TWISTED GROUP ALGEBRAS AND THEIR REPRESENTATIONS

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Introduction

Let \( \mathcal{G} \) be a finite group, \( \mathcal{F} \) a field. A twisted group algebra \( \mathcal{A}(\mathcal{G}) \) on \( \mathcal{G} \) over \( \mathcal{F} \) is an associative algebra whose elements are the formal linear combinations

\[
\sum_{A \in \mathcal{G}} a_A(A) \quad (a_A \in \mathcal{F})
\]

and in which the product \( (A)(B) \) is a non-zero multiple of \( (AB) \), where \( AB \) is the group product of \( A, B \in \mathcal{G} \):

\[
(A)(B) = f_{A,B}(AB) \quad (f_{A,B} \in \mathcal{F}, f_{A,B} \neq 0).
\]

One gets the ordinary group algebra \( \mathcal{F}(\mathcal{G}) \) by taking each \( f_{A,B} = 1 \).

Twisted group algebras play a central part in Schur's theory of the projective representations of finite groups [17], [18]. They also arise naturally in the theory of ordinary representations. Let \( \mathcal{L} \) be an irreducible \( \mathcal{F} \)-representation of a normal subgroup \( \mathcal{H} \) of \( \mathcal{G} \). Miss Tucker [21] has shown that the analysis of the induced representation \( \mathcal{L}^\mathcal{G} \) of \( \mathcal{G} \) depends on a twisted group algebra \( \mathcal{A}(\mathcal{H}) \) on a certain subgroup \( \mathcal{K} \) of \( \mathcal{G} \). Clifford [5] encountered much the same algebra in the analysis of the restriction to \( \mathcal{K} \) of an irreducible representation of \( \mathcal{G} \).

The aim of the present paper is to develop the theory of twisted group algebras by exploiting their analogy with ordinary group algebras. This approach permits a unified treatment of such problems as Miss Tucker's cited above. It will be seen that the theory of ordinary group algebras carries over in considerable detail.

In § 1, a normalization theorem is proved which brings out the multiplicative similarity between ordinary and twisted group algebras. This theorem is fundamental for the subsequent work. In § 2, a two-fold generalization of Miss Tucker's paper is given. Firstly, the ordinary group algebras of \( \mathcal{G} \) and \( \mathcal{H} \) are replaced by twisted ones. Secondly, the representation \( \mathcal{L} \) is

1 Kleppner [14] has extended the theory to infinite discrete groups.
assumed to be indecomposable rather than irreducible. As in Miss Tucker's theory, the analysis of $S_f$ depends on the decomposition of a certain twisted group algebra into indecomposable left ideals.

A first step towards such a decomposition is to obtain the decomposition into two-sided ideals. This leads to the consideration, in § 3, of the blocks of a twisted group algebra. Here we follow the treatment of Rosenberg [16] rather than the original treatment of Brauer [4]. Finally, in § 4, we develop Higman's theory of relative projectivity [9], [11] and Green's theory of vertices and sources [8] for twisted algebras.

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1. Normalization of twisted group algebras

We take a twisted group algebra $S_f$ as defined in the introduction. For $A \in S_f$, we write $C(A)$ for the centralizer of $A$ in $S_f$. Let $F^*$ denote the set of non-zero elements of $F$. Let $p$ be the characteristic of $F$; we allow $p = 0$. $E$ will be the identity element of $S_f$.

The elements $k(A)$ of $S_f$ ($k \in F^*$, $A \in S_f$) form a multiplicative subgroup $\Gamma$. The elements $k(E)$ form a multiplicative subgroup $K$ such that $\Gamma/K \cong S_f$, and the $\langle A \rangle$ are coset representatives for $K$ in $\Gamma$.

An element $A \in S_f$ is called a $\nu$-element if

$$(B)^{-1}(A)(B) = (A),$$

for all $B \in C(A)$. Thus the centralizer of $\langle A \rangle$ in $\Gamma$ consists of all multiples $k(B)$, where $k \in F^*$, $B \in C(A)$. All conjugates of $A$ are also $\nu$-elements.

The condition of associativity of $S_f$ is equivalent to

$$f_{A,B}f_{AB,C} = f_{A,BC}f_{B,C},$$

for all $A, B, C \in S_f$. Thus the set $\{f_{A,B}\}$ forms a factor system for $S_f$. If we take a new basis of $S_f$

$$(A) = d_A(A),$$

where $d_A \in F^*$, $A \in S_f$, then the $f_{A,B}$ are modified to

$$f_{A,B} = \frac{d_A d_B}{d_{AB}} f_{A,B}.$$
Transformations given by (1) correspond to taking a different choice of coset representatives in \( \Gamma/K \).

**Theorem.** Let \( n \) be the order of \( \mathcal{G} \). Let \( \mathcal{D} \) be the largest normal \( p \)-subgroup of \( \mathcal{G} \) if \( p \neq 0 \), \( \mathcal{D} = \{ E \} \) if \( p = 0 \). After making a finite number of primary radical extensions to the field \( \mathbb{F} \), if necessary, it is possible to choose the coset representatives \( (A) \) such that:

(a) \[ f_{A,B}^n = 1 \text{ (n odd)} \]
   \[ f_{A,B}^n = 1 \text{ (n even)} \]
   (for all \( A, B \in \mathcal{D} \)),

(b) the representatives \( (A), (B), \ldots, (A, B, \ldots, \in \mathcal{D}) \) form a normal subgroup of \( \Gamma \),

(c) \[ (A)^{-1} = (A^{-1}) \]
   (all \( A \in \mathcal{D} \)),

(d) \[ (X)^{-1}(A)(X) = (X^{-1}AX) \]
   whenever \( A \) is a \( u \)-element, \( X \in \mathcal{D} \).

**Proof.** (i) Since

\[ f_{A,B} f_{AB,C} = f_{B,C} f_{A,BC}, \]

we have

\[ f_{A,B}^d = h_A h_B h_{AB} \]
   (for \( A, B \in \mathcal{D} \)),

where \( d = \text{order of } \mathcal{D}, \ h_A = \prod_{C \in \mathcal{G}} f_{A,C} \). Replacing \( (A) \) by \( h_A^{-1/d}(A) \), we have

\[ f_{A,B}^d = 1 \]
   (for \( A, B \in \mathcal{D} \)).

Since \( d \) is a power of \( p, f_{A,B}^d = 1, \text{ all } A, B \in \mathcal{D} \). If \( X \in \mathcal{D}, A \in \mathcal{D} \),

\[ (X)^{-1}(A)(X) = l(X^{-1}AX), \]

where \( l \in \mathbb{F}^* \), and so

\[ (X)^{-1}(A)^d(X) = l^d(X^{-1}AX)^d. \]

Thus

\[ l^d = 1. \]

Hence

\[ l = 1, \text{ and (b) holds.} \]

(ii) Similarly,

\[ f_{A,B}^n = k_A k_B k_{AB} \]
   (all \( A, B \in \mathcal{G} \)),

where \( k_A = \prod_{C \in \mathcal{G}} f_{A,C} \). For each \( A \in \mathcal{G} \), choose a definite value for \( k_A^{-1/n} \).

Replacing \( (A) \) by \( k_A^{-1/n}(A) \), we may assume \( f_{A,B}^n = 1 \) for all \( A, B \in \mathcal{G} \).

(For \( A, B \in \mathcal{D} \),

\[ 1 = f_{A,B}^n = k_A k_B k_{AB}, \]

whence \( k_A = 1 \); choose \( 1^{-1/n} \) in \( \mathbb{F} \) as \( 1 \); then (b) still holds.)

(iii) Let \( \mathcal{X} = \{ A_1, \ldots, A_r \} \) be any conjugacy class of \( u \)-elements not in \( \mathcal{D} \).
The \( u \)-condition tells us that \( (A_1) \) has \( r \) conjugates in \( \Gamma \). Choosing \( (A_1) \) arbitrarily and taking \( (A_2), \ldots, (A_r) \) as its other conjugates in \( \Gamma \) we have condition (d) holding, and we still have \( f^n_{A, B} = 1 \).

(iv) Consider the elements in \( \mathcal{S} \) not in \( \mathcal{D} \). For such an element, \( (A)(A^{-1}) = l(E)(l \in \mathcal{F}^*) \). For one, say \( A \), out of each pair \( A, A^{-1} \) of non-involutory, non-\( u \)-elements, leave \( (A) \) as before and replace \( (A')^{-1} \) by \( L^{-1}(A) \). For each non-\( u \) involution \( A \), replace \( (A) \) by \( l^{-1}(A) \). As \( l^n = 1 \), \( (l^{-1})^n = 1 \), \( n \) odd, \( (l^{-1})^2n = 1 \), \( n \) even.

Now consider the \( u \)-class

\[ \mathcal{K} = \{A_1, \ldots, A_r\}. \]

We still have the choice of \( (A_1) \) at our disposal. If \( \mathcal{K} \neq \mathcal{K}^{-1} = \{A_1^{-1}, \ldots, A_r^{-1}\} \), we choose \( (A_1), (A_r^{-1}) \) as above in the case \( A \neq A^{-1} \). If \( (X)^{-1}(A_1)(X) = (A_1)^{-1} \), then \( (X)^{-1}(A_1)^{-1}(X) = (A_1)^{-1} \), by choice of \( (A_1), (A_r^{-1}) \).

Finally, let \( \mathcal{K} \) be self-inverse. Thus

\[ A_1^{-1} = T^{-1}A_1T, \]

and

\[ (A_1^{-1}) = (T)^{-1}(A_1)(T). \]

Replacing \( (A_1) \) by \( l^{-1}(A_1) \), and so all \( (A_i) \) by \( l^{-1}(A_i) \), we still have (*) and also \( (A_i^{-1}) = (A_i)^{-1} \).

**Remarks.**

1. \( (E) \) is now the identity element of \( \mathcal{A}(\mathcal{S}) \). Further \( (A)(A^{-1}) = (A^{-1})(A) = (E) \). If we write \( \mathcal{A}(\mathcal{D}) \) to denote the natural restriction of \( \mathcal{A}(\mathcal{S}) \) to the subspace spanned by the elements \( (D) (D \in \mathcal{D}) \), then \( \mathcal{A}(\mathcal{D}) \) is precisely the group algebra \( \mathcal{F}(\mathcal{D}) \).

