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GROUPS WITH ABELIAN SYLOW SUBGROUPS

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This paper is dedicated to Gabriella Corsi on the occasion of her 70th birthday.

Abstract

Finite groups with abelian Sylow p-subgroups for certain primes p are characterized in terms of arithmetical properties of commutators.

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1. Introduction

Let G be a group. Many properties of G, such as nilpotency of class at most c, can be defined by commutator conditions such as $[X_1, X_2, \ldots, X_{c+1}] = 1$. In this example, no powers of elements nor their orders occur. In this paper, however, we are interested in relations between arithmetical properties of commutators of elements and properties of certain subgroups of G.

For example, let π be a set of primes, and let G be finite. If the commutator of every pair of elements of G is a π-element, then the derived subgroup G' of G is a π-group (the converse is obvious). This result was proved in Brandl [3], and again in Shumyatsky [8, Lemma 3.2]. The analogue for infinite groups is true in some cases (see [8]), but not in general (see [4]). It is still open whether (infinite) groups in which $[x, y]^3 = 1$ for all $x, y \in G$ are always soluble (see [7, Problem 12.4]).

Here, we are interested in finite groups with abelian Sylow p-subgroups, where p runs through a certain set of primes. How can this property be captured by properties of commutators (see [7, Problem 11.15])?

Consider the following condition on a finite group G:

$$(\text{C-}p): \text{ the commutator of every two } p\text{-elements of } G \text{ is a } p'\text{-element.}$$

Let G satisfy (C-p) for the prime p, and let P be a Sylow p-subgroup of G. For every pair $a, b \in P$, we obviously have $[a, b] \in P$, and by Condition (C-p), this commutator is a p'-element. Hence, $[a, b] = 1$, and so P is abelian. Conversely, assume that G has abelian Sylow p-subgroups. Does G satisfy (C-p)?

If G is p-soluble, then by a result of Hall and Higman (see Huppert [6, p. 691]), there exists a series $1 \leq R \leq S \leq G$ of normal subgroups of G such that R and G/S are p'-groups, and S/R is an abelian p-group. Let a, b be p-elements of G. As G/S is a p'-group, we have $a, b \in S$. As S/R is abelian, we get $[a, b] \in R$, and hence $[a, b]$ is a p'-element. Hence, G satisfies (C-p). In Brandl [2] it was shown that the hypothesis on p-solubility can be dropped in the case when $p = 2$. Indeed, the following result holds.

THEOREM 1 (See [2]). *Let G be a finite group. Then G has abelian Sylow* 2-*subgroups if and only if it satisfies Condition (C-2).*

The object of this paper is to search for analogues of this result. In the example below, for every prime $p \neq 2$, we will construct a finite group G with abelian Sylow p-subgroups that does *not* satisfy Condition (C-p). By the remark preceding Theorem 1, such a group G is necessarily insoluble.

To extend Theorem 1, we have to consider sets of primes. The following describes finite groups with (C-p) for all odd primes p. It will turn out that these groups are necessarily soluble.

THEOREM 2. *For a finite group G, the following are equivalent:*

(i) *G is soluble with abelian Sylow p-subgroups for all odd primes p;*
(ii) *G satisfies* (C-p) *for all odd primes p.*

The following immediate consequence of Theorem 2 describes finite groups which satisfy (C-p) for all primes p.

COROLLARY 3. *For a finite group G the following are equivalent:*

(i) *G is a soluble group with all Sylow subgroups abelian;*
(ii) *for every prime p and every two p-elements x, y in G, the commutator $[x, y]$ is a p'-element.*

We use standard notation throughout. In addition, $o(x)$ denotes the order of the element x of some group. For a prime p, call x a p'-element if $p \nmid o(x)$.

2. Proofs

In view of the remark preceding Theorem 1, it suffices to show that groups with (C-p) for all odd primes p are soluble. The proof of this will require some information concerning certain (minimal) simple groups. We start with the projective linear groups.

LEMMA 4. *Let $q = p^f$ be a prime power, and let $G = \mathrm{PSL}(2, q)$.*

(a) *If $q \equiv 1 \bmod 4$, then there exist two elements x, y in G of order p such that $[x, y]$ is of order p.*

(b) *If $q \equiv 3 \bmod 4$, then for every two p-elements x, y in G, the commutator $[x, y]$ is a p'-element. In particular, G satisfies the law $[X^m, Y^m]^m = 1$ where $m = q^2 - 1$.*

(c) *If r is an odd prime dividing $q - 1$, then there exist elements x, y in G of order r such that $[x, y]$ is of order r.*

PROOF. The element x of G has order p if and only if its minimal polynomial is $(X \pm 1)^2$. This is obviously equivalent to $\mathrm{Tr}(x) = \pm 2$, where Tr denotes the trace function.

Let $x = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $y = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. A straightforward calculation using the fact that $ad - bc = 1$ shows that

$$[x, y] = \begin{pmatrix} 1 - cd & 1 - cd - d^2 \\ c^2 & 1 + cd + c^2 \end{pmatrix}. \tag{1}$$

For (a), note that $\mathrm{Tr}([x, y]) = 2 + c^2$. If $q \equiv 1 \bmod 4$, we can choose $c \in \mathrm{GF}(q)$ with $c^2 = -4$. Hence $[x, y]$ has trace -2. If we choose $a = d = 1$ and $b = 0$, then $[x, y] \neq 1$, and so this is an element of order p. This proves (a).

