TWISTED GROUP ALGEBRAS AND THEIR REPRESENTATIONS

S. B. CONLON

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Introduction

Let $\mathscr{G}$ be a finite group, $\mathscr{F}$ a field. A *twisted group algebra* $\mathscr{A}(\mathscr{G})$ on $\mathscr{G}$ over $\mathscr{F}$ is an associative algebra whose elements are the formal linear combinations

$$\sum_{A\in\mathscr{G}} a_A(A) \qquad (a_A \in \mathscr{F})$$

and in which the product $(A)(B)$ is a non-zero multiple of (AB), where AB is the group product of A, $B \in \mathscr{G}$:

$$(A)(B) = f_{A,B}(AB) \qquad (f_{A,B} \in \mathscr{F}, f_{A,B} \neq 0).$$

One gets the ordinary group algebra $\mathscr{F}(\mathscr{G})$ by taking each $f_{A,B} = 1$.

Twisted group algebras play a central part in Schur's theory of the projective representations of finite groups [17], [18]. They also arise naturally in the theory of ordinary representations. Let $\mathscr{L}$ be an irreducible $\mathscr{F}$-representation of a normal subgroup $\mathscr{H}$ of $\mathscr{G}$. Miss Tucker [21] [1] has shown that the analysis of the induced representation $\mathscr{L}^{\mathscr{G}}$ of $\mathscr{G}$ depends on a twisted group algebra $\mathscr{A}(\mathscr{K})$ on a certain subgroup $\mathscr{K}$ of $\mathscr{G}/\mathscr{H}$. Clifford [5] encountered much the same algebra in the analysis of the restriction to $\mathscr{H}$ of an irreducible representation of $\mathscr{G}$.

The aim of the present paper is to develop the theory of twisted group algebras by exploiting their analogy with ordinary group algebras. This approach permits a unified treatment of such problems as Miss Tucker's cited above. It will be seen that the theory of ordinary group algebras carries over in considerable detail.

In § 1, a normalization theorem is proved which brings out the multiplicative similarity between ordinary and twisted group algebras. This theorem is fundamental for the subsequent work. In § 2, a two-fold generalization of Miss Tucker's paper is given. Firstly, the ordinary group algebras of $\mathscr{G}$ and $\mathscr{H}$ are replaced by twisted ones. Secondly, the representation $\mathscr{L}$ is

[1] Kleppner [14] has extended the theory to infinite discrete groups.

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assumed to be indecomposable rather than irreducible. As in Miss Tucker's theory, the analysis of $\mathscr{L}^{\mathscr{G}}$ depends on the decomposition of a certain twisted group algebra into indecomposable left ideals.

A first step towards such a decomposition is to obtain the decomposition into two-sided ideals. This leads to the consideration, in § 3, of the blocks of a twisted group algebra. Here we follow the treatment of Rosenberg [16] rather than the original treatment of Brauer [4]. Finally, in § 4, we develop Higman's theory of relative projectivity [9], [11] and Green's theory of vertices and sources [8] for twisted algebras.

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1. Normalization of twisted group algebras

We take a twisted group algebra $\mathscr{A}(\mathscr{G})$ as defined in the introduction. For $A \in \mathscr{G}$, we write $\mathscr{C}(A)$ for the centralizer of A in $\mathscr{G}$. Let $\mathscr{F}^*$ denote the set of non-zero elements of $\mathscr{F}$. Let p be the characteristic of $\mathscr{F}$; we allow $p = 0$. E will be the identity element of $\mathscr{G}$.

The elements $k(A)$ of $\mathscr{A}(\mathscr{G})$ $(k \in \mathscr{F}^*,\ A \in \mathscr{G})$ form a multiplicative subgroup Γ. The elements $k(E)$ form a multiplicative subgroup K such that $\Gamma/K \cong \mathscr{G}$, and the (A) are coset representatives for K in Γ.

An element $A \in \mathscr{G}$ is called a u-element if

$$(B)^{-1}(A)(B) = (A),$$

for all $B \in \mathscr{C}(A)$. Thus the centralizer of (A) in Γ consists of all multiples $k(B)$, where $k \in \mathscr{F}^*$, $B \in \mathscr{C}(A)$. All conjugates of A are also u-elements.

The condition of associativity of $\mathscr{A}(\mathscr{G})$ is equivalent to

$$f_{A,B} f_{AB,C} = f_{A,BC} f_{B,C},$$

for all $A, B, C \in \mathscr{G}$. Thus the set $\{f_{A,B}\}$ forms a factor system[2] for $\mathscr{G}$. If we take a new basis of $\mathscr{A}(\mathscr{G})$

$$(\bar{A}) = d_A(A), \tag{1}$$

where $d_A \in \mathscr{F}^*$, $A \in \mathscr{G}$, then the $f_{A,B}$ are modified to

$$\bar{f}_{A,B} = \frac{d_A d_B}{d_{AB}} f_{A,B}.$$

[2] The factor systems $\{f_{A,B}\}$, modulo the principal factor systems $\{d_A\}$ form an abelian group, which Schur [17] has called the "Multiplikator". Further discussion of the group is found in [2] and [3].

Transformations given by (1) correspond to taking a different choice of coset representatives in Γ/K.

THEOREM. *Let n be the order of $\mathscr{G}$. Let $\mathscr{D}$ be the largest normal p-subgroup of $\mathscr{G}$ if $p \neq 0$, $\mathscr{D} = \{E\}$ if $p = 0$. After making a finite number of primary radical extensions to the field $\mathscr{F}$, if necessary, it is possible to choose the coset representatives (A) such that*:

(a)
$$\left.\begin{matrix} f^n_{A,B} = 1 \ (n \text{ odd}) \\ f^{2n}_{A,B} = 1 \ (n \text{ even}) \end{matrix}\right\} \qquad (\text{for all } A, B \in \mathscr{G}),$$

(b) *the representatives $(A), (B), \cdots, (A, B, \cdots, \in \mathscr{D})$ form a normal subgroup of Γ,*

(c)
$$(A)^{-1} = (A^{-1}) \qquad (\text{all } A \in \mathscr{G}),$$

(d) $(X)^{-1}(A)(X) = (X^{-1}AX)$ *whenever A is a u-element, $X \in \mathscr{G}$.*

PROOF. (i) Since
$$f_{A,B}f_{AB,C} = f_{B,C}f_{A,BC},$$
we have
$$f^d_{A,B} = h_A h_B/h_{AB} \qquad (\text{for } A, B \in \mathscr{D}),$$
where $d =$ order of $\mathscr{D}$, $h_A = \prod_{C\in\mathscr{D}} f_{A,C}$. Replacing (A) by $h_A^{-1/d}(A)$, we have
$$f^d_{A,B} = 1 \qquad (\text{for } A, B \in \mathscr{D}).$$

Since d is a power of p, $f_{A,B} = 1$, all $A, B \in \mathscr{D}$. If $X \in \mathscr{G}$, $A \in \mathscr{D}$,
$$(X)^{-1}(A)(X) = l(X^{-1}AX),$$
where $l \in \mathscr{F}^*$, and so
$$(X)^{-1}(A)^d(X) = l^d(X^{-1}AX)^d.$$
Thus
$$l^d = 1.$$
Hence
$$l = 1, \quad \text{and (b) holds.}$$

(ii) Similarly,
$$f^n_{A,B} = k_A k_B/k_{AB} \qquad (\text{all } A, B \in \mathscr{G}),$$
where $k_A = \prod_{C\in\mathscr{G}} f_{A,C}$. For each $A \in \mathscr{G}$, choose a definite value for $k_A^{-1/n}$. Replacing (A) by $k_A^{-1/n}(A)$, we may assume $f^n_{A,B} = 1$ for all $A, B \in \mathscr{G}$. (For $A, B \in \mathscr{D}$,
$$1 = f^n_{A,B} = k_A k_B/k_{AB},$$
whence $k_A = 1$; choose $1^{-1/n}$ in $\mathscr{F}$ as 1; then (b) still holds.)

(iii) Let $\mathscr{K} = \{A_1, \cdots, A_r\}$ be any conjugacy class of u-elements not in $\mathscr{D}$.

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The u-condition tells us that (A_1) has r conjugates *in* Γ. Choosing (A_1) arbitrarily and taking $(A_2), \cdots, (A_r)$ as its other conjugates in Γ we have condition (d) holding, and we still have $f^n_{A,B} = 1$.

(iv) Consider the elements in $\mathscr{G}$ not in $\mathscr{D}$. For such an element, $(A)(A^{-1}) = l(E)(l \in \mathscr{F}^*)$. For one, say A, out of each pair A, A^{-1} of non-involutory, non-u elements, leave (A) as before and replace (A^{-1}) by $(A)^{-1} = l^{-1}(A^{-1})$. For each non-$u$ involution A, replace (A) by $l^{-\frac{1}{2}}(A)$. As $l^n = 1$, $(l^{-1})^n = 1$, n odd, $(l^{-\frac{1}{2}})^{2n} = 1$, n even.

Now consider the u-class

$$\mathscr{K} = \{A_1, \cdots, A_r\}.$$

We still have the choice of (A_1) at our disposal. If $\mathscr{K} \neq \mathscr{K}^{-1} = \{A_1^{-1}, \cdots, A_r^{-1}\}$, we choose (A_1), (A_1^{-1}) as above in the case $A \neq A^{-1}$. If $(X)^{-1}(A_1)(X) = (A_i)$, then $(X)^{-1}(A_1)^{-1}(X) = (A_i)^{-1} = (A_i^{-1})$, by choice of (A_i), (A_i^{-1}).

Finally, let $\mathscr{K}$ be self-inverse. Thus

$$A_1^{-1} = T^{-1}A_1T,$$

and

$$(A_1^{-1}) = (T)^{-1}(A_1)(T). \tag{*}$$

Replacing (A_1) by $l^{-\frac{1}{2}}(A_1)$, and so all (A_i) by $l^{-\frac{1}{2}}(A_i)$, we still have (*) and also $(A_i^{-1}) = (A_i)^{-1}$.

Remarks. 1. (E) is now the identity element of $\mathscr{A}(\mathscr{G})$. Further $(A)(A^{-1}) = (A^{-1})(A) = (E)$. If we write $\mathscr{A}(\mathscr{D})$ to denote the natural restriction of $\mathscr{A}(\mathscr{G})$ to the subspace spanned by the elements (D) $(D \in \mathscr{D})$, then $\mathscr{A}(\mathscr{D})$ is precisely the group algebra $\mathscr{F}(\mathscr{D})$.

2. If $\mathscr{A}(\mathscr{G})$ satisfies (c) [(c), (d)] [[(b), (c), (d)]] then we shall call $\mathscr{A}(\mathscr{G})$ normalized [u-normalized] [[p-u-normalized]].

3. If A is a u-element, and if t is prime to the order of A, then A^t is a u-element. In particular A^{-1} is a u-element.