2. If \( \mathcal{A}(\mathcal{S}) \) satisfies (c) \([(c), (d)] \) and (b) \([(b), (c), (d)] \) then we shall call \( \mathcal{A}(\mathcal{S}) \) normalized \([u-normalized]\) \([[l-u-normalized]]\).

3. If \( A \) is a \( u \)-element, and if \( t \) is prime to the order of \( A \), then \( A^t \) is a \( u \)-element. In particular \( A^{-1} \) is a \( u \)-element.

4. If \( \mathcal{A}(\mathcal{S}) \) is \( u \)-normalized and \( \mathcal{X}_1, \ldots, \mathcal{X}_r \) are the \( u \)-classes, then the \( u \)-class sums \( K_s = \sum G_{\in \mathcal{X}_s} G \) form a basis for the centre \( \mathcal{Z}(\mathcal{S}) \) of \( \mathcal{A}(\mathcal{S}) \), which has dimension \( t^3 \).

5. A twisted group algebra \( \mathcal{A}(\mathcal{S}) \) is actually an (two-sided) ideal direct summand of a group algebra \( \mathcal{S} \): suppose \( \mathcal{A}(\mathcal{S}) \) has been normalized as in

\[ (X)^{-1}(Y)^{-1}(A)(Y)(X) = (X^{-1}Y^{-1})(A)(YX), \]

for all \( X, Y \in \mathcal{S} \).

6. If \( \mathcal{A}(\mathcal{S}) \) is \( u \)-normalized and \( \mathcal{K}_1, \ldots, \mathcal{K}_r \) are the \( u \)-classes, then the \( u \)-class sums \( K_s = \sum G_{\in \mathcal{K}_s} G \) form a basis for the centre \( \mathcal{Z}(\mathcal{S}) \) of \( \mathcal{A}(\mathcal{S}) \), which has dimension \( t^3 \).

\[ c. f. \text{ Satz 1, p. 83 of [20]. Tazawa's formulation is not so explicit and is confined to the non-modular case.} \]

\[ 3 \text{ I am indebted to the referee for this remark and its proof.} \]
(ii) above so that all \( f_{A,B} \) satisfy \( f_{A,B}^n = 1 \). If \( \mathcal{F} \) has characteristic \( p \), and \( n = mp^s \), \((m, p) = 1\), then in fact \( f_{A,B}^m = 1 \). Thus the \( f_{A,B} \) all belong to the multiplicative group \( W_m \) of \( m \)-th roots of unity. Let \( f \rightarrow f^* \) be an isomorphism onto some other cyclic group \( G_m \) of order \( m \), generated by \( \mu^* \), and define a central extension \( G^* \) of \( G \) by \( G_m \) in which \( G^* \) is generated by elements \( S_A(A \in G) \) and \( G_m \), with \( S_A S_B = f_{A,B}^* S_{AB} \). Then \( \mathcal{F}(G_m) \), considered as embedded in \( \mathcal{F}(G^*) \), is in the centre of \( \mathcal{F}(G^*) \); let

\[
S_E = E_1 + \cdots + E_m,
\]

where

\[
E_i = \frac{1}{m} \sum_{a=0}^{m-1} \mu^a(i) \mu^a(a),
\]

be a decomposition of the identity \( S_E \) of \( \mathcal{F}(G^*) \) into primitive idempotents of \( \mathcal{F}(G_m) \). It is readily verified that \( \mathcal{A}(G) \approx E_1 \mathcal{F}(G^*) \).

As \( \mathcal{F}(G^*) \) is symmetric \(^6\), it follows that \( \mathcal{A}(G) \) is symmetric. (This can also be seen directly without using \( \mathcal{F}(G^*) \).)

6. If \( p = 0 \), or \( p \nmid n \) (non-modular case) (thus \( p \nmid |G^*| \), \( \mathcal{F}(G^*) \) is semi-simple, and so \( \mathcal{A}(G) \) is semi-simple \(^6\). In this case there are \( t \) different irreducible representations of \( \mathcal{A}(G) \), where \( t = \) number of \( u \)-conjugacy classes.

In the modular case, the number of irreducibles is equal to the number of \( p \)-regular \( u \)-conjugacy classes of \( G \) \(^7\). (An element \( A \in G \) is \( p \)-regular if its order is prime to \( p \).) This can be proved using Brauer's Theorem 3A, p. 410 of [4].

7. From remark 1, any twisted group algebra \( \mathcal{A}(D) \) on a \( p \)-group \( D \) over a field \( \mathcal{F} \) of characteristic \( p \neq 0 \) is the group algebra \( \mathcal{F}(D) \). This is a local algebra whose radical is spanned by the elements \( (P) - (E) \), \( P \in D \), \( E \) identity of \( D \). The regular representation of \( \mathcal{F}(D) \) is indecomposable.

8. This last result can be extended a little further. Let \( G \) be a cyclic extension of a normal \( p \)-subgroup \( D \), where \( p \neq 0 \). Then \( \mathcal{A}(G) \) is the group algebra on \( G \).

PROOF. Clearly it can be assumed that \( |G/D| = m \), prime to \( p \). Take \( G \in G \) such that the coset \( G/D \) generates \( G/D \). Write

\[
G^m = K \in D, \quad (G)^m = d(K), \quad (d \in \mathcal{F}^*).
\]

Any element of \( G \) can be written uniquely in the form \( G^k D \), where \( 0 \leq k < m \), \( D \in D \).

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\(^6\) See definition of symmetric on p. 440 of [6].

\(^7\) See also p. 207 of [2].
By the theorem, \( \mathcal{A}(\mathcal{G}) \) can be supposed to be \( p\mu \)-normalized. If now we replace \( (G^kD) \) by \( d^{-k/m}(G)^k(D) \) this ensures that \( \mathcal{A}(\mathcal{G}) \) is the group algebra \( \mathcal{F}(\mathcal{G}) \).

9. If a twisted group algebra \( \mathcal{A}(\mathcal{G}) \) has one representation of degree 1, then it is the group algebra \( \mathcal{F}(\mathcal{G}) \).

2. Induced representations

Let \( \mathcal{A}(\mathcal{G}) \) be a normalized twisted group algebra and let \( \mathcal{A}(\mathcal{H}) \) be the natural restriction of \( \mathcal{A}(\mathcal{G}) \) to a subgroup \( \mathcal{H} \) of \( \mathcal{G} \). Let \( \mathcal{L} \) be a left \( \mathcal{A}(\mathcal{H}) \)-module. (Throughout this paper all modules will be taken as having finite dimension considered as vector spaces over the base field \( \mathcal{F} \).) We define \( \mathcal{L}^\mathcal{G} \) to be the left \( \mathcal{A}(\mathcal{G}) \)-module given by

\[
\mathcal{L}^\mathcal{G} = \mathcal{A}(\mathcal{G}) \otimes \mathcal{A}(\mathcal{H}) \mathcal{L},
\]

where \( \otimes \) is defined as in [6]. If \( \mathcal{M} \) is an \( \mathcal{A}(\mathcal{G}) \)-module, then we shall write \( \mathcal{A}_\mathcal{G} \) for the \( \mathcal{A}(\mathcal{H}) \)-module obtained from \( \mathcal{M} \) by simple restriction of the module multiplication to the ring \( \mathcal{A}(\mathcal{H}) \).

Let \( \mathcal{M}, \mathcal{N} \) be \( \mathcal{A}(\mathcal{H}) \)-modules. Then we write \( \text{Hom}_\mathcal{H}(\mathcal{M}, \mathcal{N}) \) for the set of \( \mathcal{A}(\mathcal{H}) \)-homomorphisms of \( \mathcal{M} \) into \( \mathcal{N} \), \( E_\mathcal{H}(\mathcal{M}) = \text{Hom}_\mathcal{H}(\mathcal{M}, \mathcal{M}) \) for the ring of \( \mathcal{A}(\mathcal{H}) \)-endomorphisms of \( \mathcal{M} \), and \( R_\mathcal{H}(\mathcal{M}) \) for the radical of \( E_\mathcal{H}(\mathcal{M}) \). Throughout this section homomorphisms will be written on the right. We quote the following simple lemma.

**Lemma.** If \( \mathcal{L} \) is an \( \mathcal{A}(\mathcal{H}) \)-module and \( \mathcal{M} \) an \( \mathcal{A}(\mathcal{G}) \)-module, then

\[
\text{Hom}_\mathcal{H}(\mathcal{L}, \mathcal{M}) \cong \text{Hom}_\mathcal{G}(\mathcal{L}^\mathcal{G}, \mathcal{M}).
\]

This correspondence \( \eta \mapsto \eta^\mathcal{G} \) is given by defining for \( \eta \in \text{Hom}_\mathcal{H}(\mathcal{L}, \mathcal{M}) \), \( \eta^\mathcal{G} \in \text{Hom}_\mathcal{G}(\mathcal{L}^\mathcal{G}, \mathcal{M}) \) by

\[
(A \otimes L)\eta^\mathcal{G} = A(L\eta) \quad (A \in \mathcal{A}(\mathcal{G}), L \in \mathcal{L}).
\]

Henceforth we take \( \mathcal{H} \) to be a normal subgroup of \( \mathcal{G} \), and \( \mathcal{L} \) to be an \( \mathcal{A}(\mathcal{H}) \)-module. The main theorem of this section concerns the structure of \( \mathcal{L}^\mathcal{G} \) and this analysis is to be made through its ring of endomorphisms \( E_\mathcal{G}(\mathcal{L}^\mathcal{G}) \).

Given an element \( G \in \mathcal{G} \), we can consider the \( \mathcal{A}(\mathcal{H}) \)-submodules of \( \mathcal{L}^\mathcal{G} \) of the form

\[
(G) \otimes \mathcal{A}(\mathcal{H}) \mathcal{L},
\]

where \( (H)(G) \otimes L = (G) \otimes (G)^{-1}(H)(G)L \) for \( H \in \mathcal{H} \), \( L \in \mathcal{L} \). \( (G) \otimes \mathcal{L} \) may or may not be \( \mathcal{A}(\mathcal{H}) \)-isomorphic to \( \mathcal{L} \). The stabilizer \( \mathcal{P} \) of \( \mathcal{L} \) is the set of elements \( S \in \mathcal{G} \) such that \( (S) \otimes \mathcal{L} \cong \mathcal{L} \). Then \( \mathcal{P} \) is a subgroup of \( \mathcal{G} \) containing \( \mathcal{H} \).