For (b), let $x_0, y_0 \in G$ have order p. Clearly, $[x_0, y_0]$ is a p'-element if and only if $[\alpha(x_0), \alpha(y_0)]$ is a p'-element for some automorphism α of G. We thus may assume that $y_0 = y$ is the element introduced above. By (1), the trace of $[x, y]$ is $2 + c^2$. If the order of $[x, y]$ were divisible by p, we would have $o([x, y]) = p$. As above, this would imply that $\mathrm{Tr}([x, y]) = 2 + c^2 = \pm 2$, and so we would have $c^2 = -4$ or $c = 0$. The first case would contradict $q \equiv 3 \bmod 4$. In the second case, it would follow that $x = \pm\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$, and so $[x, y] = 1$ is a p'-element.

For the second statement of (b), note that for all $z \in G$, the element $z^m = z^{q^2-1}$ is of order 1 or p. The first statement yields that $[x^m, y^m]$ is a p'-element, whence it is of order dividing m. The claim follows.

Finally, we prove (c). Let $\varepsilon \in \mathrm{GF}(q)$ be of multiplicative order r. Set

$$x = \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & \varepsilon \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 0 & -1 \\ 1 & \varepsilon + \varepsilon^{-1} \end{pmatrix}.$$

Then x and y have the same trace. Moreover, $o(x) = r$. As $r \neq 2$, the eigenvalues of x and y are distinct. This implies that x and y are similar, and so $o(x) = o(y) = r$.

A straightforward calculation shows that $[x, y] = \left(\begin{smallmatrix} \varepsilon^2 & * \\ * & \varepsilon^{-2} \end{smallmatrix}\right)$. Hence, the trace of $[x, y]$ is $\varepsilon^2 + \varepsilon^{-2}$, which is equal to the trace of x^2. As above, this yields $o([x, y]) = o(x^2) = r$. □

The following example shows that Theorem 2 is no longer true if we assume that condition (ii) holds for a *finite* set of odd primes only.

EXAMPLE 5. Let π be a finite set of primes such that $2 \notin \pi$. Then there exists a group with abelian Sylow p-subgroups for all $p \in \pi$, but not satisfying (C-p) for any $p \in \pi$.

PROOF. Let $\pi = \{p_1, \dots, p_t\}$. Choose f such that $2^f \equiv 1 \bmod (p_1 \cdot \dots \cdot p_t)$, and let $G = \mathrm{PSL}(2, 2^f)$. Then all Sylow subgroups of G are abelian. However, by Lemma 4(c), the group G does not satisfy (C-p_i) for any index i. $\square$

We now set out to classify groups G satisfying (C-p) for all odd primes p. For this, we need the following analogue of part (c) of Lemma 4 for the simple Suzuki groups $\mathrm{Sz}(q)$.

LEMMA 6. *Let $G = \mathrm{Sz}(q)$, where $q = 2^f \geq 8$ and f is odd. If r is a prime dividing $q - 1$, then there exist elements x, y in G of order r such that the commutator $[x, y]$ is of order r.*

PROOF. Let $\lambda \in \mathrm{GF}(q)$ be of multiplicative order r, let ϑ be the automorphism of $\mathrm{GF}(q)$ such that ϑ^2 is the Frobenius automorphism, and choose

$$x = \begin{pmatrix} \lambda^{1+\vartheta} & & & \\ & \lambda & & \\ & & \lambda^{-1} & \\ & & & \lambda^{-1-\vartheta} \end{pmatrix} \quad \text{and} \quad y = y(b) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & b \\ 1 & 0 & b & b^{\vartheta} \end{pmatrix}.$$

Then $x, y \in G$. If we choose b such that $b^{\vartheta} = \lambda^{1+\vartheta} + \lambda + \lambda^{-1} + \lambda^{-1-\vartheta}$, then x and y are of equal trace, and so Bäärnhielm [1, Lemma 2.7] implies that x and y are conjugate. In particular, $o(y) = o(x) = r$.

A simple calculation shows that $[x, y]$ has trace $\lambda^{-2-2\vartheta} + \lambda^{-2} + \lambda^{2} + \lambda^{2+2\vartheta}$. Hence $[x, y]$ and x^2 have the same trace, and by [1, Lemma 2.7] again, $[x, y]$ and x^2 are conjugate. As r is odd, we thus have $o([x, y]) = o(x^2) = o(x) = r$ as claimed. $\square$

PROOF OF THEOREM 2. It has already been shown that (i) implies (ii). For the converse, we first show that G is soluble. A counterexample of least possible order is clearly a minimal simple group. First, let $G = \mathrm{PSL}(2, q)$. If $q - 1$ is not a power of 2, then by Lemma 4(c), G does not satisfy (C-r) for a suitable odd prime divisor r of $q - 1$. Hence, q is a Fermat prime or $q = 9$ (see [5, Hilfssatz]). As $q \neq 3$, we get $q \equiv 1 \bmod 4$. But then Lemma 4(a) applies. Next, $\mathrm{PSL}(3, 3)$ has nonabelian Sylow 3-subgroups, and hence does not satisfy (C-3). Finally, let $G = \mathrm{Sz}(q)$ where $q = 2^f$. As G is minimal simple, f is odd and $f \geq 3$. Choose a prime r dividing $q - 1$. Then the elements x, y constructed in Lemma 6 show that G does not satisfy (C-r). This is a contradiction. Hence, G is soluble. Clearly, the Sylow subgroups of odd order of G are abelian. Thus, (i) holds. $\square$

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