If $p \neq 0$, and if A has order a power of p, then A is a u-element.

Even if A is non-u, (c) ensures that

$$(X)^{-1}(Y)^{-1}(A)(Y)(X) = (X^{-1}Y^{-1})(A)(YX),$$

for all $X, Y \in \mathscr{G}$.

4. If $\mathscr{A}(\mathscr{G})$ is u-normalized and $\mathscr{K}_1, \cdots, \mathscr{K}_t$ are the u-classes, then the u-class sums $K_\alpha = \sum_{G \in \mathscr{K}_\alpha} G$ form a basis for the centre $\mathscr{Z}(\mathscr{G})$ of $\mathscr{A}(\mathscr{G})$, which has dimension t.[3]

5. A twisted group algebra $\mathscr{A}(\mathscr{G})$ is actually an (two-sided) ideal direct summand of a group algebra[4]: suppose $\mathscr{A}(\mathscr{G})$ has been normalized as in

[3] c. f. Satz 1, p. 83 of [20]. Tazawa's formulation is not so explicit and is confined to the non-modular case.

[4] I am indebted to the referee for this remark and its proof.

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(ii) above so that all $f_{A,B}$ satisfy $f^n_{A,B} = 1$. If $\mathscr{F}$ has characteristic p, and $n = mp^\alpha$, $(m, p) = 1$, then in fact $f^m_{A,B} = 1$. Thus the $f_{A,B}$ all belong to the multiplicative group W_m of m-th roots of unity. Let $f \to f^*$ be an isomorphism onto some other cyclic group $\mathscr{C}_m$ of order m, generated by μ^*, and define a central extension $\mathscr{G}^*$ of $\mathscr{G}$ by $\mathscr{C}_m$ in which $\mathscr{G}^*$ is generated by elements $S_A (A \in \mathscr{G})$ and $\mathscr{C}_m$, with $S_A S_B = f^*_{A,B} S_{AB}$. Then $\mathscr{F}(\mathscr{C}_m)$, considered as embedded in $\mathscr{F}(\mathscr{G}^*)$, is in the centre of $\mathscr{F}(\mathscr{G}^*)$; let

$$S_E = E_1 + \cdots + E_m,$$

where

$$E_i = \frac{1}{m} \sum_{\alpha=0}^{m-1} \mu^{\alpha i} (\mu^*)^\alpha,$$

be a decomposition of the identity S_E of $\mathscr{F}(\mathscr{G}^*)$ into primitive idempotents of $\mathscr{F}(\mathscr{C}_m)$. It is readily verified that $\mathscr{A}(\mathscr{G}) \cong E_1 \mathscr{F}(\mathscr{G}^*)$.

As $\mathscr{F}(\mathscr{G}^*)$ is symmetric [5], it follows that $\mathscr{A}(\mathscr{G})$ is symmetric. (This can also be seen directly without using $\mathscr{F}(\mathscr{G}^*)$.)

6. If $p = 0$, or $p \nmid n$ (non-modular case) (thus $p \nmid |\mathscr{G}^*|$), $\mathscr{F}(\mathscr{G}^*)$ is semi-simple, and so $\mathscr{A}(\mathscr{G})$ is semi-simple [6]. In this case there are t different irreducible representations of $\mathscr{A}(\mathscr{G})$, where t = number of u-conjugacy classes.

In the modular case, the number of irreducibles is equal to the number of p-regular u-conjugacy classes of $\mathscr{G}$ [7]. (An element $A \in \mathscr{G}$ is p-regular if its order is prime to p.) This can be proved using Brauer's Theorem 3A, p. 410 of [4].

7. From remark 1, any twisted group algebra $\mathscr{A}(\mathscr{D})$ on a p-group $\mathscr{D}$ over a field $\mathscr{F}$ of characteristic $p \neq 0$ is the group algebra $\mathscr{F}(\mathscr{D})$. This is a local algebra whose radical is spanned by the elements $(P)-(E)$, $P \in \mathscr{D}$, E identity of $\mathscr{D}$. The regular representation of $\mathscr{F}(\mathscr{D})$ is indecomposable.

8. This last result can be extended a little further. *Let $\mathscr{G}$ be a cyclic extension of a normal p-subgroup $\mathscr{D}$, where $p \neq 0$. Then $\mathscr{A}(\mathscr{G})$ is the group algebra on $\mathscr{G}$.*

Proof. Clearly it can be assumed that $|\mathscr{G}/\mathscr{D}| = m$, prime to p. Take $G \in \mathscr{G}$ such that the coset $G\mathscr{D}$ generates $\mathscr{G}/\mathscr{D}$. Write

$$G^m = K \in \mathscr{D},$$
$$(G)^m = d(K), \qquad (d \in \mathscr{F}^*).$$

Any element of $\mathscr{G}$ can be written uniquely in the form $G^k D$, where $0 \leqq k < m$, $D \in \mathscr{D}$.

[5] See definition of symmetric on p. 440 of [6].

[6] This can also be seen by a direct calculation of the discriminant of $\mathscr{A}(\mathscr{G})$, e. g. see p. 80 of [20].

[7] See also p. 207 of [2].

By the theorem, $\mathscr{A}(\mathscr{G})$ can be supposed to be p-u-normalized. If now we replace $(G^k D)$ by $d^{-k/m}(G)^k(D)$ this ensures that $\mathscr{A}(\mathscr{G})$ is the group algebra $\mathscr{F}(\mathscr{G})$.

9. If a twisted group algebra $\mathscr{A}(\mathscr{G})$ has one representation of degree 1, then it is the group algebra $\mathscr{F}(\mathscr{G})$.

2. Induced representations

Let $\mathscr{A}(\mathscr{G})$ be a normalized twisted group algebra and let $\mathscr{A}(\mathscr{H})$ be the natural restriction of $\mathscr{A}(\mathscr{G})$ to a subgroup $\mathscr{H}$ of $\mathscr{G}$. Let $\mathscr{L}$ be a left $\mathscr{A}(\mathscr{H})$-module. (Throughout this apper all modules will be taken as having finite dimension considered as vector spaces over the base field $\mathscr{F}$.) We define $\mathscr{L}^{\mathscr{G}}$ to be the left $\mathscr{A}(\mathscr{G})$-module given by

$$\mathscr{L}^{\mathscr{G}} = \mathscr{A}(\mathscr{G}) \otimes_{\mathscr{A}(\mathscr{H})} \mathscr{L},$$

where $\otimes$ is defined as in [6]. If $\mathscr{M}$ is an $\mathscr{A}(\mathscr{G})$-module, then we shall write $\mathscr{M}_{\mathscr{H}}$ for the $\mathscr{A}(\mathscr{H})$-module obained from $\mathscr{M}$ by simple restriction of the module multiplication to the ring $\mathscr{A}(\mathscr{H})$.

Let $\mathscr{M}, \mathscr{N}$ be $\mathscr{A}(\mathscr{H})$-modules. Then we write $\mathrm{Hom}_{\mathscr{H}}(\mathscr{M}, \mathscr{N})$ for the set of $\mathscr{A}(\mathscr{H})$-homomorphisms of $\mathscr{M}$ into $\mathscr{N}$, $E_{\mathscr{H}}(\mathscr{M}) = \mathrm{Hom}_{\mathscr{H}}(\mathscr{M}, \mathscr{M})$ for the ring of $\mathscr{A}(\mathscr{H})$-endomorphisms of $\mathscr{M}$, and $R_{\mathscr{H}}(\mathscr{M})$ for the radical of $E_{\mathscr{H}}(\mathscr{M})$. Throughout this section homomorphisms will be written on the right. We quote the following simple lemma.

LEMMA. *If $\mathscr{L}$ is an $\mathscr{A}(\mathscr{H})$-module and $\mathscr{M}$ an $\mathscr{A}(\mathscr{G})$-module, then* $\mathrm{Hom}_{\mathscr{H}}(\mathscr{L}, \mathscr{M}_{\mathscr{H}}) \cong \mathrm{Hom}_{\mathscr{G}}(\mathscr{L}^{\mathscr{G}}, \mathscr{M})$. *This correspondence $\eta \to \eta^{\mathscr{G}}$ is given by defining for $\eta \in \mathrm{Hom}_{\mathscr{H}}(\mathscr{L}, \mathscr{M}_{\mathscr{H}})$, $\eta^{\mathscr{G}} \in \mathrm{Hom}_{\mathscr{G}}(\mathscr{L}^{\mathscr{G}}, \mathscr{M})$ by*

$$(A \otimes L)\eta^{\mathscr{G}} = A(L\eta) \quad (A \in \mathscr{A}(\mathscr{G}), L \in \mathscr{L}).$$

Henceforth we take $\mathscr{H}$ to be a normal subgroup of $\mathscr{G}$, and $\mathscr{L}$ to be an $\mathscr{A}(\mathscr{H})$-module. The main theorem of this section concerns the structure of $\mathscr{L}^{\mathscr{G}}$ and this analysis is to be made through its ring of endomorphisms $E_{\mathscr{G}}(\mathscr{L}^{\mathscr{G}})$.

Given an element $G \in \mathscr{G}$, we can consider the $\mathscr{A}(\mathscr{H})$-submodules of $\mathscr{L}^{\mathscr{G}}$ of the form

$$(G) \otimes_{\mathscr{A}(\mathscr{H})} \mathscr{L},$$

where $(H)((G) \otimes L) = (G) \otimes (G)^{-1}(H)(G)L$ for $H \in \mathscr{H}, L \in \mathscr{L}$. $(G) \otimes \mathscr{L}$ may or may not be $\mathscr{A}(\mathscr{H})$-isomorphic to $\mathscr{L}$. The *stabilizer* $\mathscr{S}$ of $\mathscr{L}$ is the set of elements $S \in \mathscr{G}$ such that $(S) \otimes \mathscr{L} \cong \mathscr{L}$. Then $\mathscr{S}$ is a subgroup of $\mathscr{G}$ containing $\mathscr{H}$.

Take a set $\{X_\alpha\}$ of elements of $\mathscr{G}$ such that $X_1\mathscr{H}, \cdots, X_s\mathscr{H}$ $(X_1\mathscr{H}, \cdots,$

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$X_g\mathscr{H}$) are the different cosets of $\mathscr{H}$ in $\mathscr{S}$ (of $\mathscr{H}$ in $\mathscr{G}$) with $X_1 = E$. Then we may write

$$\mathscr{L}^{\mathscr{G}} = \sum_1^g (X_\alpha) \otimes \mathscr{L} = \sum_1^g \mathscr{L}_\alpha, \tag{1}$$

$$\mathscr{L}^{\mathscr{S}} = \sum_1^s \mathscr{L}_\alpha, \tag{1'}$$

the $\sum$ meaning vector space sum over $\mathscr{F}$. We identify $\mathscr{L}_1$ and $\mathscr{L}$. If we restrict to $\mathscr{H}$, (1) and (1′) then become $\mathscr{A}(\mathscr{H})$-direct decompositions of $(\mathscr{L}^{\mathscr{G}})_{\mathscr{H}}$ and $(\mathscr{L}^{\mathscr{S}})_{\mathscr{H}}$ respectively.

