dimensional Banach space. Assume that $G = Z \oplus_2 G_0$, with $\dim Z \geq \alpha n$. If every αn-dimensional subspace E of G has the basis constant $\mathrm{bc}(E) \leq M$, then the weak cotype 2 constant of G_0 satisfies $wC_2(G_0) \leq C(\alpha)(M \inf_H \mathrm{d}_H)^4$, where the infimum runs over all αn-dimensional subspaces $H \subset Z$.*

PROOF. Let $H \subset Z$ be an αn-dimensional subspace. We shall show that every subspace G_1 of G_0 contains a subspace $\tilde{E}$, with $\dim \tilde{E} \geq 2^{-12} \min(\dim G_1, \alpha n)$ such that $\mathrm{d}_{\tilde{E}} \leq C(M\mathrm{d}_H)^4$. By (2.1), this will imply $wC_2(G_0) \leq C(\alpha)(M\mathrm{d}_H)^4$, hence the conclusion will follow by passing to the infimum over H. Obviously it is enough to consider the case $\dim G_1 \leq \alpha n$ only.

Fix an arbitrary subspace G_1 of G_0 with $k = \dim G_1 \leq \alpha n$ and let $F = G_1^*$. By the remark following Corollary 3.3 there exists a quotient F_1 of F with $k' = \dim F_1 \geq (1+2^{-12})^{-1}k$ satisfying (3.6) with $\delta = 0$. Fix an arbitrary $H_1 \subset H$ with $\dim H_1 = \alpha n - k'$. We have $\mathrm{d}_{H_1^*} = \mathrm{d}_{H_1} \leq \mathrm{d}_H$ and H_1^* is a quotient of Z. Thus

$$\mathrm{bc}(F_1 \oplus_2 H_1^*) \geq c\mathrm{d}_H^{-1} V_{2^{-10}k'}(F)^{-1/2}.$$

Observe that $(F_1 \oplus_2 H_1^*)^*$ is an αn-dimensional subspace of G. Hence $\mathrm{bc}(F_1 \oplus_2 H_1^*) = \mathrm{bc}\big((F_1 \oplus_2 H_1^*)^*\big) \leq M$. Combining the last two estimates we get

$$V_{2^{-10}k'}(F) \geq c'(M\mathrm{d}_H)^{-2}.$$

By Lemma 2.1 (with $\sigma = 1$) we obtain a Euclidean quotient of F of dimension $2^{-11}k' \geq 2^{-12}k$ and so we complete the proof by passing to the dual. ∎

Let us note that the one-sided boundedness assumption alone still yields the existence of some Euclidean subspaces, but this time on a proportional level only.

LEMMA 5.3. *Let $0 < \alpha < (1+2^{-12})^{-1}$ and let $M \geq 1$. Let G be an n-dimensional Banach space such that every αn-dimensional subspace E of G has the basis constant* $\mathrm{bc}(E) \leq M$. *Then for every $(1+2^{-12})\alpha < \lambda \leq 1$, every λn-dimensional subspace G_0 of G contains a subspace H with $\dim H = (\lambda - \alpha)n \geq 2^{-13}\lambda n$ such that $\mathrm{d}_H \leq C(\alpha, \lambda)M^2$.*

PROOF. A similar argument as in the proposition above, in which the use of (3.6) is replaced by (3.5), shows that every subspace G_1 of G_0 with $\dim G_1 = \xi n = (1+2^{-12})\alpha n$ contains a CM^2-Euclidean subspace of dimension $2^{-11}\alpha n$, where C is a universal constant. The proof is then concluded by applying Lemma 2.4, with $\delta = (1-2^{-12})\alpha$ and $\eta = 2^{-12}\alpha$, to any λn-dimensional subspace G_0 of G. ∎

An *a priori* argument which we are going to use is based on a finite-dimensional version of one of properties characterizing weak type 2 spaces. A known argument (*cf.* [P.3], Chapter 11), localized to a fixed proportional-dimensional level, gives the following lemma. Recall that for a Banach space G, $K(G)$ stands for the K-convexity constant of G.