Take a set \( \{X_s\} \) of elements of \( \mathcal{G} \) such that \( X_1 \mathcal{H}, \ldots, X_s \mathcal{H} (X_1 \mathcal{H}, \ldots, \)
$X \cdot \mathcal{H}$ are the different cosets of $\mathcal{H}$ in $\mathcal{S}$ (of $\mathcal{H}$ in $\mathcal{S}$) with $X_1 = E$. Then we may write

\[(1) \quad \mathcal{L}^g = \sum_1^g (X_a) \otimes \mathcal{L} = \sum_1^g \mathcal{L}_a,\]

\[(1') \quad \mathcal{L}^y = \sum_1^g \mathcal{L}_a,\]

the $\sum$ meaning vector space sum over $\mathcal{F}$. We identify $\mathcal{L}_1$ and $\mathcal{L}$.

If we restrict to $\mathcal{H}$, $(1)$ and $(1')$ then become $\mathcal{A}(\mathcal{H})$-direct decompositions of $(\mathcal{L}^g)_\mathcal{H}$ and $(\mathcal{L}^y)_\mathcal{H}$ respectively.

Let

$$\omega_a : \mathcal{L}_a \to (\mathcal{L}^g)_\mathcal{H}, \quad \chi_a : (\mathcal{L}^y)_\mathcal{H} \to \mathcal{L}_a$$

be the inclusion and projection $\mathcal{A}(\mathcal{H})$-homomorphisms according to $(1)$. (We use the same symbols for the decomposition in $(1')$ and regard $(\mathcal{L}^y)_\mathcal{H} \subset (\mathcal{L}^y)_\mathcal{H}^\prime$ naturally.) Thus the identity $\iota$ of $E_\mathcal{H}(\mathcal{L}^g)$\(^8\) may be written

$$\iota = \sum_1^g \chi_a \omega_a.$$  

If $\eta \in \text{Hom}_\mathcal{H}(\mathcal{L}, \mathcal{L}^y)$, then

$$\eta = \sum_1^g \eta \chi_a \omega_a = \sum_1^g \eta_a \omega_a,$$

where $\eta_a = \eta \chi_a \in \text{Hom}_\mathcal{H}(\mathcal{L}, \mathcal{L}_a)$. Similarly if $\zeta \in E_\mathcal{H}(\mathcal{L}^y)$, we write

$$\zeta = \sum_1^g \sum_1^g \chi_a \omega_a \chi_\beta \omega_\beta = \sum_1^g \chi_a \chi_\beta \omega_\beta,$$

where

$$\chi_\beta = \omega_\beta \chi_\beta \in \text{Hom}_\mathcal{H}(\mathcal{L}_a, \mathcal{L}_\beta).$$

Suppose

$$L \eta_\beta = (X_\beta) \otimes L_\beta, \quad (L \in \mathcal{L}),$$

$$(X_a)(X_\beta) = (X_\gamma)H_{a, \beta}$$

where $X_a X_\beta \in X_\gamma \mathcal{H}$, $H_{a, \beta} \in \mathcal{A}(\mathcal{H})$. Then

$$(X_a) \otimes L) \eta^g = \sum_\beta (X_a)(X_\beta) \otimes L_\beta.$$  

Thus $(\eta^g)_\mathcal{H}$ maps $(X_a) \otimes L$ to $(X_\gamma) \otimes H_{a, \beta} L_\beta$, where $\beta$ is determined by $X_a X_\beta \in X_\gamma \mathcal{H}$.

From this point onwards we shall take $\mathcal{L}$ to be an indecomposable $\mathcal{A}(\mathcal{H})$-module. Hence $E_\mathcal{H}(\mathcal{L})$ is a completely-primary ring.

\(^8\) Here $E_\mathcal{H}(\mathcal{L}^g)$ means $E_\mathcal{H}((\mathcal{L}^g)_\mathcal{H})$. Similarly $\text{Hom}_\mathcal{H}(\mathcal{L}, \mathcal{L}^g)$ means $\text{Hom}_\mathcal{H}(\mathcal{L}, (\mathcal{L}^g)_\mathcal{H})$ etc.
**Lemma 1.** Let \( \eta \in \text{Hom}_\mathcal{R}(\mathcal{L}, \mathcal{L}^\#) \). Then \( \eta^\# \in R_\mathcal{R}(\mathcal{L}^\#) \) if, and only if, none of \( \eta_1, \cdots, \eta_s \) is an \( \mathcal{H} \)-isomorphism.

**Proof.** By Jacobson [13], p. 60, \( \eta^\# \in R_\mathcal{R}(\mathcal{L}^\#) \) if, and only if, no \( (\eta^\#)_\beta \) is an \( \mathcal{H} \)-isomorphism. By the above, this is the case if, and only if, no \( \eta_\beta (\beta = 1, \cdots, g) \) is an \( \mathcal{H} \)-isomorphism. No \( \eta_\beta (\beta > s) \) is an \( \mathcal{H} \)-isomorphism because, by the definition of \( \mathcal{L}, \mathcal{L} \) is not \( \mathcal{H} \)-isomorphic to \( \mathcal{L} \). This gives the lemma.

There is of course the analogous 1–1 correspondence \( \eta \leftrightarrow \eta^\# \) between the \( \mathcal{H} \)-isomorphisms \( \eta \) of \( \mathcal{L} \) into \( \mathcal{L}^\# \) and \( \mathcal{L} \)-endomorphisms \( \eta^\# \) of \( \mathcal{L}^\# \), where \( \eta^\# \) is defined by

\[
(A \otimes L)\eta^\# = A (L\eta) \quad (A \in \mathcal{A}(\mathcal{S}), L \in \mathcal{L}).
\]

**Corollary.** Let \( \eta \in \text{Hom}_\mathcal{R}(\mathcal{L}, \mathcal{L}^\#) \). Then \( \eta^\# \in R_\mathcal{R}(\mathcal{L}^\#) \) if, and only if, \( \eta^\# \in R_\mathcal{R}(\mathcal{L}^\#) \). (Here \( \text{Hom}_\mathcal{R}(\mathcal{L}, \mathcal{L}^\#) \) is considered in the natural way as a subset of \( \text{Hom}_\mathcal{R}(\mathcal{L}, \mathcal{L}^\#) \).

If \( \mu \in E_\mathcal{R}(\mathcal{L}^\#) \), the equations

\[
(A \otimes \mathcal{M}(\mathcal{S}))\mu^\# = A \otimes \mathcal{M}(\mathcal{S}) (\mu) \quad (A \in \mathcal{A}(\mathcal{S}), \mathcal{M} \in \mathcal{L}^\#)
\]

define an element \( \mu^\# \) of \( E_\mathcal{R}(\mathcal{L}^\#) \). Moreover, the mapping \( \mu \rightarrow \mu^\# \) of \( E_\mathcal{R}(\mathcal{L}^\#) \) into \( E_\mathcal{R}(\mathcal{L}^\#) \) is a ring monomorphism.

**Lemma 2.**

\[
E_\mathcal{R}(\mathcal{L}^\#)^* + \hat{R}_\mathcal{R}(\mathcal{L}^\#) = E_\mathcal{R}(\mathcal{L}^\#),
\]

\[
E_\mathcal{R}(\mathcal{L}^\#)^* \cap \hat{R}_\mathcal{R}(\mathcal{L}^\#) = \hat{R}_\mathcal{R}(\mathcal{L}^\#)^*
\]

where

\[
\hat{R}_\mathcal{R}(\mathcal{L}^\#) = E_\mathcal{R}(\mathcal{L}^\#) \cap R_\mathcal{R}(\mathcal{L}^\#),
\]

\[
\hat{R}_\mathcal{R}(\mathcal{L}^\#)^* = E_\mathcal{R}(\mathcal{L}^\#)^* \cap R_\mathcal{R}(\mathcal{L}^\#).
\]

**Proof.** Let \( \mu \in E_\mathcal{R}(\mathcal{L}^\#) \). Then \( \mu = \eta^\#, \eta \in \text{Hom}_\mathcal{R}(\mathcal{L}, \mathcal{L}^\#) \). Also \( \eta^\# = \mu^\# \).

By lemma 1, corollary, \( \mu \in R_\mathcal{R}(\mathcal{L}^\#) \) if, and only if, \( \mu^\# \in R_\mathcal{R}(\mathcal{L}^\#) \). This gives the second relation.

Now let \( \rho \in E_\mathcal{R}(\mathcal{L}^\#) \). Then \( \rho = \zeta^\#, \zeta \in \text{Hom}_\mathcal{R}(\mathcal{L}, \mathcal{L}^\#) \). Write

\[
\zeta = \zeta' + \zeta''
\]

where \( \zeta' = \sum_{k=1}^l \zeta_k \omega_k \). Since \( \zeta' \in \text{Hom}_\mathcal{R}(\mathcal{L}, \mathcal{L}^\#) \), \( \zeta'^\# = (\zeta')^* \in E_\mathcal{R}(\mathcal{L}^\#)^* \).

Also, by lemma 1, \( \zeta''^\# \in R_\mathcal{R}(\mathcal{L}^\#) \). Hence

\[
\rho = \zeta'^\# + \zeta''^\# \in E_\mathcal{R}(\mathcal{L}^\#)^* + \hat{R}_\mathcal{R}(\mathcal{L}^\#).
\]

This proves the first relation.

**Corollary 1.** If \( e = \sum \epsilon_\lambda \) is a decomposition of the identity of \( E_\mathcal{R}(\mathcal{L}^\#) \) into indecomposable idempotents in \( E_\mathcal{R}(\mathcal{L}^\#) \), then \( e^\# = \sum \epsilon_\lambda^\# \) is a similar decomposition in \( E_\mathcal{R}(\mathcal{L}^\#) \).
COROLLARY 2. \( E_\varphi(\mathcal{L}^\varphi)/\tilde{R}_\varphi(\mathcal{L}^\varphi) \approx E_\varphi(\mathcal{L}^\varphi) / \tilde{R}_\varphi(\mathcal{L}^\varphi) \).

COROLLARY 3. \( E_\varphi(\mathcal{L}^\varphi)/R_\varphi(\mathcal{L}^\varphi) \approx E_\varphi(\mathcal{L}^\varphi)/\tilde{R}_\varphi(\mathcal{L}^\varphi) \).