Let

$$\omega_\alpha : \mathscr{L}_\alpha \to (\mathscr{L}^{\mathscr{G}})_{\mathscr{H}}, \quad \chi_\alpha : (\mathscr{L}^{\mathscr{G}})_{\mathscr{H}} \to \mathscr{L}_\alpha$$

be the inclusion and projection $\mathscr{A}(\mathscr{H})$-homomorphisms according to (1). (We use the same symbols for the decomposition in (1′) and regard $(\mathscr{L}^{\mathscr{S}})_{\mathscr{H}} \subset (\mathscr{L}^{\mathscr{G}})_{\mathscr{H}}$ naturally.) Thus the identity ι of $E_{\mathscr{H}}(\mathscr{L}^{\mathscr{G}})$ [8] may be written

$$\iota = \sum_1^g \chi_\alpha \omega_\alpha.$$

If $\eta \in \mathrm{Hom}_{\mathscr{H}}(\mathscr{L}, \mathscr{L}^{\mathscr{G}})$, then

$$\eta = \sum_1^g \eta\chi_\alpha\omega_\alpha = \sum \eta_\alpha\omega_\alpha,$$

where $\eta_\alpha = \eta\chi_\alpha \in \mathrm{Hom}_{\mathscr{H}}(\mathscr{L}, \mathscr{L}_\alpha)$. Similarly if $\zeta \in E_{\mathscr{H}}(\mathscr{L}^{\mathscr{G}})$, we write

$$\zeta = \sum_1^g \sum_1^g \chi_\alpha\omega_\alpha\zeta\chi_\beta\omega_\beta = \sum_1^g \sum_1^g \chi_\alpha\zeta_{\alpha\beta}\omega_\beta,$$

where

$$\zeta_{\alpha\beta} = \omega_\alpha\zeta\chi_\beta \in \mathrm{Hom}_{\mathscr{H}}(\mathscr{L}_\alpha, \mathscr{L}_\beta).$$

Suppose

$$L\eta_\beta = (X_\beta)\otimes L_\beta \qquad (L \in \mathscr{L}),$$
$$(X_\alpha)(X_\beta) = (X_\gamma)H_{\alpha,\beta}$$

where $X_\alpha X_\beta \in X_\gamma\mathscr{H}$, $H_{\alpha,\beta} \in \mathscr{A}(\mathscr{H})$. Then

$$((X_\alpha) \otimes L)\eta^{\mathscr{G}} = \sum_\beta (X_\alpha)(X_\beta) \otimes L_\beta.$$

Thus $(\eta^{\mathscr{G}})_{\alpha\gamma}$ maps $(X_\alpha)\otimes L$ to $(X_\gamma) \otimes H_{\alpha,\beta}L_\beta$, where β is determined by $X_\alpha X_\beta \in X_\gamma\mathscr{H}$.

From this point onwards we shall take $\mathscr{L}$ to be an indecomposable $\mathscr{A}(\mathscr{H})$-module. Hence $E_{\mathscr{H}}(\mathscr{L})$ is a completely primary ring.

[8] Here $E_{\mathscr{H}}(\mathscr{L}^{\mathscr{G}})$ means $E_{\mathscr{H}}((\mathscr{L}^{\mathscr{G}})_{\mathscr{H}})$. Similarly $\mathrm{Hom}_{\mathscr{H}}(\mathscr{L}, \mathscr{L}^{\mathscr{G}})$ means $\mathrm{Hom}_{\mathscr{H}}(\mathscr{L}, (\mathscr{L}^{\mathscr{G}})_{\mathscr{H}})$ etc.

LEMMA 1. *Let* $\eta \in \mathrm{Hom}_{\mathscr{H}}(\mathscr{L}, \mathscr{L}^{\mathscr{G}})$. *Then* $\eta^{\mathscr{G}} \in R_{\mathscr{H}}(\mathscr{L}^{\mathscr{G}})$ *if, and only if, none of* $\eta_1, \cdots, \eta_s$ *is an* $\mathscr{H}$*-isomorphism.*

PROOF. By Jacobson [13], p. 60, $\eta^{\mathscr{G}} \in R_{\mathscr{H}}(\mathscr{L}^{\mathscr{G}})$ if, and only if, no $(\eta^{\mathscr{G}})_{\alpha\beta}$ is an $\mathscr{H}$-isomorphism. By the above, this is the case if, and only if, no η_β $(\beta = 1, \cdots, g)$ is an $\mathscr{H}$-isomorphism. No η_β $(\beta > s)$ is an $\mathscr{H}$-isomorphism because, by the definition of $\mathscr{S}$, $\mathscr{L}$ is not $\mathscr{H}$-isomorphic to $\mathscr{L}$. This gives the lemma.

There is of course the analogous $1-1$ correspondence $\eta \leftrightarrow \eta^{\mathscr{S}}$ between the $\mathscr{H}$-isomorphisms η of $\mathscr{L}$ into $\mathscr{L}^{\mathscr{S}}$ and $\mathscr{S}$-endomorphisms $\eta^{\mathscr{S}}$ of $\mathscr{L}^{\mathscr{S}}$, where $\eta^{\mathscr{S}}$ is defined by

$$(A \otimes L)\eta^{\mathscr{S}} = A(L\eta) \quad (A \in \mathscr{A}(\mathscr{S}), L \in \mathscr{L}).$$

COROLLARY. *Let* $\eta \in \mathrm{Hom}_{\mathscr{H}}(\mathscr{L}, \mathscr{L}^{\mathscr{S}})$. *Then* $\eta^{\mathscr{G}} \in R_{\mathscr{H}}(\mathscr{L}^{\mathscr{G}})$ *if, and only if,* $\eta^{\mathscr{S}} \in R_{\mathscr{H}}(\mathscr{L}^{\mathscr{S}})$. (*Here* $\mathrm{Hom}_{\mathscr{H}}(\mathscr{L}, \mathscr{L}^{\mathscr{S}})$ *is considered in the natural way as a subset of* $\mathrm{Hom}_{\mathscr{H}}(\mathscr{L}, \mathscr{L}^{\mathscr{G}})$.)

If $\mu \in E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})$, the equations

$$(A \otimes_{\mathscr{A}(\mathscr{S})} M)\mu^* = A \otimes_{\mathscr{A}(\mathscr{S})} (M\mu) \qquad (A \in \mathscr{A}(\mathscr{G}), M \in \mathscr{L}^{\mathscr{S}})$$

define an element μ^* of $E_{\mathscr{G}}(\mathscr{L}^{\mathscr{G}})$. Moreover, the mapping $\mu \to \mu^*$ of $E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})$ into $E_{\mathscr{G}}(\mathscr{L}^{\mathscr{G}})$ is a ring monomorphism.

LEMMA 2.

$$E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})^* + \bar{R}_{\mathscr{G}}(\mathscr{L}^{\mathscr{G}}) = E_{\mathscr{G}}(\mathscr{L}^{\mathscr{G}}),$$
$$E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})^* \cap \bar{R}_{\mathscr{G}}(\mathscr{L}^{\mathscr{G}}) = \bar{R}_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})^*$$

where

$$\bar{R}_{\mathscr{G}}(\mathscr{L}^{\mathscr{G}}) = E_{\mathscr{G}}(\mathscr{L}^{\mathscr{G}}) \cap R_{\mathscr{H}}(\mathscr{L}^{\mathscr{G}}),$$
$$\bar{R}_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}}) = E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}}) \cap R_{\mathscr{H}}(\mathscr{L}^{\mathscr{S}}).$$

PROOF. Let $\mu \in E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})$. Then $\mu = \eta^{\mathscr{S}}$, $\eta \in \mathrm{Hom}_{\mathscr{H}}(\mathscr{L}, \mathscr{L}^{\mathscr{S}})$. Also $\eta^{\mathscr{G}} = \mu^*$. By lemma 1, corollary, $\mu \in R_{\mathscr{H}}(\mathscr{L}^{\mathscr{S}})$ if, and only if, $\mu^* \in R_{\mathscr{H}}(\mathscr{L}^{\mathscr{G}})$. This gives the second relation.

Now let $\rho \in E_{\mathscr{G}}(\mathscr{L}^{\mathscr{G}})$. Then $\rho = \zeta^{\mathscr{G}}$, $\zeta \in \mathrm{Hom}_{\mathscr{H}}(\mathscr{L}, \mathscr{L}^{\mathscr{G}})$. Write

$$\zeta = \zeta' + \zeta'',$$

where $\zeta' = \sum_{\alpha=1}^{s} \zeta_\alpha \omega_\alpha$. Since $\zeta' \in \mathrm{Hom}_{\mathscr{H}}(\mathscr{L}, \mathscr{L}^{\mathscr{S}})$, $\zeta'^{\mathscr{G}} = (\zeta'^{\mathscr{S}})^* \in E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})^*$. Also, by lemma 1, $\zeta''^{\mathscr{G}} \in R_{\mathscr{H}}(\mathscr{L}^{\mathscr{G}})$. Hence

$$\rho = \zeta^{\mathscr{G}} = \zeta'^{\mathscr{G}} + \zeta''^{\mathscr{G}} \in E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})^* + \bar{R}_{\mathscr{G}}(\mathscr{L}^{\mathscr{G}}).$$

This proves the first relation.

COROLLARY 1. *If* $\varepsilon = \sum \varepsilon_\lambda$ *is a decomposition of the identity of* $E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})$ *into indecomposable idempotents in* $E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})$, *then* $\varepsilon^* = \sum \varepsilon_\lambda^*$ *is a similar decomposition in* $E_{\mathscr{G}}(\mathscr{L}^{\mathscr{G}})$.

COROLLARY 2. $E_{\mathfrak{g}}(\mathscr{L}^{\mathfrak{g}})/\tilde{R}_{\mathfrak{g}}(\mathscr{L}^{\mathfrak{g}}) \approx E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})/\tilde{R}_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})$.

COROLLARY 3. $E_{\mathfrak{g}}(\mathscr{L}^{\mathfrak{g}})/R_{\mathfrak{g}}(\mathscr{L}^{\mathfrak{g}}) \approx E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})/R_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})$.

(Notice here that $\tilde{R}_{\mathfrak{g}}$, $\tilde{R}_{\mathscr{S}}$ are nilpotent ideals of $E_{\mathfrak{g}}$, $E_{\mathscr{S}}$, so that $\tilde{R}_{\mathfrak{g}} \subseteqq R_{\mathfrak{g}}$, $\tilde{R}_{\mathscr{S}} \subseteqq R_{\mathscr{S}}$).