LEMMA 5.4. *Let $0 < \delta < \beta < 1$, $D \geq 1$, and let G be an n-dimensional Banach space. Assume that every βn-dimensional subspace F of G^* contains a subspace F_1 with $\dim F_1 = \delta n$ and $d_{F_1} \leq D$. Then for every βn-dimensional subspace E of G there exist a subspace H with $\dim H = \delta n/2$ and a projection $Q: G \longrightarrow H$ such that $\|Q\| \leq C(\beta,\delta)K(G)Dd_E$.*

PROOF. Let E be a βn-dimensional subspace of G and let $w: E \longrightarrow l_2^{\beta n}$ be an isomorphism such that $\|w\|\,\|w^{-1}\| = d_E$. We will show that there exist an orthogonal rank $(\delta/2)n$ projection P in $l_2^{\beta n}$ and an operator $\tilde{w}: G \longrightarrow l_2^{\beta n}$, such that $\tilde{w} = Pw$ and $\|\tilde{w}\| \leq C(\beta,\delta)DK(G)\|w\|$. Then $H = w^{-1}P(l_2^{\beta n})$ and $Q = w^{-1}P\tilde{w}$ will satisfy the requirements of the lemma.

The argument requires the definition of the l-norm of an operator $u: l_2^k \longrightarrow Y$, for any Banach space Y, which is provided *e.g.* in [P.3], Chapter 3. Similarly as in the proof of Theorem 11.6 in [P.3], consider the operator $w^*: l_2^{\beta n} \longrightarrow G^*/E^{\perp}$. By Lemma 11.7 in [P.3], there exists $\tilde{v}: l_2^{\beta n} \longrightarrow G^*$ such that $q\tilde{v} = w^*$, where $q: G^* \longrightarrow G^*/E^{\perp}$ it the quotient map, and that $l(\tilde{v}) \leq 2K(G)l(w^*) \leq 2(\beta n)^{1/2}K(G)\|w^*\|$. Note that $\tilde{v}$ is one-to-one. Choose a subspace F_1 of $\tilde{v}(l_2^{\beta n})$ with $\dim F_1 = \delta n$ and $d_{F_1} \leq D$ and let $F_2 = \tilde{v}^{-1}(F_1)$. By well-known properties of operators acting in Hilbert spaces (*cf.*, *e.g.*, [P.3] Proposition 3.13), there is a $(\delta/2)n$-dimensional subspace $F_3 \subset F_2$ and an orthogonal projection P onto F_3 such that

$$\|\tilde{v}P\| \leq (\delta n/2)^{-1/2}Dl(\tilde{v}) \leq 2^{3/2}(\beta/\delta)^{-1/2}K(G)D\|w\|.$$

Therefore the required operator is $\tilde{w} = P\tilde{v}^*$. ∎

We are finally ready for the proof of the theorem.

PROOF OF THEOREM 5.1. Fix $0 < \alpha < 1/4$ and set $\alpha' = (1+2^{-12})\alpha$. By Lemma 5.3 with $\lambda = 1$, pick a subspace H of G with $\dim H \geq (1-\alpha)n$ and $d_H \leq C(\alpha)M^2$. Since all αn-dimensional quotients F of G satisfy $\mathrm{bc}(F) \leq M$, applying the same lemma for G^* and $\lambda = (1-\alpha) > \alpha'$, it follows that G^* satisfies the assumptions of Lemma 5.4 for $\beta = 1-\alpha$, $\delta = 1-2\alpha$ and $D = C(\alpha)M^2$. Therefore there exist a subspace H_0 of H and a projection Q from G onto H_0 such that $\dim H_0 = (1/2-\alpha)n$ and $\|Q\| \leq C'(\alpha)K(G)M^4$.