(Notice here that \( R_\varphi, \tilde{R}_\varphi \) are nilpotent ideals of \( E_\varphi, \tilde{E}_\varphi \), so that \( \tilde{R}_\varphi \subseteq R_\varphi, \tilde{R}_\varphi \subseteq R_\varphi \).

Now consider \( E_\varphi(\mathcal{L}^\varphi) \). We remark first that \( \eta \to \eta^\varphi \) gives a ring monomorphism of \( E_\varphi(\mathcal{L}) \) into \( E_\varphi(\mathcal{L}^\varphi) \). (Here, and in what follows, we consider \( E_\varphi(\mathcal{L}) = \text{Hom}_\varphi(\mathcal{L}, \mathcal{L}) \) and \( \text{Hom}_\varphi(\mathcal{L}, \mathcal{L}_a) (a \leq s) \) as subsets of \( \text{Hom}_\varphi(\mathcal{L}, \mathcal{L}^\varphi) \). We denote the image of \( E_\varphi(\mathcal{L}) \) in \( E_\varphi(\mathcal{L}^\varphi) \) by \( E_\varphi(\mathcal{L})^\varphi \).

Write \( T_a = X_a \varphi_a \) \((a = 1, \ldots, s)\) for the elements of \( \mathcal{S}|\mathcal{H} \). For each \( T_a \), choose an \( \mathcal{H} \)-isomorphism \( \xi_\mathcal{H}_a : \mathcal{L} \to \mathcal{L}_a \), and form

\[ (T_a) = \xi_\mathcal{H}_a. \]

Clearly, if \( T, T' \in \mathcal{S}|\mathcal{H}, (T)(T')(TT')^{-1} \) maps \( \mathcal{L} \) onto \( \mathcal{L} \) and so belongs to \( E_\varphi(\mathcal{L})^\varphi \):

\[ (T)(T') = \eta_{T,T'}^\varphi(TT'). \]

Similarly, if \( \eta \in E_\varphi(\mathcal{L}), T \in \mathcal{S}|\mathcal{H}, (T)^{-1} \eta^\varphi(T) \in E_\varphi(\mathcal{L})^\varphi \) and we write

\[ (T)^{-1} \eta^\varphi(T) = (\eta(T)^\varphi). \]

Clearly, \( \eta \to \eta(T) \) is an \( \mathcal{F} \)-algebra automorphism of \( E_\varphi(\mathcal{L}) \); and in fact, if \( T = \xi^\varphi \), \( \xi^{-1} \eta \xi = \eta(T) \).

Finally, since an arbitrary element \( \zeta \) of \( \text{Hom}_\varphi(\mathcal{L}, \mathcal{L}^\varphi) \) has the form

\[ \zeta = \sum_{a=1}^s \eta_a \omega_a = \sum_{a=1}^s \xi_a \omega_a, \quad \eta_a \in E_\varphi(\mathcal{L}), \]

each element of \( E_\varphi(\mathcal{L}^\varphi) \) can be uniquely expressed in the form

\[ \zeta^\varphi = \sum_{T \in \mathcal{S}|\mathcal{H}} \eta_T^\varphi(T), \quad \eta_T \in E_\varphi(\mathcal{L}). \]

Thus \( E_\varphi(\mathcal{L}^\varphi) \) is a kind of twisted group algebra on \( \mathcal{S}|\mathcal{H} \) over \( E_\varphi(\mathcal{L}) \), though the \( (T) \) do not commute with the coefficients \( \eta^\varphi \).

By lemma 1, \( \zeta^\varphi \in \tilde{R}_\varphi(\mathcal{L}^\varphi) \) if, and only if, no \( \eta_T \) is an \( \mathcal{H} \)-isomorphism, i.e.

\[ \text{if, and only if, all } \eta_T \in R_\varphi(\mathcal{L}). \]

Thus to get \( E_\varphi(\mathcal{L}^\varphi)/\tilde{R}_\varphi(\mathcal{L}^\varphi) \), we simply replace all the \( \eta \)'s in all above by their canonical images \( \eta = \eta + \tilde{R}_\varphi(\mathcal{L}) \) in \( E_\varphi(\mathcal{L})/\tilde{R}_\varphi(\mathcal{L}) \). Thus \( E_\varphi(\mathcal{L}^\varphi)/\tilde{R}_\varphi(\mathcal{L}^\varphi) \) appears as a generalized twisted group algebra over the division algebra \( E_\varphi(\mathcal{L})/\tilde{R}_\varphi(\mathcal{L}) \). The operations \( \eta \to \eta(T) \) are \( \mathcal{F} \)-algebra automorphisms of \( E_\varphi(\mathcal{L})/\tilde{R}_\varphi(\mathcal{L}) \). From now on we assume \( \mathcal{F} \) algebraically closed. Thus \( E_\varphi(\mathcal{L})/\tilde{R}_\varphi(\mathcal{L}) \) is the 1-dimensional \( \mathcal{F} \)-algebra \( \mathcal{F} \) itself, so \( \eta = \eta(T) \), all \( T \). Here \( E_\varphi(\mathcal{L}^\varphi)/\tilde{R}_\varphi(\mathcal{L}^\varphi) \) becomes a genuine twisted group algebra \( \mathcal{A}(\mathcal{S}|\mathcal{H}) \) on \( \mathcal{S}|\mathcal{H} \) over \( \mathcal{F} \).

The following lemma by Fitting [7] provides the link between a module
and its ring of endomorphisms. We use the term “component” to mean “indecomposable direct summand”.

**Lemma 3.** Let $\mathfrak{A}$ be a finite dimensional algebra (with a 1) over $\mathcal{F}$ and let $\mathfrak{M}$ be an $\mathfrak{A}$-module (finite dimensional) with $\mathcal{S}$ as its ring of $\mathfrak{A}$-endomorphisms. Let

$$\mathcal{S} = \mathcal{S}e_{11} \oplus \cdots \oplus \mathcal{S}e_{1n_1} \oplus \cdots \oplus \mathcal{S}e_{mn_m}$$

be a decomposition of $\mathcal{S}$ into left ideal components, where $\mathcal{S}e_{ij} \cong \mathcal{S}e_{i'}$ if, and only if, $i = i'$. Let

$$\mathcal{M} = \mathcal{M}_{11} \oplus \cdots \oplus \mathcal{M}_{1n_1} \oplus \cdots \oplus \mathcal{M}_{mn_m}$$

be a decomposition of $\mathcal{M}$ into components, with $\mathcal{M}_{ij} \cong \mathcal{M}_{i'}$, if and only if, $i = i'$. Then $m = m'$, $n = n'$, and one possible choice of $\mathcal{M}_{ab} = \mathcal{M}_{a'b'}$.

Let

$$\mathcal{L}^a = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_1$$

be a decomposition of $\mathcal{L}^a$ into $\mathcal{S}$-components. We can further write

$$\mathcal{L}^a = \mathcal{L}^a \oplus \cdots \oplus \mathcal{L}^a, \text{ where each of the } \mathcal{L}^a \cong \mathcal{S}, \text{ by the Krull-Schmidt theorem. Let } \epsilon = \sum_{a=1}^s \epsilon_a$$

be a decomposition of the identity of $E_{\mathcal{S}}(\mathcal{L}^a)$ according to (4). Then each $\epsilon_a$ can be further decomposed by (5) in the form

$$\epsilon_a = \sum_{a=1}^{n_a} \epsilon_{a'b'}, \quad \epsilon_{a'b'} \in \text{Hom}_{\mathcal{S}}(\mathcal{L}, \mathcal{L}^a),$$

and any element $\pi$ of $E_{\mathcal{S}}(\mathcal{L}^a)$ has a unique expression in the form

$$\pi = \sum_{a,b} \pi_{a'b'} \epsilon_{a'b'}, \quad \pi_{a'b'} \in E_{\mathcal{S}}(\mathcal{L}).$$

Clearly $\sum k_a = s$, and the left ideal $E_{\mathcal{S}}(\mathcal{L}^a)$, considered as a module over $E_{\mathcal{S}}(\mathcal{L})$, is the direct sum of $k_a$ copies of $E_{\mathcal{S}}(\mathcal{L})$. Hence the dimension over $\mathcal{F}$ of the corresponding left ideal in $\mathfrak{A}(\mathcal{S}/\mathcal{H}) (= E_{\mathcal{S}}(\mathcal{L}^a)/R_{\mathcal{S}}(\mathcal{L}^a))$ is precisely $k_a$. Moreover, as $R_{\mathcal{S}}(\mathcal{L}^a)$ is nilpotent, the images of the two left ideal components in the quotient ring are isomorphic if, and only if, the corresponding left ideal components of the original ring $E_{\mathcal{S}}(\mathcal{L}^a)$ are isomorphic. Combining these results we have that the decomposition of $\mathcal{L}^a$ is entirely reflected by the decomposition of $\mathfrak{A}(\mathcal{S}/\mathcal{H})$ into left ideals.

Now $\mathcal{L}^a \cong (\mathcal{L}^a)^a \cong \mathcal{M}_1^a \oplus \cdots \oplus \mathcal{M}_1^a$. Further, by corollary 3 to lemma 2 each $\mathcal{M}_a^a$ must remain indecomposable. Moreover, as $R_{\mathcal{S}}(\mathcal{L}^a)$

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* This was noted in § I of Nakayama [15] for the case where the kernel is actually the radical of $E_{\mathcal{S}}(\mathcal{L}^a)$. 
is nilpotent, the multiplicities of the different isomorphism types of left ideal components of $E_{\phi}(L^a)$ are the same as in $E_{\phi}(L^a)/R_{\phi}(L^a)$, i.e. as in $E_{\phi}(L^a)/R_{\phi}(L^a)$ (by lemma 2, corollary 3), i.e. as in $E_{\phi}(L^a)$ (since $R_{\phi}(L^a)$ is nilpotent). Hence we have proved the following theorem.