Now consider $E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})$. We remark first that $\eta \to \eta^{\mathscr{S}}$ gives a ring monomorphism of $E_{\mathscr{H}}(\mathscr{L})$ into $E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})$. (Here, and in what follows, we regard $E_{\mathscr{H}}(\mathscr{L}) = \mathrm{Hom}_{\mathscr{H}}(\mathscr{L}, \mathscr{L})$ and $\mathrm{Hom}_{\mathscr{H}}(\mathscr{L}, \mathscr{L}_\alpha)$ ($\alpha \leqq s$) as subsets of $\mathrm{Hom}_{\mathscr{H}}$ $(\mathscr{L}, \mathscr{L}^{\mathscr{S}})$.) We denote the image of $E_{\mathscr{H}}(\mathscr{L})$ in $E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})$ by $E_{\mathscr{H}}(\mathscr{L})^{\mathscr{S}}$.

Write $T_\alpha = X_\alpha \mathscr{H}$ ($\alpha = 1, \cdots, s$) for the elements of $\mathscr{S}/\mathscr{H}$. For each T_α, choose an $\mathscr{H}$-isomorphism $\xi_\alpha : \mathscr{L} \to \mathscr{L}_\alpha$, and form

$$(2) \qquad (T_\alpha) = \xi_\alpha^{\mathscr{S}}.$$

Clearly, if $T, T' \in \mathscr{S}/\mathscr{H}$, $(T)(T')(TT')^{-1}$ maps $\mathscr{L}$ onto $\mathscr{L}$ and so belongs to $E_{\mathscr{H}}(\mathscr{L})^{\mathscr{S}}$:

$$(3) \qquad (T)(T') = \eta_{T,T'}^{\mathscr{S}}(TT') \qquad (\eta_{T,T'} \in E_{\mathscr{H}}(\mathscr{L})).$$

Similarly, if $\eta \in E_{\mathscr{H}}(\mathscr{L})$, $T \in \mathscr{S}/\mathscr{H}$, $(T)^{-1}\eta^{\mathscr{S}}(T) \in E_{\mathscr{H}}(\mathscr{L})^{\mathscr{S}}$ and we write

$$(T)^{-1}\eta^{\mathscr{S}}(T) = (\eta^{(T)})^{\mathscr{S}}, \qquad \eta^{(T)} \in E_{\mathscr{H}}(\mathscr{L}).$$

Clearly, $\eta \to \eta^{(T)}$ is an $\mathscr{F}$-algebra automorphism of $E_{\mathscr{H}}(\mathscr{L})$; and in fact, if $(T) = \xi^{\mathscr{S}}$, $\xi^{-1}\eta\xi = \eta^{(T)}$.

Finally, since an arbitrary element ζ of $\mathrm{Hom}_{\mathscr{H}}(\mathscr{L}, \mathscr{L}^{\mathscr{S}})$ has the form

$$\zeta = \sum_{\alpha=1}^{s} \zeta_\alpha \omega_\alpha = \sum_{\alpha=1}^{s} \eta_\alpha \xi_\alpha \omega_\alpha, \qquad \eta_\alpha \in E_{\mathscr{H}}(\mathscr{L}),$$

each element of $E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})$ can be uniquely expressed in the form

$$\zeta^{\mathscr{S}} = \sum_{T \in \mathscr{S}/\mathscr{H}} \eta_T^{\mathscr{S}}(T), \qquad \eta_T \in E_{\mathscr{H}}(\mathscr{L}).$$

Thus $E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})$ is a kind of twisted group algebra on $\mathscr{S}/\mathscr{H}$ over $E_{\mathscr{H}}(\mathscr{L})$, though the (T) do not commute with the coefficients $\eta^{\mathscr{S}}$.

By lemma 1, $\zeta^{\mathscr{S}} \in \tilde{R}_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})$ if, and only if, no η_T is an $\mathscr{H}$-isomorphism, i.e. if, and only if, all $\eta_T \in R_{\mathscr{H}}(\mathscr{L})$. Thus to get $E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})/\tilde{R}_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})$, we simply replace all the η's in all above by their canonical images $\bar{\eta} = \eta + R_{\mathscr{H}}(\mathscr{L})$ in $E_{\mathscr{H}}(\mathscr{L})/R_{\mathscr{H}}(\mathscr{L})$. Thus $E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})/\tilde{R}_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})$ appears as a generalized twisted group algebra over the division algebra $E_{\mathscr{H}}(\mathscr{L})/R_{\mathscr{H}}(\mathscr{L})$. The operations $\eta \to \eta^{(T)}$ are $\mathscr{F}$-algebra automorphisms of $E_{\mathscr{H}}(\mathscr{L})/R_{\mathscr{H}}(\mathscr{L})$. From now on we assume $\mathscr{F}$ algebraically closed. Thus $E_{\mathscr{H}}(\mathscr{L})/R_{\mathscr{H}}(\mathscr{L})$ is the 1-dimensional $\mathscr{F}$-algebra $\mathscr{F}$ itself, so $\bar{\eta} = \bar{\eta}^{(T)}$ $(= \overline{\eta^{(T)}})$, all T. Here $E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})/\tilde{R}_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})$ becomes a genuine twisted group algebra $\mathscr{A}(\mathscr{S}/\mathscr{H})$ on $\mathscr{S}/\mathscr{H}$ over $\mathscr{F}$.

The following lemma by Fitting [7] provides the link between a module and its ring of endomorphisms. We use the term "component" to mean "indecomposable direct summand".

Lemma 3. *Let $\mathscr{A}$ be a finite dimensional algebra (with a 1) over $\mathscr{F}$ and let $\mathscr{M}$ be an $\mathscr{A}$-module (finite dimensional) with $\mathscr{E}$ as its ring of $\mathscr{A}$-endomorphisms. Let*

$$\mathscr{E} = \mathscr{E}\varepsilon_{11} \oplus \cdots \oplus \mathscr{E}\varepsilon_{1n_1} \oplus \cdots \oplus \mathscr{E}\varepsilon_{mn_m}$$

be a decomposition of $\mathscr{E}$ into left ideal components, where $\mathscr{E}\varepsilon_{ij} \approx \mathscr{E}\varepsilon_{i'j'}$ if, and only if, $i = i'$. Let

$$\mathscr{M} = \mathscr{M}_{11} \oplus \cdots \oplus \mathscr{M}_{1n'_1} \oplus \cdots \oplus \mathscr{M}_{m'n'_{m'}}$$

be a decomposition of $\mathscr{M}$ into components, with $\mathscr{M}_{ij} \approx \mathscr{M}_{i'j'}$ if and only if, $i = i'$. Then $m = m'$, $n = n'$, and one possible choice of $\mathscr{M}_{\alpha\beta}$ is given by $\mathscr{M}_{\alpha\beta} = \mathscr{M}\varepsilon_{\alpha\beta}$.

Let

$$\mathscr{L}^{\mathscr{S}} = \mathscr{M}_1 \oplus \cdots \oplus \mathscr{M}_l \tag{4}$$

be a decomposition of $\mathscr{L}^{\mathscr{S}}$ into $\mathscr{S}$-components. We can further write

$$(\mathscr{M}_\alpha)_{\mathscr{H}} = \mathscr{M}_{\alpha 1} \oplus \cdots \oplus \mathscr{M}_{\alpha k_\alpha}, \tag{5}$$

where each of the $\mathscr{M}_{\alpha\beta} \approx \mathscr{L}$, by the Krull-Schmidt theorem. Let $\varepsilon = \sum_{\alpha=1}^{l} \varepsilon_\alpha$ be a decomposition of the identity of $E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})$ according to (4). Then each ε_α can be further decomposed by (5) in the form

$$\varepsilon_\alpha = \sum_{\beta=1}^{k_\alpha} \varepsilon_{\alpha\beta}^{\mathscr{S}}, \qquad \varepsilon_{\alpha\beta} \in \operatorname{Hom}_{\mathscr{H}}(\mathscr{L}, \mathscr{L}^{\mathscr{S}}),$$

and any element π of $E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})$ has a unique expression in the form

$$\pi = \sum_{\alpha,\beta} \pi_{\alpha\beta}^{\mathscr{S}} \varepsilon_{\alpha\beta}^{\mathscr{S}}, \qquad \pi_{\alpha\beta} \in E_{\mathscr{H}}(\mathscr{L}).$$

Clearly $\sum k_\alpha = s$, and the left ideal $E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})\varepsilon_\alpha$, considered as a module over $E_{\mathscr{H}}(\mathscr{L})$, is the direct sum of k_α copies of $E_{\mathscr{H}}(\mathscr{L})$. Hence the dimension over $\mathscr{F}$ of the corresponding left ideal in $\mathscr{A}(\mathscr{S}/\mathscr{H})$ $(= E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})/\tilde{R}_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}}))$ is precisely k_α. Moreover, as $\tilde{R}_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})$ is nilpotent, the images of the two left ideal components in the quotient ring are isomorphic if, and only if, the corresponding left ideal components of the original ring $E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})$ are isomorphic [9]. Combining these results we have that the decomposition of $\mathscr{L}^{\mathscr{S}}$ is entirely reflected by the decomposition of $\mathscr{A}(\mathscr{S}/\mathscr{H})$ into left ideals.

Now $\mathscr{L}^{\mathscr{g}} \approx (\mathscr{L}^{\mathscr{S}})^{\mathscr{g}} \approx \mathscr{M}_1^{\mathscr{g}} \oplus \cdots \oplus \mathscr{M}_1^{\mathscr{g}}$. Further, by corollary 3 to lemma 2 each $\mathscr{M}_\alpha^{\mathscr{g}}$ must remain indecomposable. Moreover, as $R_{\mathscr{g}}(\mathscr{L}^{\mathscr{g}})$

[9] This was noted in § 1 of Nakayama [15] for the case where the kernel is actually the radical of $E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})$.

is nilpotent, the multiplicities of the different isomorphism types of left ideal components of $E_{\mathscr{G}}(\mathscr{L}^{\mathscr{G}})$ are the same as in $E_{\mathscr{G}}(\mathscr{L}^{\mathscr{G}})/R_{\mathscr{G}}(\mathscr{L}^{\mathscr{G}})$, i.e. as in $E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})/R_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})$ (by lemma 2, corollary 3), i.e. as in $E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})$ (since $R_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})$ is nilpotent). Hence we have proved the following theorem.