Notice that $\dim H_0 \geq \alpha n$. Setting $G_0 = \ker Q$ we get that G is isomorphic, up to $2\|Q\|$, to the space $H_0 \oplus_2 G_0$. In particular, the basis constant of any αn-dimensional subspace of $H_0 \oplus_2 G_0$ does not exceed $2\|Q\|M$. Therefore, Proposition 5.2 yields

$$wC_2(G_0) \leq C''(\alpha)(2\|Q\|M\mathrm{d}_{H_0})^4 \leq C'''(\alpha)K(G)^4M^{27}.$$

It is obvious from any of the definitions discussed in [P.3], Chapter 10, that $wC_2(X \oplus_2 Y) \leq c\big(wC_2(X) + wC_2(Y)\big)$, for arbitrary weak cotype 2 spaces X and Y, where c is a numerical constant. Thus we finally get

$$wC_2(G) \leq 2\|Q\|wC_2(H_0 \oplus_2 G_0) \leq \tilde{C}(\alpha)K(G)^5M^{31}. \tag{5.1}$$

Since the assumption of the theorem hold for G^* as well, $wC_2(G^*)$ also admits the same upper bound as in (5.1). This in turn, by (2.2), yields

$$wT_2(G) \leq K(G)wC_2(G^*) \leq C''(\alpha)K(G)^6M^{31}. \tag{5.2}$$

Now we use the result of Pisier stated in (2.3), for *e.g.*, $\theta = 1/22$, to get $K(G) \leq C'''(\alpha)K(G)^{1/2}M^{31/11}$. Thus

$$K(G) \leq C_0(\alpha)M^{62/11}.$$

The proof is then concluded by combining this inequality with (5.1) and (5.2). ■

REMARK. Applying Lemma 5.3 with λ arbitrarily close to 1 and Corollary 3.3 with ε efficiently small, and using (2.3) more carefully, yield more civilized powers of M.

6. **Random quotients.** Fix a probability space $(\Omega, \mathbf{P})$ and let $g_1, \ldots, g_{\varepsilon n}$ be independent standard Gaussian vectors in $\mathbb{R}^n$ with the density $(n/2\pi)^{n/2}e^{-n\|x\|_2^2/2}$, with respect to the standard Lebesgue measure in $\mathbb{R}^n$.

For $\omega \in \Omega$, define a Gaussian projection $Q_\omega: \mathbb{R}^N \longrightarrow \mathbb{R}^n$ by

$$Q_\omega(e_i) = \begin{cases} e_i & \text{for } i = 1, 2, \ldots, n \\ g_{i-n}(\omega) & \text{for } i = n+1, n+2, \ldots, N. \end{cases}$$

In the theorem below, we denote by $G_{\gamma n,n}$ the set of all γn-dimensional subspaces of $\mathbb{R}^n$. For $H \in G_{\gamma n,n}$ we denote by Q_H the orthogonal projection with $\ker Q_H = H$.

THEOREM 6.1. *For an arbitrary* $0 < \delta < 1$, $0 < \eta < 3/8$ *and* $0 < \varepsilon < 2^{-5}\eta$, *set* $\gamma = 2^{-5}\delta\varepsilon\eta$, *and for* $n \in \mathbb{N}$ *set* $N = (1+\varepsilon)n$. *Let* $E = (\mathbb{R}^N, \|\cdot\|)$ *be an N-dimensional Banach space such that* $B_E \subset B_2^N$. *Let* $\rho \geq 1$ *and* $0 < a \leq 1$ *satisfy*

$$\mathrm{vr}(E) \leq \rho \quad aB_2^N \subset B_E.$$

There exists $0 < c = c(\varepsilon, \eta, \rho) < 1$ *such that if* $\tilde{\Omega}$ *denotes the set*

$$\tilde{\Omega} = \{\omega \in \Omega \mid \|Q_H T: Q_\omega(l_1^N) \longrightarrow Q_H Q_\omega(E)\| \geq cV_{\eta n/4}(E)^{-1}a^\delta,$$
$$\text{for every } T \in \mathrm{Mix}_n(\eta n, 1) \text{ and every } H \in G_{\gamma n,n}\},$$

then $\mathbf{P}(\tilde{\Omega}) \geq 1 - c_1^{n^2}$, *where* $0 < c_1 < 1$ *is an absolute constant.*