**Theorem.** Let $A(\mathcal{H})$ be the restriction of a normalized twisted group algebra $A(\mathcal{I})$ over an algebraically closed field $F$ to a normal subgroup $\mathcal{H}$ of $\mathcal{G}$, and let $L$ be an indecomposable $A(\mathcal{H})$-module with stabilizer $\mathcal{H}$ in $\mathcal{G}$. Then the decomposition of $L^a$ is entirely determined by the decomposition of a certain twisted group algebra $A(\mathcal{I}/\mathcal{H})$ into left ideals, there being a 1–1 correspondence between left ideal components $I_a$ and components $N_a$ of $L^a$, such that the left ideals are isomorphic if, and only if, the corresponding summands are. Further

$$\dim_{\phi} N_a = \dim_{\phi}(I_a) \cdot \dim_{\phi}(L) \cdot (\mathcal{I} : \mathcal{I}).$$

A decomposition of $L^a$ is obtained from one of $A(\mathcal{I}/\mathcal{H})$ as follows: The decomposition of $A(\mathcal{I}/\mathcal{H}) \approx E_{\phi}(L^a)/R_{\phi}(L^a)$ is raised to one of $E_{\phi}(L^a)$ by the algorithm used in the proof of theorem 9.3c in [1]. A decomposition of $L^a = \sum M_a$ is obtained as in lemma 3. Finally we may take $N_a = M^a_a$.

If $L$ is irreducible, then $E_{\phi}(L^a)$ is the twisted group algebra $A(\mathcal{I}/\mathcal{H})$, as $E_{\phi}(L) \approx F$.

**Corollary 1.** If $L$ is not indecomposable, say

$$L = L_1 \oplus \cdots \oplus L_h,$$

then

$$L^a = L^a_1 \oplus \cdots \oplus L^a_h,$$

as tensor product $\otimes$ is distributive over direct sum $\oplus$. We apply the theorem to each $L^a_i$ to obtain the decomposition of $L^a$.

The problem of inducing up from a subnormal subgroup is equivalent to the decomposition of a series of twisted group algebras. For, if $\mathcal{H} \leq \mathcal{H}_1 \leq \mathcal{G}$, we have $(L^a)^{\mathcal{H}_1} \approx L^a$.

**Corollary 2.** If $\mathcal{H}$ is a subnormal subgroup of $\mathcal{G}$ of prime power index $p^v$ in $\mathcal{G}$, with $F$ of characteristic $p \neq 0$, then $L^a$ is indecomposable if $L$ is.

**Proof.** Clearly the factor groups are $p$-groups and so the twisted group algebras involved are on $p$-groups. Hence by § 1, remark 7, these are indecomposable. (c.f. Theorem 8 of Green [8]).

In decomposing a twisted group algebra $A(\mathcal{I})$ into left ideals, we may make use of a composition series of $\mathcal{I}$ and consider $A(\mathcal{I}) = (\mathcal{F}_{\{E\}})^a$, where $\mathcal{F}_{\{E\}}$ is the trivial representation of the group $\{E\}$. This leaves only
the problem of the decomposition of twisted group algebras on simple groups.

A detailed analysis will now be given of the decomposition of \( L^\varphi \). Let \( H \to \lambda(H) \) be the linear representation afforded by the module \( L \). All such linear mappings will be written on the left. In particular an element of \( E_\varphi(L) \) will be represented by a linear mapping \( \theta \) written on the left.

Corresponding to each \( \alpha = 1, \cdots, s \) we have a non-singular linear transformation \( D_\alpha \) such that the \( \mathcal{A}(\mathcal{H}) \)-isomorphism \( \xi_\alpha \) of equation (2) is given by

\[
(6) \quad \xi_\alpha : L \to \langle X_\alpha \rangle \otimes D_\alpha L \quad (L \in L).
\]

If we make a second choice of isomorphisms, say \( \xi'_\alpha : L \to L_\alpha \), and if \( D'_\alpha \) are the corresponding linear mappings, then

\[
D_\alpha = \theta D'_\alpha,
\]

where \( \theta \) is a linear mapping representing an automorphism in \( E_\varphi(L) \).

We choose \( D_1 = I \), the identity map. If \( X_\alpha X_\beta = X_\gamma H \), then corresponding to equation (3) we have

\[
(7) \quad D_\alpha D_\beta = \frac{1}{X_\alpha H} \theta_{\alpha,\beta} D_\gamma \lambda(H),
\]

where \( \theta_{\alpha,\beta} \) represents an automorphism in \( E_\varphi(L) \), and where this equation may be taken as defining \( \theta_{\alpha,\beta} \). As \( D_1 = I \), it follows that \( \theta_{\alpha,1} = \theta_{1,\alpha} = I \) also.

We now define \( D_S \) for \( S = X_\alpha H \in \mathcal{S} \):

\[
(8) \quad D_S = \frac{1}{X_\alpha H} D_\alpha \lambda(H),
\]

and so \( D_{X_\alpha} = D_\alpha \), \( D_H = D_1 = I \). Then from these definitions it follows that if \( S \in X_\alpha \mathcal{H} \), \( S' \in X_\alpha \mathcal{H} \),

\[
(9) \quad D_SD_{S'} = \frac{1}{S, S'} \theta_{S, S'} D_{SS'}.
\]

Thus the correspondence \( S \to D_S \) gives rise to an extension of \( L \) to \( \mathcal{A}(\mathcal{S}) \) if, and only if, \( \theta_{\alpha,\beta} = 1 \), all \( \alpha, \beta \).

For the case of \( \mathcal{L} \) irreducible the analysis of Clifford in the proof of his theorem 3 in [5] (although not starting from the same point of view) can be adopted to get an explicit view of \( \mathcal{L}^\varphi \).

**Proposition 1.** Let \( \mathcal{L} \) be an irreducible \( \mathcal{A}(\mathcal{H}) \)-module. Then any direct summand \( \mathcal{M} \) of \( \mathcal{L}^\varphi \) affords a linear representation \( S \to \varphi(S) \) of \( \mathcal{A}(\mathcal{S}) \), which is the product of a fixed projective linear representation \( S \to D_S \) of \( \mathcal{A}(\mathcal{S}) \) (independent of \( \mathcal{M} \)) together with a certain direct summand \( \pi(S, \mathcal{H}) \) of the linear representation afforded by considering \( \mathcal{A}(\mathcal{S}|\mathcal{H}) \) as a left module over itself ("regular representation" of \( \mathcal{A}(\mathcal{S}|\mathcal{H}) \), i.e.,
Thus $\mathcal{M}$ must decompose just as $\pi$ does. For $\mathcal{M} = \mathcal{L}^\circ$, the decomposition of $\mathcal{L}^\circ$ is related directly to that of $A(\mathcal{I}/\mathcal{H})$ into left ideals.

Again following Clifford's line of argument, we have:

**Proposition 2.** In the situation of proposition 1, if $\pi$ is an irreducible linear representation of $A(\mathcal{I}/\mathcal{H})$, then the linear representation of $A(\mathcal{I})$ given by (10) is irreducible.

The analysis in the proof of Clifford's theorem 2 in [5] provides an explicit relation between the decomposition of $\mathcal{L}^\circ$ and that of $\mathcal{L}^\circ$.

Finally we consider certain problems on extensions of $\mathcal{L}$.

**Proposition 3.** Let $\mathcal{I}/\mathcal{H}$ be cyclic of order $m$ and suppose that either $p = 0$, or $(m, p) = 1$. Let $\mathcal{L}$ (indecomposable) have stabilizer the whole of $\mathcal{I}$. Then there exist exactly $m$ extensions of $\mathcal{L}$ to be an $A(\mathcal{I})$-module to within $A(\mathcal{I})$-isomorphism.

**Proof.** By the theorem $\mathcal{L}^\circ$ decomposes just as $A(\mathcal{I}/\mathcal{H})$ does. By § 1, remark 8, this must be the group algebra $A(\mathcal{I}/\mathcal{H})$ and so decomposes into $m$ non-isomorphic one-dimensional left ideals. Hence $\mathcal{L}^\circ$ consists of the direct sum of $m$ non-isomorphic extensions of $\mathcal{L}$.

Furthermore these are the only possible extensions of $\mathcal{L}$. For, say

$$G \rightarrow D_G,$$

where

$$D_H = \lambda(H) \quad (H \in \mathcal{H}),$$

is the linear representation afforded by any other extension of $\mathcal{L}$ as an $A(\mathcal{I})$-module. $D_a = D_{xa}$, is then a possible choice of $D$'s in (6); it follows that $\theta_a, b = I$, from (9). If $G_1(\mathcal{I}) \subseteq \mathcal{I}$ generates $\mathcal{I}/\mathcal{H}$, then all $D_G(G \in \mathcal{I})$ are determined in terms of $D_{G_1}$, by equations (7), (8) and (11). A calculation shows that the $m$ extensions of $\mathcal{L}$ contained in $\mathcal{L}^\circ$ have the linear representations determined by

$$G_1 \rightarrow \omega^t D_{G_1},$$

where $\omega$ is a primitive $m$-th root of unity in $\mathcal{F}$.

**Proposition 4.** Let $\mathcal{I}/\mathcal{H}$ be a cyclic extension of a $p$-subgroup, where $\mathcal{F}$ has characteristic $p \neq 0$. Let $|\mathcal{I}/\mathcal{H}| = mp^3$, $(m, p) = 1$ and let $\mathcal{L}$ be an irreducible $A(\mathcal{H})$-module, which has stabilizer the whole of $\mathcal{I}$. Then there exist exactly $m$ extensions of $\mathcal{L}$ to be an $A(\mathcal{I})$-module to within $A(\mathcal{I})$-isomorphism.

\[10\] Here $\times$ denotes the Kronecker or tensor product.

\[11\] Propositions 3 and 4 are generalizations of lemmas 1 and 2 of Srinivasan [19].
Twisted group algebras and their representations

PROOF. As \( \mathcal{L} \) is irreducible, \( \mathcal{F} \) algebraically closed, \( E_\mathcal{g}(\mathcal{L}^*) = \mathcal{A}(\mathcal{H}/\mathcal{H}) \), and \( E_\mathcal{g}(\mathcal{L}) \cong \mathcal{F} \). The \( D_\alpha \) of (6) are then determined to within a factor in \( \mathcal{F}^* \), and the \( \theta_{\alpha, \beta} \) are elements of \( \mathcal{F}^* \). A different choice of \( D_\alpha \)'s gives a basis transformation of type \( \S1 \), (1) on \( \mathcal{A}(\mathcal{H}/\mathcal{H}) \). By \( \S1 \), remark 9, \( \mathcal{A}(\mathcal{H}/\mathcal{H}) \) is the group algebra on \( \mathcal{H}/\mathcal{H} \) and so the \( \theta_{\alpha, \beta} \) may be considered equal to 1. Then \( G \to D_\alpha \) is a linear representation of an extension of \( \mathcal{L} \) to \( \mathcal{A}(\mathcal{F}) \) by (9).