THEOREM. *Let $\mathscr{A}(\mathscr{H})$ be the restriction of a normalized twisted group algebra $\mathscr{A}(\mathscr{G})$ over an algebraically closed field $\mathscr{F}$ to a normal subgroup $\mathscr{H}$ of $\mathscr{G}$, and let $\mathscr{L}$ be an indecomposable $\mathscr{A}(\mathscr{H})$-module with stabilizer $\mathscr{S}$ in $\mathscr{G}$. Then the decomposition of $\mathscr{L}^{\mathscr{G}}$ is entirely determined by the decomposition of a certain twisted group algebra $\mathscr{A}(\mathscr{S}/\mathscr{H})$ into left ideals, there being a 1—1 correspondence between left ideal components $\mathscr{I}_\alpha$ and components $\mathscr{N}_\alpha$ of $\mathscr{L}^{\mathscr{G}}$, such that the left ideals are isomorphic if, and only if, the corresponding summands are. Further*

$$\dim_{\mathscr{F}}\mathscr{N}_\alpha = \dim_{\mathscr{F}}(\mathscr{I}_\alpha)\cdot\dim_{\mathscr{F}}(\mathscr{L})\cdot(\mathscr{G}:\mathscr{S}).$$

A decomposition of $\mathscr{L}^{\mathscr{G}}$ is obtained from one of $\mathscr{A}(\mathscr{S}/\mathscr{H})$ as follows: The decomposition of $\mathscr{A}(\mathscr{S}/\mathscr{H}) \approx E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})/\bar{R}_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})$ is raised to one of $E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})$ by the algorithm used in the proof of theorem 9.3c in [1]. A decomposition of $\mathscr{L}^{\mathscr{S}} = \sum \mathscr{M}_\alpha$ is obtained as in lemma 3. Finally we may take $\mathscr{N}_\alpha = \mathscr{M}_\alpha^{\mathscr{G}}$.

If $\mathscr{L}$ is irreducible, then $E_{\mathscr{S}}(\mathscr{L}^{\mathscr{S}})$ is the twisted group algebra $\mathscr{A}(\mathscr{S}/\mathscr{H})$, as $E_{\mathscr{H}}(\mathscr{L}) \approx \mathscr{F}$.

COROLLARY 1. *If $\mathscr{L}$ is not indecomposable, say*

$$\mathscr{L} = \mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_h,$$

then

$$\mathscr{L}^{\mathscr{G}} = \mathscr{L}_1^{\mathscr{G}} \oplus \cdots \oplus \mathscr{L}_h^{\mathscr{G}},$$

as tensor product $\otimes$ is distributive over direct sum $\oplus$. We apply the theorem to each $\mathscr{L}_\alpha^{\mathscr{G}}$ to obtain the decomposition of $\mathscr{L}^{\mathscr{G}}$.

The problem of inducing up from a subnormal subgroup is equivalent to the decomposition of a series of twisted group algebras. For, if $\mathscr{H} \leqq \mathscr{H}_1 \leqq \mathscr{G}$, we have $(\mathscr{L}^{\mathscr{H}_1})^{\mathscr{G}} \cong \mathscr{L}^{\mathscr{G}}$.

COROLLARY 2. *If $\mathscr{H}$ is a subnormal subgroup of $\mathscr{G}$ of prime power index p^v in $\mathscr{G}$, with $\mathscr{F}$ of characteristic $p \neq 0$, then $\mathscr{L}^{\mathscr{G}}$ is indecomposable if $\mathscr{L}$ is.*

PROOF. Clearly the factor groups are p-groups and so the twisted group algebras involved are on p-groups. Hence by § 1, remark 7, these are indecomposable. (c.f. Theorem 8 of Green [8]).

In decomposing a twisted group algebra $\mathscr{A}(\mathscr{G})$ into left ideals, we may make use of a composition series of $\mathscr{G}$ and consider $\mathscr{A}(\mathscr{G}) = (\mathscr{F}_{\{E\}})^{\mathscr{G}}$, where $\mathscr{F}_{\{E\}}$ is the trivial representation of the group $\{E\}$. This leaves only

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the problem of the decomposition of twisted group algebras on simple groups.

A detailed analysis will now be given of the decomposition of $\mathscr{L}^{\mathscr{S}}$. Let $H \to \lambda(H)$ be the linear representation afforded by the module $\mathscr{L}$. All such linear mappings will be written on the left. In particular an element of $E_{\mathscr{H}}(\mathscr{L})$ will be represented by a linear mapping θ written on the left.

Corresponding to each $\alpha = 1, \cdots, s$ we have a non-singular linear transformation D_α such that the $\mathscr{A}(\mathscr{H})$-isomorphism ξ_α of equation (2) is given by

$$\xi_\alpha : L \to (X_\alpha) \otimes D_\alpha L \qquad (L \in \mathscr{L}). \tag{6}$$

If we make a second choice of isomorphisms, say $\xi'_\alpha : \mathscr{L} \to \mathscr{L}_\alpha$, and if D'_α are the corresponding linear mappings, then

$$D_\alpha = \theta D'_\alpha,$$

where θ is a linear mapping representing an automorphism in $E_{\mathscr{H}}(\mathscr{L})$. We choose $D_1 = I$, the identity map. If $X_\alpha X_\beta = X_\gamma H$, then corresponding to equation (3) we have

$$D_\alpha D_\beta = \frac{f_{X_\alpha, X_\beta}}{f_{X_\gamma, H}} \theta_{\alpha,\beta} D_\gamma \lambda(H), \tag{7}$$

where $\theta_{\alpha,\beta}$ represents an automorphism in $E_{\mathscr{H}}(\mathscr{L})$, and where this equation may be taken as defining $\theta_{\alpha,\beta}$. As $D_1 = I$, it follows that $\theta_{\alpha,1} = \theta_{1,\alpha} = I$ also.

We now define D_S for $S = X_\alpha H \in \mathscr{S}$:

$$D_S = f^{-1}_{X_\alpha, H} D_\alpha \lambda(H), \tag{8}$$

and so $D_{X_\alpha} = D_\alpha$, $D_E = D_1 = I$. Then from these definitions it follows that if $S \in X_\alpha \mathscr{H}$, $S' \in X_{\alpha'} \mathscr{H}$,

$$D_S D_{S'} = f_{S,S'} \theta_{\alpha,\alpha'} D_{SS'}. \tag{9}$$

Thus the correspondence $S \to D_S$ gives rise to an extension of $\mathscr{L}$ to $\mathscr{A}(\mathscr{S})$ if, and only if, $\theta_{\alpha,\beta} = 1$, all α, β.

For the case of $\mathscr{L}$ irreducible the analysis of Clifford in the proof of his theorem 3 in [5] (although not starting from the same point of view) can be adopted to get an explicit view of $\mathscr{L}^{\mathscr{S}}$.

Proposition 1. *Let $\mathscr{L}$ be an irreducible $\mathscr{A}(\mathscr{H})$-module. Then any direct summand $\mathscr{M}$ of $\mathscr{L}^{\mathscr{S}}$ affords a linear representation $S \to \psi(S)$ of $\mathscr{A}(\mathscr{S})$, which is the product of a fixed projective linear representation $S \to D_S$ of $\mathscr{A}(\mathscr{S})$ (independent of $\mathscr{M}$) together with a certain direct summand $\pi(S\mathscr{H})$ of the linear representation afforded by considering $\mathscr{A}(\mathscr{S}|\mathscr{H})$ as a left module over itself ("regular representation" of $\mathscr{A}(\mathscr{S}|\mathscr{H})$), i.e.,*

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(10) $$\psi(S) = D_S \times \pi(S\mathscr{H}).$$ [10]

Thus $\mathscr{M}$ must decompose just as π does. For $\mathscr{M} = \mathscr{L}^{\mathscr{S}}$, the decomposition of $\mathscr{L}^{\mathscr{S}}$ is related directly to that of $\mathscr{A}(\mathscr{S}/\mathscr{H})$ into left ideals.

Again following Clifford's line of argument, we have:

PROPOSITION 2. *In the situation of proposition* 1, *if π is an irreducible linear representation of $\mathscr{A}(\mathscr{S}/\mathscr{H})$, then the linear representation of $\mathscr{A}(\mathscr{S})$ given by* (10) *is irreducible.*

The analysis in the proof of Clifford's theorem 2 in [5] provides an explicit relation between the decomposition of $\mathscr{L}^{\mathscr{S}}$ and that of $\mathscr{L}^{\mathscr{G}}$.

Finally we consider certain problems on extensions of $\mathscr{L}$.

PROPOSITION 3. *Let $\mathscr{G}/\mathscr{H}$ be cyclic of order m and suppose that either $p = 0$, or $(m, p) = 1$. Let $\mathscr{L}$ (indecomposable) have stabilizer the whole of $\mathscr{G}$. Then there exist exactly m extensions of $\mathscr{L}$ to be an $\mathscr{A}(\mathscr{G})$-module to within $\mathscr{A}(\mathscr{G})$-isomorphism.*

PROOF. By the theorem $\mathscr{L}^{\mathscr{G}}$ decomposes just as $\mathscr{A}(\mathscr{G}/\mathscr{H})$ does. By § 1, remark 8, this must be the group algebra $\mathscr{F}(\mathscr{G}/\mathscr{H})$ and so decomposes into m non-isomorphic one-dimensional left ideals. Hence $\mathscr{L}^{\mathscr{G}}$ consists of the direct sum of m non-isomorphic extensions of $\mathscr{L}$.

Furthermore these are the only possible extensions of $\mathscr{L}$. For, say

$$G \to D_G,$$

where

(11) $$D_H = \lambda(H) \qquad (H \in \mathscr{H}),$$

is the linear representation afforded by any other extension of $\mathscr{L}$ as an $\mathscr{A}(\mathscr{G})$-module. $D_\alpha = D_{X_\alpha}$ is then a possible choice of D's in (6); it follows that $\theta_{\alpha,\beta} = I$, from (9). If $G_1\mathscr{H} (G_1 \in \mathscr{G})$ generates $\mathscr{G}/\mathscr{H}$, then all $D_G (G \in \mathscr{G})$ are determined in terms of D_{G_1}, by equations (7), (8) and (11). A calculation shows that the m extensions of $\mathscr{L}$ contained in $\mathscr{L}^{\mathscr{G}}$ have the linear representations determined by

(12) $$G_1 \to \omega^j D_{G_1}$$

where ω is a primitive m-th root of unity in $\mathscr{F}$.

PROPOSITION 4 [11]. *Let $\mathscr{G}/\mathscr{H}$ be a cyclic extension of a p-subgroup, where $\mathscr{F}$ has characteristic $p \neq 0$. Let $|\mathscr{G}/\mathscr{H}| = mp^a$, $(m, p) = 1$ and let $\mathscr{L}$ be an irreducible $\mathscr{A}(\mathscr{H})$-module, which has stabilizer the whole of $\mathscr{G}$. Then there exist exactly m extensions of $\mathscr{L}$ to be an $\mathscr{A}(\mathscr{G})$-module to within $\mathscr{A}(\mathscr{G})$-isomorphism.*

[10] Here $\times$ denotes the Kronecker or tensor product.

[11] Propositions 3 and 4 are generalizations of lemmas 1 and 2 of Srinivasan [19].