To deduce Theorem 3.1 pick $\omega \in \tilde{\Omega}$ and set $F = Q_\omega(E)$. Now it is enough to observe that since $\kappa B_1^N \subset B_E$ then

$$\|Q_H T: Q_\omega(E) \to Q_H Q_\omega(E)\| \geq \kappa \|Q_H T: Q_\omega(l_1^N) \to Q_H Q_\omega(E)\|.$$

For $E = l_1^N$, Theorem 6.1 was proved recently in [M-T.2] Theorem 4. In the general case the argument follows the steps from [M-T.2] blended with a technique which enables to pass from quotients of l_1^N to quotients of arbitrary Banach spaces, as presented in [M-T.3], Section 5. Therefore we shall only briefly discuss the main points, referring the reader to [M-T.2] and [M-T.3] for the details.

Passing to the description of the proof of Theorem 6.1, we require additional notation. For every $\omega \in \Omega$ let $H_\omega = \operatorname{span}[g_1(\omega), g_2(\omega), \dots, g_{\varepsilon n}(\omega)]$. If $H \in G_{\gamma n,n}$, let $Q_{\omega,H}$ be the orthogonal projection in $\mathbb{R}^n$ with $\ker Q_{\omega,H} = H + H_\omega$.

Let

$$\Omega_0 = \{\omega \in \Omega \mid 1/2 \leq \|g_i(\omega)\|_2 \leq 2 \text{ for all } i = 1, \dots, \varepsilon n\}.$$

Fix $T \in \operatorname{Mix}_n(2\eta n/3, 1)$. By the definition of the mixing class, there is $G \subset \mathbb{R}^n$, $\dim G = 2\eta n/3$ such that $\|P_{G^\perp} Tx\|_2 \geq \|x\|_2$ for every $x \in G$. The well-known argument on half-dimensional circular sections of an ellipsoid yields that there exists $G_0 \subset G$ with $\dim G_0 = \eta n/3$ and $\lambda \geq 1$ such that $\|P_{G^\perp} Tx\|_2 = \lambda \|x\|_2$ for every $x \in G_0$. For every $\omega \in \Omega$ and $H \in G_{\gamma n,n}$ fix an orthogonal projection Q_{ω,H,G_0} in $\mathbb{R}^n$ with

$$\ker Q_{\omega,H,G_0} \supset H_\omega + H + G + P_{G^\perp} T P_{G_0^\perp}(H_\omega),$$

and $\operatorname{rank} Q_{\omega,H,G_0} = \eta n/4$.

Set

$$\Omega_{T,H} = \{\omega \in \Omega_0 \mid Q_{\omega,H,G_0} T P_{G_0} g_j \in 4\lambda\alpha\sqrt{\eta} a^\delta V_{\eta n/4}^{-1} Q_{\omega,H,G_0} Q_\omega(B_E) \text{ for } j = 1, 2, \dots, \varepsilon n\}. \tag{6.1}$$

LEMMA 6.2. *Let $H \in G_{\gamma n,n}$. Then*

$$\mathbf{P}(\Omega_{T,H}) \leq \left(C_0 \alpha \sqrt{\eta} a^\delta\right)^{\varepsilon \eta n^2/4},$$

where $C_0 \geq 1$ is an absolute constant.

PROOF. Set $\tilde{Q}_\omega = Q_{\omega,H,G_0}$ for $\omega \in \Omega$. For $j = 1, 2, \dots, \varepsilon n$ define $g_j' = P_{G_0} g_j$ and $g_j'' = P_{G_0^\perp} g_j$. Similarly as in [M-T.2], Lemma 7, $\tilde{Q}_\omega$ is independent of the g_j''s. For every fixed $j = 1, 2, \dots, \varepsilon n$ we have

$$\{\omega \in \Omega \mid \tilde{Q}_\omega T P_{G_0} g_j \in 4\lambda\alpha\sqrt{\eta} a^\delta V_{\eta n/4}^{-1} \tilde{Q}_\omega Q_\omega(B_E)\}$$
$$= \{\omega \in \Omega \mid \tilde{Q}_\omega T g_j' \in 4\lambda\alpha\sqrt{\eta} a^\delta V_{\eta n/4}^{-1} \tilde{Q}_\omega Q_\omega(B_E)\}.$$