Write \( \mathcal{P} \) for the subgroup of \( \mathcal{L} \), such that \( \mathcal{P}/\mathcal{H} \) is the Sylow \( p \)-group of \( \mathcal{L}/\mathcal{H} \). Restricting our attention to \( \mathcal{A}(\mathcal{P}) \) and \( \mathcal{A}(\mathcal{P}/\mathcal{H}) \), we see that if \( \theta_{\alpha, \beta} = 1 \), then the choice of \( D_P \ (P \in \mathcal{P}) \) is uniquely determined, for the only basis transformation of type \( \S1 \) (1) on the group algebra of a \( p \)-group, keeping the multiplication constants all 1, is the identity transformation. Let \( \mathcal{M} \) be this unique extension of \( \mathcal{L} \) to \( \mathcal{A}(\mathcal{P}) \).

By proposition 3, \( \mathcal{M} \) has exactly \( m \) different extensions to \( \mathcal{A}(\mathcal{F}) \) to within isomorphism.

3. Blocks and centres of twisted group algebras

The decomposition of a finite dimensional algebra \( \mathcal{A} \) into the direct sum of two sided ideals is determined by the corresponding decomposition of the centre \( \mathcal{Z} \). This in turn is determined by the decomposition of the identity element \( (E) \) as the sum of primitive central idempotents:

\[
E = I_1 + \cdots + I_n.
\]

The term block will be used to describe either an \( I_\alpha \) or the corresponding two sided ideal of \( \mathcal{Z} \) or \( \mathcal{A} \).

Rosenberg's analysis [16] of blocks of group algebras can be adapted to the twisted case by using the normalization theorem of \( \S1 \).

If \( \mathcal{A}(\mathcal{F}) \) is \( u \)-normalized, then a basis for its centre \( \mathcal{Z}(\mathcal{F}) \) is provided by the \( u \)-class sums \( K_\alpha \), as in \( \S1 \), remark 4. Then any block can be expressed as:

\[
I = \sum \alpha K_\alpha.
\]

Let us assume that the field characteristic \( p \neq 0 \). Consider the centralizers \( \mathcal{C}(A) \) in \( \mathcal{F} \) of elements \( A \) of \( \mathcal{I} \) which have non-zero coefficients in (2). The largest among the Sylow \( p \)-subgroups of these \( \mathcal{C}(A) \) is well defined up to conjugacy in \( \mathcal{I} \) and is the defect group \( \mathcal{D} \) of \( I \). If \( |\mathcal{D}| = p^d \), \( d \) is called the defect of \( I \).

If \( \mathcal{D} \) is any subgroup of \( \mathcal{I} \), write \( \mathcal{N}(\mathcal{D}) \) for the normalizer of \( \mathcal{D} \) in \( \mathcal{I} \) and \( \mathcal{C}(\mathcal{D}) \) for the centralizer of \( \mathcal{D} \) in \( \mathcal{I} \).

Take \( \mathcal{D} \) to be a \( p \)-group and write \( \mathcal{H} = \mathcal{N}(\mathcal{D}) \). Let \( \mathcal{I}(\mathcal{H}) \) be the centre of \( \mathcal{A}(\mathcal{H}) \). Consider a \( u \)-class \( \mathcal{K} \) of elements of \( \mathcal{I} \) with \( u \)-class sum \( K \) and write

\[
\sigma(K) = \text{sum of elements } (A),
\]
where $A \in \mathcal{X} \cap \mathcal{C}(\mathcal{D})$, if such elements exist, 0 otherwise. $\sigma$ can be extended to the whole of $\mathcal{L}(\mathcal{D})$ by linearity and is verified to be an $\mathcal{F}$-algebra homomorphism,

$$\sigma : \mathcal{L}(\mathcal{D}) \to \mathcal{L}(\mathcal{H}).$$

In the case of group algebras, Brauer's first theorem on blocks may be stated as follows:

$\sigma$ gives a $1-1$ correspondence between the blocks of $\mathcal{L}(\mathcal{D})$ with $\mathcal{D}$ as one of their defect groups and the blocks of $\mathcal{L}(\mathcal{H})$ of defect $d$. The latter have $\mathcal{D}$ as their unique defect group.

However, in the twisted case a complication arises as an element $H(e) \in \mathcal{H}$ may be a $u$-element in $\mathcal{A}(\mathcal{H})$ but not in $\mathcal{A}(\mathcal{D})$. To overcome this difficulty we define $\mathcal{U}(\mathcal{D})$ to be the subspace of $\mathcal{L}(\mathcal{H})$ spanned by those $u$-class sums of $\mathcal{A}(\mathcal{H})$ which have defect group $\mathcal{D}$ and whose elements are $u$-elements in $\mathcal{A}(\mathcal{D})$. Then $\mathcal{U}(\mathcal{D})$ is a subalgebra of $\mathcal{L}(\mathcal{H})$. The theorem for blocks in the twisted case can now be stated as follows:

$\sigma$ gives a $1-1$ correspondence between the blocks of $\mathcal{L}(\mathcal{D})$ with $\mathcal{D}$ as one of their defect groups and primitive idempotents of $\mathcal{U}(\mathcal{D})$. Each such idempotent is the sum of primitive idempotents of $\mathcal{L}(\mathcal{H})$ with $\mathcal{D}$ as their unique defect group.

Since this last theorem has reduced (to a certain extent) the problem to the case of blocks $I$ with a normal defect group $\mathcal{D}$ (which must then be unique), this special case warrants more attention. As $\mathcal{D}$ is normal in $\mathcal{D}$, it is certainly contained in the maximal normal $p$-subgroup $\overline{\mathcal{D}}$ of $\mathcal{D}$. Let us suppose then that $\mathcal{A}(\mathcal{D})$ has been $p$-$u$-normalized. Then the natural homomorphism $\mathcal{D} \rightarrow \mathcal{D}/\mathcal{D}$ gives rise to an algebra homomorphism

$$\tau : \mathcal{A}(\mathcal{D}) \rightarrow \mathcal{A}(\mathcal{D}/\mathcal{D}),$$

where $\mathcal{A}(\mathcal{D}/\mathcal{D})$ is a twisted group algebra on $\mathcal{D}/\mathcal{D}$. Ker $\tau$ is spanned by the elements $(A)((D)-(E))$, $A \in \mathcal{D}$, $D \in \mathcal{D}$, and is a nilpotent ideal of $\mathcal{A}(\mathcal{D})$. Further if $K$ is a $u$-class sum of $\mathcal{A}(\mathcal{D})$, such that $\mathcal{X} \cap \mathcal{C}(\mathcal{D}) = 0$, then $\tau(K) = 0$, and so $K$ is nilpotent. As ker $\tau$ is nilpotent, $\tau$ provides a $1-1$ correspondence between idempotents of $\mathcal{L}(\mathcal{D})$ and those of $\mathcal{L}(\mathcal{D}/\mathcal{D})$; thus the problem of blocks is further reduced to the case of defect $d = 0$.

Finally we have the following theorem for blocks of maximum defect, which we prove in full as the $u$-property needs careful attention.

**Theorem.** Let $\mathcal{D}$ have order $p^a m$, $(m, p) = 1$. Let $\mathcal{A}(\mathcal{D})$ be a twisted group algebra over an algebraically closed field $\mathcal{F}$ of characteristic $p \neq 0$. Then the number of blocks of defect $a$ equals the number of $p$-regular $u$-classes of defect $12 a$.

12 The defect group of a conjugacy class is any one of the Sylow $p$-subgroups of the centralizers in $\mathcal{D}$ of its elements.
Twisted group algebras and their representations

PROOF. A block of \( \mathcal{A}(\mathcal{D}) \) of defect \( a \) has the Sylow \( p \)-subgroups as its defect groups. Let \( \mathcal{D} \) be any such and write \( \mathcal{H} = \mathcal{N}(\mathcal{D}) \). Then the above theorem tells us that the number of blocks of defect \( a \) is the same as the number of primitive idempotents of \( \mathcal{U}(\mathcal{D}) \).

The homomorphism \( \tau \),

\[
\tau : \mathcal{A}(\mathcal{H}) \to \mathcal{A}(\mathcal{H}/\mathcal{D}),
\]

is defined as above. \( \mathcal{U}(\mathcal{D}) \) contains the identity element of \( \mathcal{A}(\mathcal{H}) \) and so, as \( \text{ker} \tau \) is nilpotent, the restriction of \( \tau \) to \( \mathcal{U}(\mathcal{D}) \) gives a 1—1 correspondence between idempotents of \( \mathcal{U}(\mathcal{D}) \) and those of \( \tau(\mathcal{U}(\mathcal{D})) \). \( \mathcal{A}(\mathcal{H}/\mathcal{D}) \) is semi-simple by § 1, remark 6, and so its centre \( \mathcal{Z}(\mathcal{H}/\mathcal{D}) \) is the direct sum of copies of \( \mathcal{F} \). As \( \tau(\mathcal{U}(\mathcal{D})) \) is a subalgebra of \( \mathcal{Z}(\mathcal{H}/\mathcal{D}) \), it is also semi-simple and hence the number of blocks of defect \( a \) in \( \mathcal{A}(\mathcal{D}) \) is equal to the dimension of \( \tau(\mathcal{U}(\mathcal{D})) \).

We may assume that \( \mathcal{A}(\mathcal{D}), \mathcal{A}(\mathcal{H}) \) and \( \mathcal{A}(\mathcal{H}/\mathcal{D}) \) are (separately) \( p \)-un-normalized. Write \( (G), [H] \) for the basis elements of \( \mathcal{A}(\mathcal{D}), \mathcal{A}(\mathcal{H}) \) respectively, where \( G \in \mathcal{G}, H \in \mathcal{H} \) and \( \{H\} \) for the basis element of \( \mathcal{A}(\mathcal{H}/\mathcal{D}) \) corresponding to the coset \( HD \) of \( \mathcal{H}/\mathcal{D} \). Thus \( \{H\} = \{HD\} \), for all \( D \in \mathcal{D} \).