PROOF. As $\mathscr{L}$ is irreducible, $\mathscr{F}$ algebraically closed, $E_{\mathscr{G}}(\mathscr{L}^{\mathscr{G}}) = \mathscr{A}(\mathscr{G}/\mathscr{H})$, and $E_{\mathscr{H}}(\mathscr{L}) \approx \mathscr{F}$. The D_α of (6) are then determined to within a factor in $\mathscr{F}^*$, and the $\theta_{\alpha,\beta}$ are elements of $\mathscr{F}^*$. A different choice of D_α's gives a basis transformation of type § 1, (1) on $\mathscr{A}(\mathscr{G}/\mathscr{H})$. By § 1, remark 9, $\mathscr{A}(\mathscr{G}/\mathscr{H})$ is the group algebra on $\mathscr{G}/\mathscr{H}$ and so the $\theta_{\alpha,\beta}$ may be considered equal to 1. Then $G \to D_G$ is a linear representation of an extension of $\mathscr{L}$ to $\mathscr{A}(\mathscr{G})$ by (9).

Write $\mathscr{P}$ for the subgroup of $\mathscr{G}$, such that $\mathscr{P}/\mathscr{H}$ is the Sylow p-group of $\mathscr{G}/\mathscr{H}$. Restricting our attention to $\mathscr{A}(\mathscr{P})$ and $\mathscr{A}(\mathscr{P}/\mathscr{H})$, we see that if $\theta_{\alpha,\beta} = 1$, then the choice of D_P $(P \in \mathscr{P})$ is uniquely determined, for the only basis transformation of type § 1 (1) on the group algebra of a p-group, keeping the multiplication constants all 1, is the identity transformation. Let $\mathscr{M}$ be this unique extension of $\mathscr{L}$ to $\mathscr{A}(\mathscr{P})$.

By proposition 3, $\mathscr{M}$ has exactly m different extensions to $\mathscr{A}(\mathscr{G})$ to within isomorphism.

3. Blocks and centres of twisted group algebras

The decomposition of a finite dimensional algebra $\mathscr{A}$ into the direct sum of two sided ideals is determined by the corresponding decomposition of the centre $\mathscr{Z}$. This in turn is determined by the decomposition of the identity element (E) as the sum of primitive central idempotents:

(1) $$(E) = I_1 + \cdots + I_s.$$

The term *block* will be used to describe either an I_λ or the corresponding two sided ideal of $\mathscr{Z}$ or $\mathscr{A}$.

Rosenberg's analysis [16] of blocks of group algebras can be adapted to the twisted case by using the normalization theorem of § 1.

If $\mathscr{A}(\mathscr{G})$ is u-normalized, then a basis for its centre $\mathscr{Z}(\mathscr{G})$ is provided by the u-class sums K_α, as in § 1, remark 4. Then any block can be expressed as:

(2) $$I = \sum f_\alpha K_\alpha.$$

Let us assume that the field characteristic $p \neq 0$. Consider the centralizers $\mathscr{C}(A)$ in $\mathscr{G}$ of elements A of $\mathscr{G}$ which have non-zero coefficients in (2). The largest among the Sylow p-subgroups of these $\mathscr{C}(A)$ is well defined up to conjugacy in $\mathscr{G}$ and is the *defect group* $\mathscr{D}$ of I. If $|\mathscr{D}| = p^d$, d is called the *defect* of I.

If $\mathscr{D}$ is any subgroup of $\mathscr{G}$, write $\mathscr{N}(\mathscr{D})$ for the normalizer of $\mathscr{D}$ in $\mathscr{G}$ and $\mathscr{C}(\mathscr{D})$ for the centralizer of $\mathscr{D}$ in $\mathscr{G}$.

Take $\mathscr{D}$ to be a p-group and write $\mathscr{H} = \mathscr{N}(\mathscr{D})$. Let $\mathscr{Z}(\mathscr{H})$ be the centre of $\mathscr{A}(\mathscr{H})$. Consider a u-class $\mathscr{K}$ of elements of $\mathscr{G}$ with u-class sum K and write

$$\sigma(K) = \text{sum of elements } (A),$$

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where $A \in \mathscr{K} \cap \mathscr{C}(\mathscr{D})$, if such elements exist, 0 otherwise. σ can be extended to the whole of $\mathscr{Z}(\mathscr{G})$ by linearity and is verified to be an $\mathscr{F}$-algebra homomorphism,

$$\sigma : \mathscr{Z}(\mathscr{G}) \to \mathscr{Z}(\mathscr{H}).$$

In the case of group algebras, Brauer's first theorem on blocks may be stated as follows:

σ gives a $1-1$ correspondence between the blocks of $\mathscr{Z}(\mathscr{G})$ with $\mathscr{D}$ as one of their defect groups and the blocks of $\mathscr{Z}(\mathscr{H})$ of defect d. The latter have $\mathscr{D}$ as their unique defect group.

However, in the twisted case a complication arises as an element H $(\in \mathscr{H})$ may be a u-element in $\mathscr{A}(\mathscr{H})$ but not in $\mathscr{A}(\mathscr{G})$. To overcome this difficulty we define $\mathscr{U}(\mathscr{D})$ to be the subspace of $\mathscr{Z}(\mathscr{H})$ spanned by those u-class sums of $\mathscr{A}(\mathscr{H})$ which have defect group $\mathscr{D}$ and whose elements are u-elements in $\mathscr{A}(\mathscr{G})$. Then $\mathscr{U}(\mathscr{D})$ is a subalgebra of $\mathscr{Z}(\mathscr{H})$. The theorem for blocks in the twisted case can now be stated as follows:

σ gives a $1-1$ correspondence between the blocks of $\mathscr{Z}(\mathscr{G})$ with $\mathscr{D}$ as one of their defect groups and primitive idempotents of $\mathscr{U}(\mathscr{D})$. Each such idempotent is the sum of primitive idempotents of $\mathscr{Z}(\mathscr{H})$ with $\mathscr{D}$ as their unique defect group.

Since this last theorem has reduced (to a certain extent) the problem to the case of blocks I with a normal defect group $\mathscr{D}$ (which must then be unique), this special case warrants more attention. As $\mathscr{D}$ is normal in $\mathscr{G}$, it is certainly contained in the maximal normal p-subgroup $\bar{\mathscr{D}}$ of $\mathscr{G}$. Let us suppose then that $\mathscr{A}(\mathscr{G})$ has been p-u-normalized. Then the natural homomorphism $\mathscr{G} \to \mathscr{G}/\bar{\mathscr{D}}$ gives rise to an algebra homomorphism

$$\tau : \mathscr{A}(\mathscr{G}) \to \mathscr{A}(\mathscr{G}/\bar{\mathscr{D}}),$$

where $\mathscr{A}(\mathscr{G}/\bar{\mathscr{D}})$ is a twisted group algebra on $\mathscr{G}/\bar{\mathscr{D}}$. Ker τ is spanned by the elements $(A)((D)-(E))$, $A \in \mathscr{G}$, $D \in \bar{\mathscr{D}}$, and is a nilpotent ideal of $\mathscr{A}(\mathscr{G})$. Further if K is a u-class sum of $\mathscr{A}(\mathscr{G})$, such that $\mathscr{K} \cap \mathscr{C}(\bar{\mathscr{D}}) = \emptyset$, then $\tau(K) = 0$, and so K is nilpotent. As ker τ is nilpotent, τ provides a $1-1$ correspondence between idempotents of $\mathscr{Z}(\mathscr{G})$ and those of $\mathscr{Z}(\mathscr{G}/\bar{\mathscr{D}})$; thus the problem of blocks is further reduced to the case of defect $d = 0$.

Finally we have the following theorem for blocks of maximum defect, which we prove in full as the u-property needs careful attention.

THEOREM. *Let $\mathscr{G}$ have order $p^a m$, $(m, p) = 1$. Let $\mathscr{A}(\mathscr{G})$ be a twisted group algebra over an algebraically closed field $\mathscr{F}$ of characteristic $p \neq 0$. Then the number of blocks of defect a equals the number of p-regular u-classes of defect[12] a.*

[12] The defect group of a conjugacy class is any one of the Sylow p-subgroups of the centralizers in $\mathscr{G}$ of its elements.

Proof. A block of $\mathscr{A}(\mathscr{G})$ of defect a has the Sylow p-subgroups as its defect groups. Let $\mathscr{D}$ be any such and write $\mathscr{H} = \mathscr{N}(\mathscr{D})$. Then the above theorem tells us that the number of blocks of defect a is the same as the number of primitive idempotents of $\mathscr{U}(\mathscr{D})$.

The homomorphism τ,

$$\tau : \mathscr{A}(\mathscr{H}) \to \mathscr{A}(\mathscr{H}/\mathscr{D}),$$

is defined as above. $\mathscr{U}(\mathscr{D})$ contains the identity element of $\mathscr{A}(\mathscr{H})$ and so, as $\ker \tau$ is nilpotent, the restriction of τ to $\mathscr{U}(\mathscr{D})$ gives a 1—1 correspondence between idempotents of $\mathscr{U}(\mathscr{D})$ and those of $\tau(\mathscr{U}(\mathscr{D}))$. $\mathscr{A}(\mathscr{H}/\mathscr{D})$ is semi-simple by § 1, remark 6, and so its centre $\mathscr{Z}(\mathscr{H}/\mathscr{D})$ is the direct sum of copies of $\mathscr{F}$. As $\tau(\mathscr{U}(\mathscr{D}))$ is a subalgebra of $\mathscr{Z}(\mathscr{H}/\mathscr{D})$, it is also semi-simple and hence the number of blocks of defect a in $\mathscr{A}(\mathscr{G})$ is equal to the dimension of $\tau(\mathscr{U}(\mathscr{D}))$.

We may assume that $\mathscr{A}(\mathscr{G})$, $\mathscr{A}(\mathscr{H})$ and $\mathscr{A}(\mathscr{H}/\mathscr{D})$ are (separately) p-u-normalized. Write (G), $[H]$ for the basis elements of $\mathscr{A}(\mathscr{G})$, $\mathscr{A}(\mathscr{H})$ respectively, where $G \in \mathscr{G}$, $H \in \mathscr{H}$ and $\{H\}$ for the basis element of $\mathscr{A}(\mathscr{H}/\mathscr{D})$ corresponding to the coset $H\mathscr{D}$ of $\mathscr{H}/\mathscr{D}$. Thus $\{H\} = \{HD\}$, for all $D \in \mathscr{D}$.