Since $G \subset \ker Q_\omega$, the definition of λ implies that $\lambda^{-1} \tilde{Q}_\omega T$ is a contraction in the

Euclidean norm on $\mathbb{R}^n$. Moreover, it has k s-numbers equal to 1, with $k \geq \eta n/3 - \gamma n - 2\varepsilon n \geq \eta n/4$. Hence, using Claim 6.2 in [Sz.1] (with $n/3$ replaced by $\eta n/4$), and the fact that $\sqrt{3/\eta}g_j'$ is a standard Gaussian variable in G_0, (note that $3/\eta = n/\dim G_0$), *cf.*, *e.g.*, [Sz.1] (3.3), we have, by the definition of $V_{\eta n/4}$,

$$\begin{aligned}\mathbf{P}\big(\{\omega \in \Omega \mid \tilde{Q}_\omega T g_j' \in 4\lambda\alpha\sqrt{\eta}a^\delta V_{\eta n/4}^{-1}\tilde{Q}_\omega Q_\omega(B_E)\}\big)\\ \leq \mathbf{P}\big(\{\omega \in \Omega \mid (\lambda^{-1}\tilde{Q}_\omega T)(\sqrt{3/\eta}g_j') \in 4\sqrt{3}\alpha a^\delta V_{\eta n/4}^{-1}\tilde{Q}_\omega Q_\omega(B_E)\}\big)\\ \leq (c'4\sqrt{3}\alpha a^\delta)^{\eta n/4}.\end{aligned}$$

where c' is an absolute constant. Hence

$$\mathbf{P}(\Omega_{T,H}) \leq (c'\alpha a^\delta)^{\varepsilon\eta n^2/4},$$

which concludes the proof of the lemma. ∎

The next lemma is a restatement of Lemma 7.3 in [Sz.1].

LEMMA 6.3. *For every* $0 < \sigma < 1$, *the set*

$$\mathcal{P}_{k,n} = \{P\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n \mid P \text{ an orthogonal projection with } \operatorname{rank} P = n-k\}$$

admits a σ*-net* $\mathcal{M}$ *in the operator norm in* l_2^n *with the cardinality* $|\mathcal{M}| \leq C^{n^2}\sigma^{-nk}$, *where* $C > 1$ *is an absolute constant.*

Using Lemma 6.2 and Lemma 6.3 with $\sigma = \alpha a^{1+\delta}/4$ and $k = \gamma n$, the same argument as in the proof of Proposition 5 in [M-T.2] yields.

PROPOSITION 6.4. *Let* $0 < \alpha, \delta < 1$ *and* $0 < \eta < 3/8$. *For an operator* $T \in \operatorname{Mix}_n(2\eta n/3, 1)$ *set*

$$\Omega_T = \{\omega \in \Omega_0 \mid \|Q_{\omega,H}T\colon Q_\omega(l_1^N) \longrightarrow Q_{\omega,H}Q_\omega(E)\| \leq 2\alpha\sqrt{\eta}a^\delta V_{\eta n/4}^{-1} \textit{for some } H \in G_{\gamma n,n}\}. \tag{6.2}$$

Then for every $T \in \operatorname{Mix}_n(2\eta n/3, 1)$ *one has*

$$\mathbf{P}(\Omega_T) \leq C^{n^2}(4/a^2\alpha)^{\gamma n^2}\left(C_0\alpha\sqrt{\eta}a^\delta\right)^{\varepsilon\eta n^2/4}, \tag{6.3}$$

where $C \geq 1$ *and* $C_0 \leq 1$ *are absolute constants.*

The rest of the proof of Theorem 6.1 is essentially the same as of Theorem 4 in [M-T.2]. One has to replace Proposition 6 there by Lemma 5.3 from [M-T.3], with $A = 2\alpha\sqrt{\eta}a^\delta V_{\eta n/4}^{-1}$ and choosing $\alpha > 0$ sufficiently small as to ensure that $\left(4c(\rho)\right)^{1+\varepsilon^2} C C_0^{\varepsilon\eta/4}(\alpha)^{\varepsilon\eta/8} \leq 1/2$.

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