Let \( G \) be a \( u \)-element of \( \mathcal{A}(\mathcal{D}) \) such that \( \mathcal{D} \) is a Sylow \( p \)-subgroup of \( \mathcal{U}(G) \). Write \( G = PR \), where \( P, R \) are powers of \( G \), \( P \) has order a power of \( p \), \( R \) is \( p \)-regular. Then \( \mathcal{D} \) is a Sylow \( p \)-subgroup of \( \mathcal{U}(R) \). Let \( \mathcal{H} \) be the \( u \)-class containing \( G \), and write \( \mathcal{L} = \mathcal{H} \cap \mathcal{U}(\mathcal{D}) \); then \( \mathcal{L} \) is a complete \( 13 \) conjugacy class in \( \mathcal{H} \). Thus

\[
\sigma(K) = dL,
\]

where \( K, L \) are the \( u \)-class sums of \( \mathcal{H}, \mathcal{L} \). (The factor \( d (\in \mathcal{F}^*) \) has to be introduced because of the possibly different normalizations of \( \mathcal{A}(\mathcal{D}), \mathcal{A}(\mathcal{H}). \)) Then

\[
\tau(\sigma(K)) = d\tau(L) \in \mathcal{A}(\mathcal{H}/\mathcal{D}).
\]

If \( \tau(\sigma(K)) \neq 0 \), it will now be proved that \( R \) is also a \( u \)-element in \( \mathcal{A}(\mathcal{D}) \).

If \( H \in \mathcal{H} \), write \( \mathcal{C}(H) = \text{centralizer of } H \text{ in } \mathcal{H}, \)

\[
\mathcal{C}(H) \cap \mathcal{H}.
\]

\( \mathcal{D} \) is the Sylow \( p \)-subgroup of \( \mathcal{U}(G) \). Further \( P \in \mathcal{C}(R) \) and so \( P \in \mathcal{D} \). Thus \( \{G\} = \{R\} \). As \( \tau(\sigma(K)) \neq 0 \), and \( \tau(\sigma(K)) \in \mathcal{A}(\mathcal{H}/\mathcal{D}), GD = RD \) must be a \( u \)-element in \( \mathcal{A}(\mathcal{H}/\mathcal{D}) \) (see § 1, remark 4). Take \( N \in \mathcal{C}(R) \) and write

\[
[N][R][N^{-1}] = b[R],
\]

\[
\tau([R]) = c[R],
\]

This is proved in Rosenberg's paper [16].
where \( b, c \in \mathfrak{F}^* \). Then
\[
\tau([N][R][N^{-1}]) = b\tau([R]) = bc\{R\}.
\]
On the other hand this is equal to
\[
\tau([N])\tau([R])\tau([N^{-1}]),
\]
\[
= \{N\}c\{R\}\{N^{-1} \} \quad \text{as both } \mathcal{A}(\mathcal{H}), \mathcal{A}(\mathcal{H}/\mathcal{D}) \text{ are normalized},
\]
\[
= c\{R\} \quad \text{as } R\mathcal{D} \text{ is a } \nu \text{-element in } \mathcal{A}(\mathcal{H}/\mathcal{D}),
\]
and so \( b = 1 \), i.e., \( R \) is a \( \nu \)-element in \( \mathcal{A}(\mathcal{H}) \). Hence we have
\[
(N)(R)(N^{-1}) = (R)
\]
in \( \mathcal{A}(\mathcal{D}) \), for all \( N \in \mathcal{C}(R) \cap \mathcal{N}(\mathcal{D}) \), as both \( \mathcal{A}(\mathcal{D}), \mathcal{A}(\mathcal{H}) \) are normalized.

Let \( \mathcal{D}' \) be any other Sylow \( p \)-subgroup of \( \mathcal{C}(R) \); then there exists \( T \in \mathcal{C}(R) \) such that \( \mathcal{D}' = T\mathcal{D}T^{-1} \). Thus
\[
T(\mathcal{C}(R) \cap \mathcal{N}(\mathcal{D}))T^{-1} = \mathcal{C}(R) \cap \mathcal{N}(\mathcal{D}'),
\]
\[
TGT^{-1} = R(TPT^{-1}).
\]
Take \( TNT^{-1} \in \mathcal{C}(R) \cap \mathcal{N}(\mathcal{D}') \), where \( N \in \mathcal{C}(R) \cap \mathcal{N}(\mathcal{D}) \). From (3) we get
\[
((T)(N)(T^{-1}))((T)(R)(T^{-1}))((T)(N^{-1})(T^{-1})) = (T)(R)(T^{-1}),
\]
i.e.
\[
((T)(N)(T^{-1}))(R)((T)(N^{-1})(T^{-1})) = (R).
\]
Using § 1, remark 3, we get
\[
(TNT^{-1})(R)(TN^{-1}T^{-1}) = (R),
\]
and so
\[
(M)(R)(M^{-1}) = (R),
\]
for all \( M \in \mathcal{C}(R) \cap \mathcal{N}(\mathcal{D}') \).

Let \( \mathcal{D}_1 = \mathcal{D}, \mathcal{D}_2, \ldots, \mathcal{D}_\alpha \) be all the Sylow \( p \)-subgroups of \( \mathcal{C}(R) \) and let \( \mathcal{D} \) be the group union of the subgroups \( \mathcal{C}(R) \cap \mathcal{N}(\mathcal{D}_\alpha) \). Then \( \mathcal{C}(R) = \mathcal{D} \), \( \mathcal{D} \) is normal in \( \mathcal{C}(R) \) and \( \mathcal{D} \) contains the normalizer of a Sylow \( p \)-subgroup of \( \mathcal{C}(R) \). Any element of \( \mathcal{C}(R) \) has the form \( C = A_1A_2 \cdots A_m \), where \( A_\alpha \in \) some \( \mathcal{C}(R) \cap \mathcal{N}(\mathcal{D}_\alpha) \). Thus if \( \tau(\sigma(K)) \neq 0 \), then
\[
(C)(R)(C^{-1}) = (A_1 \cdots A_m)(R)(A_m^{-1} \cdots A_1^{-1}),
\]
\[
= (A_1) \cdots (A_m)(R)(A_m^{-1}) \cdots (A_1^{-1}) \quad \text{by } \text{§ 1, remark 3},
\]
\[
= (R) \quad \text{by repeated use of } (4),
\]
and so \( R \) is a \( \nu \)-element of \( \mathcal{A}(\mathcal{D}) \).

Let \( \mathcal{K}_\alpha (\alpha = 1, \ldots, r) \) be the \( p \)-regular \( \nu \)-classes of defect \( \alpha \) in \( \mathcal{A}(\mathcal{D}) \) with corresponding \( \nu \)-class sums \( K_\alpha \). The \( \mathcal{L}_\alpha = \mathcal{K}_\alpha \cap \mathcal{C}(\mathcal{D}) \) consist of single
conjugacy classes in $\mathcal{H}$, and so the $\sigma(K_a)$ are multiples of the class sums $L_a$. Write $\mathcal{P} = \bigcup_a L_a$ (set union). Then the \{H\} ($H \in \mathcal{P}$) are all distinct in $\mathcal{A}(\mathcal{H} \otimes \mathcal{D})$. For say \{H\} = \{H'\}. Then $H = H'D$, for some $D \in \mathcal{D}$. But each $L_a$ has defect group $\mathcal{D}$ and so $D \in \mathcal{D} \subset \mathcal{C}(H')$. Further, the orders of $H, H'$ are prime to $p$ and so $D = E$, or $H = H'$. Hence the $\tau(L_a)$ are all non-zero and linearly independent. But $\tau(L_a) \in \tau(\mathcal{U}(\mathcal{D}))$ and so dim $\tau(\mathcal{U}(\mathcal{D})) \geq r$. It remains to show that the $\tau(L_a)$ actually span $\tau(\mathcal{U}(\mathcal{D}))$.

It is clear that the $L_a$ exhaust all the $p$-regular conjugacy classes of $\mathcal{H}$ of defect group $\mathcal{D}$ which consist of $u$-elements in $\mathcal{A}(\mathcal{P})$. Let then $\mathcal{P}$ be any $p$-singular class of $\mathcal{H}$ of defect group $\mathcal{D}$ and consisting of $u$-elements in $\mathcal{A}(\mathcal{D})$, i.e. $L$ is a $p$-singular $u$-class sum in $\mathcal{U}(\mathcal{D})$. Take $G \in \mathcal{L}$, and write $G = PR$ as before. Then if $\tau(L) \neq 0$, $R$ is a $u$-element of $\mathcal{A}(\mathcal{P})$ and $\tau(L)$ is equal to a multiple of $\tau(M)$, where $M$ is the class sum of the conjugacy class $\mathcal{M}$ of $R$ in $\mathcal{A}(\mathcal{H})$. But $\mathcal{M}$ must be one of the classes $\mathcal{L}_a$ and so the $\tau(L_a)$ do in fact span $\tau(\mathcal{U}(\mathcal{D}))$.

Thus the number of blocks of $\mathcal{A}(\mathcal{P})$ of highest defect = dim $\tau(\mathcal{U}(\mathcal{D})) = r$, the number of $p$-regular $u$-classes of highest defect $a$.

4. Vertices and sources

The results of Higman [9] [10] and Green [8] can also be carried over to the twisted case. Here the generalization is even more direct than in § 3 and for most of the results we need only insist that the algebras be normalized. As before all modules will be assumed to have finite dimension over $\mathcal{F}$.

Let $\mathcal{H}$ be a subgroup of $\mathcal{J}$. An $\mathcal{A}(\mathcal{J})$-module $\mathcal{M}$ is said to be $\mathcal{H}$-projective if there exists an $\mathcal{A}(\mathcal{H})$-module $\mathcal{R}$ such that $\mathcal{M}$ is isomorphic to an $\mathcal{A}(\mathcal{J})$-direct summand of $\mathcal{R}^\mathcal{H}$. This definition is equivalent to $\mathcal{M}$ being $(\mathcal{A}(\mathcal{F}), \mathcal{A}(\mathcal{H}))$-projective or $(\mathcal{A}(\mathcal{J}), \mathcal{A}(\mathcal{H}))$-injective in the sense of Hochschild [12] or Higman [11].

When $\mathcal{F}$ has characteristic $p = 0$, or $p \nmid |\mathcal{G}|$, by § 1, remark 6, $\mathcal{A}(\mathcal{J})$ is semi-simple. Hence all $\mathcal{A}(\mathcal{J})$-indecomposables occur in the regular representation. Thus all $\mathcal{A}(\mathcal{J})$-modules are $\{E\}$-projective and the theory is trivial. From now on we assume $p \neq 0$.