Let G be a u-element of $\mathscr{A}(\mathscr{G})$ such that $\mathscr{D}$ is a Sylow p-subgroup of $\mathscr{C}(G)$. Write $G = PR$, where P, R are powers of G, P has order a power of p, R is p-regular. Then $\mathscr{D}$ is a Sylow p-subgroup of $\mathscr{C}(R)$. Let $\mathscr{K}$ be the u-class of $\mathscr{G}$ containing G, and write $\mathscr{L} = \mathscr{K} \cap \mathscr{C}(\mathscr{D})$; then $\mathscr{L}$ is a complete[13] conjugacy class in $\mathscr{H}$. Thus

$$\sigma(K) = dL,$$

where K, L are the u-class sums of $\mathscr{K}$, $\mathscr{L}$. (The factor d ($\in \mathscr{F}^*$) has to be introduced because of the possibly different normalizations of $\mathscr{A}(\mathscr{G})$, $\mathscr{A}(\mathscr{H})$.) Then

$$\tau(\sigma(K)) = d\tau(L) \in \mathscr{Z}(\mathscr{H}/\mathscr{D}).$$

If $\tau(\sigma(K)) \neq 0$, it will now be proved that R is also a u-element in $\mathscr{A}(\mathscr{G})$.
If

$$H \in \mathscr{H}, \text{ write } \bar{\mathscr{C}}(H) = \text{centralizer of } H \text{ in } \mathscr{H}, \\ = \mathscr{C}(H) \cap \mathscr{H}.$$

$\mathscr{D}$ is the Sylow p-subgroup of $\bar{\mathscr{C}}(G)$. Further $P \in \bar{\mathscr{C}}(R)$ and so $P \in \mathscr{D}$. Thus $\{G\} = \{R\}$. As $\tau(\sigma(K)) \neq 0$, and $\tau(\sigma(K)) \in \mathscr{Z}(\mathscr{H}/\mathscr{D})$, $G\mathscr{D} = R\mathscr{D}$ must be a u-element in $\mathscr{A}(\mathscr{H}/\mathscr{D})$ (see § 1, remark 4). Take $N \in \bar{\mathscr{C}}(R)$ and write

$$[N][R][N^{-1}] = b[R], \\ \tau([R]) = c\{R\},$$

[13] This is proved in Rosenberg's paper [16].

where b, $c \in \mathscr{F}^*$. Then

$$\tau([N][R][N^{-1}]) = b\tau([R]) = bc\{R\}.$$

On the other hand this is equal to

$$\begin{aligned} &\tau([N])\tau([R])\tau([N^{-1}]), \\ &= \{N\}c\{R\}\{N^{-1}\} \quad \text{(as both } \mathscr{A}(\mathscr{H}),\ \mathscr{A}(\mathscr{H}/\mathscr{D}) \text{ are normalized)}, \\ &= c\{R\} \quad \text{(as } R\mathscr{D} \text{ is a } u\text{-element in } \mathscr{A}(\mathscr{H}/\mathscr{D})), \end{aligned}$$

and so $b = 1$, i.e., R is a u-element in $\mathscr{A}(\mathscr{H})$. Hence we have

$$(3) \qquad (N)(R)(N^{-1}) = (R)$$

in $\mathscr{A}(\mathscr{G})$, for all $N \in \mathscr{C}(R) \cap \mathscr{N}(\mathscr{D})$, as both $\mathscr{A}(\mathscr{G})$, $\mathscr{A}(\mathscr{H})$ are normalized.

Let $\mathscr{D}'$ be any other Sylow p-subgroup of $\mathscr{C}(R)$; then there exists $T \in \mathscr{C}(R)$ such that $\mathscr{D}' = T\mathscr{D}T^{-1}$. Thus

$$T(\mathscr{C}(R) \cap \mathscr{N}(\mathscr{D}))T^{-1} = \mathscr{C}(R) \cap \mathscr{N}(\mathscr{D}'),$$
$$TGT^{-1} = R(TPT^{-1}).$$

Take $TNT^{-1} \in \mathscr{C}(R) \cap \mathscr{N}(\mathscr{D}')$, where $N \in \mathscr{C}(R) \cap \mathscr{N}(\mathscr{D})$. From (3) we get

$$((T)(N)(T^{-1}))((T)(R)(T^{-1}))((T)(N^{-1})(T^{-1})) = (T)(R)(T^{-1}),$$

i.e.

$$((T)(N)(T^{-1}))(R)((T)(N^{-1})(T^{-1})) = (R).$$

Using § 1, remark 3, we get

$$(TNT^{-1})(R)(TN^{-1}T^{-1}) = (R),$$

and so

$$(4) \qquad (M)(R)(M^{-1}) = (R),$$

for all $M \in \mathscr{C}(R) \cap \mathscr{N}(\mathscr{D}')$.

Let $\mathscr{D}_1 = \mathscr{D}, \mathscr{D}_2, \cdots, \mathscr{D}_q$ be all the Sylow p-subgroups of $\mathscr{C}(R)$ and let $\mathscr{Q}$ be the group union of the subgroups $\mathscr{C}(R) \cap \mathscr{N}(\mathscr{D}_\alpha)$. Then $\mathscr{C}(R) = \mathscr{Q}$, for $\mathscr{Q}$ is normal in $\mathscr{C}(R)$ and $\mathscr{Q}$ contains the normalizer of a Sylow p-subgroup of $\mathscr{C}(R)$. Any element of $\mathscr{C}(R)$ has the form $C = A_1A_2 \cdots A_m$, where $A_\alpha \in$ some $\mathscr{C}(R) \cap \mathscr{N}(\mathscr{D}_\beta)$. Thus if $\tau(\sigma(K)) \neq 0$, then

$$\begin{aligned} (C)(R)(C^{-1}) &= (A_1 \cdots A_m)(R)(A_m^{-1} \cdots A_1^{-1}), \\ &= (A_1) \cdots (A_m)(R)(A_m^{-1}) \cdots (A_1^{-1}) \quad \text{(by § 1, remark 3)}, \\ &= (R) \quad \text{(by repeated use of (4))}, \end{aligned}$$

and so R is a u-element of $\mathscr{A}(\mathscr{G})$.

Let $\mathscr{K}_\alpha$ $(\alpha = 1, \cdots, r)$ be the p-regular u-classes of defect a in $\mathscr{A}(\mathscr{G})$ with corresponding u-class sums K_α. The $\mathscr{L}_\alpha = \mathscr{K}_\alpha \cap \mathscr{C}(\mathscr{D})$ consist of single conjugacy classes in $\mathscr{H}$, and so the $\sigma(K_\alpha)$ are multiples of the class sums L_α. Write $\mathscr{P} = \bigcup_\alpha \mathscr{L}_\alpha$ (set union). Then the $\{H\}$ ($H \in \mathscr{P}$) are all distinct in $\mathscr{A}(\mathscr{H}/\mathscr{D})$. For say $\{H\} = \{H'\}$. Then $H = H'D$, for some $D \in \mathscr{D}$. But each $\mathscr{L}_\alpha$ has defect group $\mathscr{D}$ and so $D \in \mathscr{D} \subset \mathscr{C}(H')$. Further, the orders of H, H' are prime to p and so $D = E$, or $H = H'$. Hence the $\tau(L_\alpha)$ are all non-zero and linearly independent. But $\tau(L_\alpha) \in \tau(\mathscr{U}(\mathscr{D}))$ and so $\dim \tau(\mathscr{U}(\mathscr{D})) \geqq r$. It remains to show that the $\tau(L_\alpha)$ actually span $\tau(\mathscr{U}(\mathscr{D}))$.

It is clear that the $\mathscr{L}_\alpha$ exhaust all the p-regular conjugacy classes of $\mathscr{H}$ of defect group $\mathscr{D}$ which consist of u-elements in $\mathscr{A}(\mathscr{G})$. Let then $\mathscr{L}$ be any p-singular class of $\mathscr{H}$ of defect group $\mathscr{D}$ and consisting of u-elements in $\mathscr{A}(\mathscr{G})$, i.e. L is a p-singular u-class sum in $\mathscr{U}(\mathscr{D})$. Take $G \in \mathscr{L}$, and write $G = PR$ as before. Then if $\tau(L) \neq 0$, R is a u-element of $\mathscr{A}(\mathscr{G})$ and $\tau(L)$ is equal to a multiple of $\tau(M)$, where M is the class sum of the conjugacy class $\mathscr{M}$ of R in $\mathscr{A}(\mathscr{H})$. But $\mathscr{M}$ must be one of the classes $\mathscr{L}_\alpha$ and so the $\tau(L_\alpha)$ do in fact span $\tau(\mathscr{U}(\mathscr{D}))$.

Thus the number of blocks of $\mathscr{A}(\mathscr{G})$ of highest defect $= \dim \tau(\mathscr{U}(\mathscr{D})) = r$, the number of p-regular u-classes of highest defect a.

4. Vertices and sources

The results of Higman [9] [10] and Green [8] can also be carried over to the twisted case. Here the generalization is even more direct than in § 3 and for most of the results we need only insist that the algebras be normalized. As before all modules will be assumed to have finite dimension over $\mathscr{F}$.

Let $\mathscr{H}$ be a subgroup of $\mathscr{G}$. An $\mathscr{A}(\mathscr{G})$-module $\mathscr{M}$ is said to be *$\mathscr{H}$-projective* if there esists an $\mathscr{A}(\mathscr{H})$-module $\mathscr{R}$ such that $\mathscr{M}$ is isomorphic to an $\mathscr{A}(\mathscr{G})$-direct summand of $\mathscr{R}^{\mathscr{G}}$. This definition is equivalent to $\mathscr{M}$ being $(\mathscr{A}(\mathscr{G}), \mathscr{A}(\mathscr{H}))$-projective or $(\mathscr{A}(\mathscr{G}), \mathscr{A}(\mathscr{H}))$-injective in the sense of Hochschild [12] or Higman [11].

When $\mathscr{F}$ has characteristic $p = 0$, or $p \nmid |G|$, by § 1, remark 6, $\mathscr{A}(\mathscr{G})$ is semi-simple. Hence all $\mathscr{A}(\mathscr{G})$-indecomposables occur in the regular representation. Thus all $\mathscr{A}(\mathscr{G})$-modules are $\{E\}$-projective and the theory is trivial. From now on we assume $p \neq 0$.

Higman's criterion [14] for $\mathscr{M}$ to be $\mathscr{H}$-projective can be written down immediately. Further, taking $\mathscr{H} = \mathscr{P}$, a Sylow p-subgroup of $\mathscr{G}$, we find that every indecomposable $\mathscr{A}(\mathscr{G})$-module $\mathscr{M}$ is a component of a module induced from some $\mathscr{A}(\mathscr{P})$-module. But by § 1, remark 7, if $\mathscr{F}$ is large enough, $\mathscr{A}(\mathscr{P})$ is the group algebra $\mathscr{F}(\mathscr{P})$ and so all indecomposable $\mathscr{A}(\mathscr{G})$-modules can be obtained by inducing from ordinary group representations of p-groups. $\mathscr{A}(\mathscr{G})$ has a finite number of different indecomposable $\mathscr{A}(\mathscr{G})$-modules if,

[14] c.f. theorem 1, p. 371 of [9].

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and only if, $\mathscr{P}$ is cyclic, and as in [10] a rough upper bound for the number of indecomposables is

$$\tfrac{1}{2}p^a(m(p^a+1)-p^a+1),$$

where $|\mathscr{G}| = mp^a$, $(m, p) = 1$.