Higman’s criterion$^{14}$ for $\mathcal{M}$ to be $\mathcal{H}$-projective can be written down immediately. Further, taking $\mathcal{H} = \mathcal{P}$, a Sylow $p$-subgroup of $\mathcal{J}$, we find that every indecomposable $\mathcal{A}(\mathcal{J})$-module $\mathcal{M}$ is a component of a module induced from some $\mathcal{A}(\mathcal{P})$-module. But by § 1, remark 7, if $\mathcal{F}$ is large enough, $\mathcal{A}(\mathcal{P})$ is the group algebra $\mathcal{F}(\mathcal{P})$ and so all indecomposable $\mathcal{A}(\mathcal{J})$-modules can be obtained by inducing from ordinary group representations of $p$-groups. $\mathcal{A}(\mathcal{J})$ has a finite number of different indecomposable $\mathcal{A}(\mathcal{J})$-modules if,

$^{14}$ c.f. theorem 1, p. 371 of [9].
and only if, \( \mathcal{D} \) is cyclic, and as in [10] a rough upper bound for the number of indecomposables is
\[
\frac{1}{2} p^a (m(p^a + 1) - p^a + 1),
\]
where \( |\mathcal{G}| = mp^a, (m, p) = 1 \).

If \( \mathcal{D}, \mathcal{L} \) are subgroups of \( \mathcal{G} \) we shall write \( \mathcal{D} \subseteq \mathcal{L} \) if there exists a \( T \in \mathcal{G} \) such that \( \mathcal{D} \subseteq T \mathcal{L} T^{-1} \), and \( \mathcal{D} = \mathcal{L} \), if \( \mathcal{D} = T \mathcal{L} T^{-1} \). If \( \mathcal{M} \) is an indecomposable \( \mathfrak{A}(\mathcal{G}) \)-module, then a subgroup \( \mathcal{V} \) of \( \mathcal{G} \) is a called a vertex of \( \mathcal{M} \) if
(a) \( \mathcal{M} \) is \( \mathcal{V} \)-projective, and
(b) if \( \mathcal{M} \) is \( \mathcal{H} \)-projective, then \( \mathcal{V} \subseteq \mathcal{H} \). \( \mathcal{V} \) is then determined up to conjugacy in \( \mathcal{G} \) and is a \( p \)-subgroup. When \( p \nmid |\mathcal{G}| \) (or \( p = 0 \)), all vertices coincide with \( \{E\} \).

We may also look at the various \( \mathfrak{A}(\mathcal{G}) \)-modules \( \mathcal{I} \) such that \( \mathcal{I}^w \) contains \( \mathcal{M} \) as a component. As the process of inducing (i.e. \( \otimes \)) is distributive over direct sum and \( \mathcal{M} \) is indecomposable, it is sufficient to consider \( \mathcal{I} \) indecomposable. If \( \mathcal{I}' \) is a second such indecomposable \( \mathfrak{A}(\mathcal{G}) \)-module, then there exists an element \( X \in \mathfrak{N}(\mathcal{G}) \) such that
\[
\mathcal{I}' \cong (X) \otimes_{\mathfrak{A}(\mathcal{G})} \mathcal{I},
\]
considered as \( \mathfrak{A}(\mathcal{G}) \)-modules. Thus \( \mathcal{I}' \) is called a source of \( \mathcal{M} \).

As in the corollary to theorem 6 of [8], the problem of determining the vertex and source of a given indecomposable \( \mathfrak{A}(\mathcal{G}) \)-module \( \mathcal{M} \) can be reduced to the same problem for \( \mathfrak{A}(\mathcal{D}) \), where \( \mathcal{D} \) is a Sylow \( p \)-subgroup of \( \mathcal{G} \), i.e. to the same problem for \( p \)-group representations. Hence Green's discussion of induced modules in \( p \)-groups (§ 4 of [8]) is relevant.

The existence of the vertex and source of a given indecomposable \( \mathcal{M} \) can also be inferred from the non-twisted case by means of the group algebra \( \mathfrak{I}(\mathcal{G}^*) \) defined in § 1, remark 5.

The notion of blocks of § 3 can be extended further to embrace indecomposable \( \mathfrak{A}(\mathcal{G}) \)-modules \( \mathcal{M} \). If \( (E) \) is decomposed as in § 3 (1), then
\[
\mathcal{M} = (E) \mathcal{M} \cong I_1 \mathcal{M} \oplus \cdots \oplus I_s \mathcal{M},
\]
this being an \( \mathfrak{A}(\mathcal{G}) \)-direct sum decomposition. But \( \mathcal{M} \) is indecomposable and so there is one and only one \( I_i \) such that \( I_i \mathcal{M} = \mathcal{M} \). We say that \( \mathcal{M} \) is in the block \( I_i \).

Let then \( \mathcal{M} \) be an indecomposable \( \mathfrak{A}(\mathcal{G}) \)-module of vertex \( \mathcal{V} \), and in the block \( I \) of defect group \( \mathcal{D} \). Then \( \mathcal{V} \subseteq \mathcal{D} \). On the other hand we shall prove the existence of an \( \mathfrak{A}(\mathcal{G}) \)-module in the block \( I \) with vertex \( \mathcal{V} \) and so the defect group \( \mathcal{D} \) of a block \( I \) may be characterised as being the "supremum" of the vertices of indecomposable modules in the block.

The following proposition helps in the construction of the above indecomposable.
PROPOSITION. Let $I$ be a block of $A(D)$ of defect group $D$. Let $\sigma$ be defined with respect to $D$ and write

$$\sigma(I) = J_1 + \cdots + J_r,$$

where $J_i$ are primitive idempotents (blocks) of $A(H)$ ($H = N(D)$). Let $R$ be an indecomposable $A(H)$-module belonging to one of the above blocks, $J_1$ say. Then there is a component $M$ of $R^\sigma$ belonging to the block $I$ such that $R$ is isomorphic to a component of $M$.

PROOF. Let $X_r H$ be the cosets of $H$ in $\mathcal{G}(X_r \in \mathcal{G})$, with $X_1 = E$. Then

$$(R^\sigma)_r \approx ((E) \otimes a(H) R) \oplus \left( \sum_{\alpha > 1} (X_\alpha) \otimes a(H) R \right)$$

is an $A(H)$-direct decomposition. We write $R = \sum_{\alpha > 1} (X_\alpha) \otimes R$ and we identify $(E) \otimes R$ with $R$. Let $\pi$ denote the $A(H)$-projection:

$$\pi : (R^\sigma)_r \rightarrow (E) \otimes R = R.$$

We write

$$I = \sigma(I) + T_1 + T_2,$$

where $T_1$ is the sum of terms in $A(H)$ but not in $A(C(D))$, and $T_2$ is the sum of the remaining terms not in $A(H)$. For each $w$-class sum $L$ in $T_1$, $L \cap C(D) = \emptyset$ and so $\tau(L) = 0$ ($\tau$ is defined in § 3). Hence $\tau(T_1) = 0$, and $T_1$ is nilpotent.

For $A \in A(H)$, we write $\rho(A)$ for the linear transformation representing $A$ in the representation afforded by $(E) \otimes R = R$. Clearly $\sigma(I)$ acts identically on $R$, and so $\rho(\sigma(I) + T_1)$, being the sum of the identity transformation and a nilpotent one, is non-singular. Hence the map

$$R \rightarrow IR = \rho(\sigma(I) + T_1) R \oplus (T_2 \otimes R) \quad (R \in \mathcal{G})$$

is an $A(H)$-homomorphism, the decomposition on the right hand side being that of (1). On the other hand

$$\pi(IR) = \rho(\sigma(I) + T_1) R$$

and so $\pi I$ is an $A(H)$-automorphism of $(E) \otimes R = R$. Hence $R \cong I(R)$ and $I(R)$ is an $A(H)$-component of $(I(R^\sigma))_r$. By the Krull-Schmidt theorem there is a component $M$ of $I(R^\sigma) (\subseteq R^\sigma)$ such that $M_\sigma$ has a component isomorphic to $R$. $M$ must also be in the block $I$.

The construction of the required indecomposable in block $I$ of vertex $\mathcal{G}$ is now simple. Suppose first of all that $D$ is normal in $\mathcal{G}$. As ker $\tau$ is nilpotent, $\tau(I)$ must be a non-zero idempotent of $\mathcal{G}(\mathcal{G}/D)$. Write

$\text{This follows from the lemma: If } U, V \text{ are modules and there exist homomorphisms } \alpha : U \rightarrow V, \beta : V \rightarrow U \text{ such that } \beta \alpha (\alpha \text{ followed by } \beta) \text{ is an automorphism, then } V = \text{Im} \alpha \oplus \ker \beta.$
as a decomposition into blocks of $B(D)$. Let $A$ be any principal component of $A(D)$ in block $J_1$, say. $J_1$ has defect group $\{e\}$ in $D$ and $A$ has vertex $\{e\}$ in $D$. By means of the homomorphism $\tau$, $A$ can be considered as an $A(D)$-module, and as such it will be in the block $I$ and will have vertex $D$.

For the case where $D$ is not necessarily normal we first write
$$\sigma(I) = J'_1 + \cdots + J'_r,$$
where the $J'_r$ are primitive idempotents in $B(H)$, each having defect group $D$ by the main theorem on blocks. By the previous paragraph there is an indecomposable $A(H)$-module $A$ in block $J'_1$, say, with vertex $D$. By the proposition there is a component $M$ of $\mathfrak{M}$ in block $I$ with a component of $\mathfrak{M}_x$ isomorphic to $A$. As the defect group of $I$ is $D$ and as $M$ is in the block $I$, the vertex $V$ of $M$ satisfies
$$V \subseteq D.$$
On the other hand as $M$ is $V$-projective each of the components of $\mathfrak{M}_x$ has vertex $\{e\} \subseteq V$. In particular the vertex $D$ of the component isomorphic to $A$ satisfies
$$D \subseteq V.$$
Hence $D = V$, and so $M$ is in block $I$ with vertex $D$.

References


This follows as in theorem 6 of [8]
Twisted group algebras and their representations

15 Nakayama, T., Some studies on regular representations, induced representations and modular representations, Ann. of Math. (2) 39 (2) (1938), 361—369.

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