If $\mathscr{P}$, $\mathscr{Q}$ are subgroups of $\mathscr{G}$ we shall write $\mathscr{P} \subseteqq_{\mathscr{G}} \mathscr{Q}$ if there exists a $T \in \mathscr{G}$ such that $\mathscr{P} \subseteqq T\mathscr{Q}T^{-1}$, and $\mathscr{P} =_{\mathscr{G}} \mathscr{Q}$, if $\mathscr{P} = T\mathscr{Q}T^{-1}$. If $\mathscr{M}$ is an indecomposable $\mathscr{A}(\mathscr{G})$-module, then a subgroup $\mathscr{V}$ of $\mathscr{G}$ is called a *vertex* of $\mathscr{M}$ if

(a) $\mathscr{M}$ is $\mathscr{V}$-projective, and

(b) if $\mathscr{M}$ is $\mathscr{H}$-projective, then $\mathscr{V} \subseteqq_{\mathscr{G}} \mathscr{H}$. $\mathscr{V}$ is then determined up to conjugacy in $\mathscr{G}$ and is a p-subgroup. When $p \nmid |\mathscr{G}|$ (or $p = 0$), all vertices coincide with $\{E\}$.

We may also look at the various $\mathscr{A}(\mathscr{V})$-modules $\mathscr{S}$ such that $\mathscr{S}^{\mathscr{G}}$ contains $\mathscr{M}$ as a component. As the process of inducing (i.e. $\otimes$) is distributive over direct sum and $\mathscr{M}$ is indecomposable, it is sufficient to consider $\mathscr{S}$ indecomposable. If $\mathscr{S}'$ is a second such indecomposable $\mathscr{A}(\mathscr{V})$-module, then there exists an element $X \in \mathscr{N}(\mathscr{V})$ such that

$$\mathscr{S}' \approx (X) \otimes_{\mathscr{A}(\mathscr{V})} \mathscr{S},$$

considered as $\mathscr{A}(\mathscr{V})$-modules. Thus $\mathscr{S}$ is called a *source* of $\mathscr{M}$.

As in the corollary to theorem 6 of [8], the problem of determining the vertex and source of a given indecomposable $\mathscr{A}(\mathscr{G})$-module $\mathscr{M}$ can be reduced to the same problem for $\mathscr{A}(\mathscr{P})$, where $\mathscr{P}$ is a Sylow p-subgroup of $\mathscr{G}$, i.e. to the same problem for p-group representations. Hence Green's discussion of induced modules in p-groups (§ 4 of [8]) is relevant.

The existence of the vertex and source of a given indecomposable $\mathscr{M}$ can also be inferred from the non-twisted case by means of the group algebra $\mathscr{F}(\mathscr{G}^*)$ defined in § 1, remark 5.

The notion of blocks of § 3 can be extended further to embrace indecomposable $\mathscr{A}(\mathscr{G})$-modules $\mathscr{M}$. If (E) is decomposed as in § 3 (1), then

$$\mathscr{M} = (E)\mathscr{M} \approx I_1\mathscr{M} \oplus \cdots \oplus I_s\mathscr{M},$$

this being an $\mathscr{A}(\mathscr{G})$-direct sum decomposition. But $\mathscr{M}$ is indecomposable and so there is one and only one I_i such that $I_i\mathscr{M} = \mathscr{M}$. We say that *$\mathscr{M}$ is in the block I_i*.

Let then $\mathscr{M}$ be an indecomposable $\mathscr{A}(\mathscr{G})$-module of vertex $\mathscr{V}$, and in the block I of defect group $\mathscr{D}$. Then $\mathscr{V} \subseteqq_{\mathscr{G}} \mathscr{D}$. On the other hand we shall prove the existence of an $\mathscr{A}(\mathscr{G})$-module in the block I with vertex $\mathscr{D}$ and so the defect group $\mathscr{D}$ of a block I may be characterised as being the "supremum" of the vertices of indecomposable modules in the block.

The following proposition helps in the construction of the above indecomposable.

PROPOSITION. *Let I be a block of $\mathscr{A}(\mathscr{G})$ of defect group $\mathscr{D}$. Let σ be defined with respect to $\mathscr{D}$ and write*

$$\sigma(I) = J_1 + \cdots + J_t,$$

where J_α are primitive idempotents (blocks) of $\mathscr{Z}(\mathscr{H})$ $(\mathscr{H} = \mathscr{N}(\mathscr{D}))$. Let $\mathscr{R}$ be an indecomposable $\mathscr{A}(\mathscr{H})$-module belonging to one of the above blocks, J_1 say. Then there is a component $\mathscr{M}$ of $\mathscr{R}^{\mathscr{G}}$ belonging to the block I such that $\mathscr{R}$ is isomorphic to a component of $\mathscr{M}_{\mathscr{H}}$.

PROOF. Let $X_\alpha \mathscr{H}$ be the cosets of $\mathscr{H}$ in $\mathscr{G}(X_\alpha \in \mathscr{G})$, with $X_1 = E$. Then

$$(1) \qquad (\mathscr{R}^{\mathscr{G}})_{\mathscr{H}} \approx \left((E) \otimes_{\mathscr{A}(\mathscr{H})} \mathscr{R}\right) \oplus \left(\sum_{\alpha > 1} (X_\alpha) \otimes_{\mathscr{A}(\mathscr{H})} \mathscr{R}\right)$$

is an $\mathscr{A}(\mathscr{H})$-direct decomposition. We write $\mathscr{Q} = \sum_{\alpha>1}(X_\alpha) \otimes \mathscr{R}$ and we identify $(E) \otimes \mathscr{R}$ with $\mathscr{R}$. Let π denote the $\mathscr{A}(\mathscr{H})$-projection:

$$\pi : (\mathscr{R}^{\mathscr{G}})_{\mathscr{H}} \to (E) \otimes \mathscr{R} = \mathscr{R}.$$

We write

$$I = \sigma(I) + T_1 + T_2,$$

where T_1 is the sum of terms in $\mathscr{A}(\mathscr{H})$ but not in $\mathscr{A}(\mathscr{C}(\mathscr{D}))$, and T_2 is the sum of the remaining terms not in $\mathscr{A}(\mathscr{H})$. For each u-class sum L in T_1, $\mathscr{L} \cap \mathscr{C}(\mathscr{D}) = \emptyset$ and so $\tau(L) = 0$ (τ is defined in § 3). Hence $\tau(T_1) = 0$, and T_1 is nilpotent.

For $A \in \mathscr{A}(\mathscr{H})$, we write $\rho(A)$ for the linear transformation representing A in the representation afforded by $(E) \otimes \mathscr{R} = \mathscr{R}$. Clearly $\sigma(I)$ acts identically on $\mathscr{R}$, and so $\rho(\sigma(I)+T_1)$, being the sum of the identity transformation and a nilpotent one, is non-singular. Hence the map

$$R \to IR = \rho(\sigma(I)+T_1)R \oplus (T_2 \otimes R) \qquad (R \in \mathscr{R})$$

is an $\mathscr{A}(\mathscr{H})$-homomorphism, the decomposition on the right hand side being that of (1). On the other hand

$$\pi(IR) = \rho(\sigma(I)+T_1)R$$

and so πI is an $\mathscr{A}(\mathscr{H})$-automorphism of $(E) \otimes \mathscr{R} = \mathscr{R}$. Hence $\mathscr{R} \cong I(\mathscr{R})$ and $I(\mathscr{R})$ is an $\mathscr{A}(\mathscr{H})$-component of $(I(\mathscr{R}^{\mathscr{G}}))_{\mathscr{H}}$[15]. By the Krull-Schmidt theorem there is a component $\mathscr{M}$ of $I(\mathscr{R}^{\mathscr{G}})$ $(\subseteq \mathscr{R}^{\mathscr{G}})$ such that $\mathscr{M}_{\mathscr{H}}$ has a component isomorphic to $\mathscr{R}$. $\mathscr{M}$ must also be in the block I.

The construction of the required indecomposable in block I of vertex $\mathscr{V}$ is now simple. Suppose first of all that $\mathscr{D}$ is normal in $\mathscr{G}$. As $\ker \tau$ is nilpotent, $\tau(I)$ must be a non-zero idempotent of $\mathscr{Z}(\mathscr{G}/\mathscr{D})$. Write

[15] This follows from the lemma: If U, V are modules and there exist homomorphisms $\alpha : U \to V$, $\beta : V \to U$ such that $\beta\alpha$ (α followed by β) is an automorphism, then $V = \operatorname{Im}\alpha \oplus \ker\beta$.

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$$\tau(I) = J_1 + \cdots + J_t$$

as a decomposition into blocks of $\mathscr{Z}(\mathscr{G}/\mathscr{D})$. Let $\mathscr{R}$ be any principal component of $\mathscr{A}(\mathscr{G}/\mathscr{D})$ in block J_1, say. J_1 has defect group $\{E\}$ in $\mathscr{G}/\mathscr{D}$ and $\mathscr{R}$ has vertex $\{E\}$ in $\mathscr{G}/\mathscr{D}$. By means of the homomorphism τ, $\mathscr{R}$ can be considered as an $\mathscr{A}(\mathscr{G})$-module, and as such it will be in the block I and will have vertex $\mathscr{D}$.

For the case where $\mathscr{D}$ is not necessarily normal we first write

$$\sigma(I) = J'_1 + \cdots + J'_s,$$

where the J'_α are primitive idempotents in $\mathscr{Z}(\mathscr{H})$, each having defect group $\mathscr{D}$ by the main theorem on blocks. By the previous paragraph there is an indecomposable $\mathscr{A}(\mathscr{H})$-module $\mathscr{R}$ in block J'_1, say, with vertex $\mathscr{D}$. By the proposition there is a component $\mathscr{M}$ of $\mathscr{R}^{\mathscr{G}}$ in block I with a component of $\mathscr{M}_{\mathscr{H}}$ isomorphic to $\mathscr{R}$. As the defect group of I is $\mathscr{D}$ and as $\mathscr{M}$ is in the block I, the vertex $\mathscr{V}$ of $\mathscr{M}$ satisfies

$$\mathscr{V} \underset{\mathscr{G}}{\subseteqq} \mathscr{D}.$$

On the other hand as $\mathscr{M}$ is $\mathscr{V}$-projective each of the components of $\mathscr{M}_{\mathscr{H}}$ has vertex[16] $\subseteqq_{\mathscr{G}} \mathscr{V}$. In particular the vertex $\mathscr{D}$ of the component isomorphic to $\mathscr{R}$ satisfies

$$\mathscr{D} \underset{\mathscr{G}}{\subseteqq} \mathscr{V}.$$

Hence $\mathscr{D} =_{\mathscr{G}} \mathscr{V}$, and so $\mathscr{M}$ is in block I with vertex $\mathscr{D}$.

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University of Sydney.