1

Basic Group Theory and Representation Theory

In this chapter, we present the basic theory of finite groups and their representations as preparation for the discussion of continuous groups that starts from Chapter 3. It is assumed that readers know the basics of set theory, vector spaces, transformations, linear operators, matrix representations, inner products and such. These will be called upon as and when needed.

1.1 Definition of a Group

A group G is a set of elements $a, b, c, \cdots, g, g', \cdots, e, \cdots$ along with a composition (or 'multiplication') law obeying four conditions:

(i) Closure: $a, b \in G \to ab =$ unique product element $\in G$.

(ii) Associativity: for any $a, b, c \in G$,

$$a(bc) = (ab)c = abc \in G.$$

(iii) Identity: there is a unique element $e \in G$ such that

$$ae = ea = a, \qquad \text{for any } a \in G.$$

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(iv) Inverses: for each $a \in G$, there is a unique $a^{-1} \in G$, the inverse of a, such that

$$a^{-1}a = aa^{-1} = e. \tag{1.1}$$

The composition rule or law can be called a binary law as the product is defined for any *pair* of elements. The conditions in (1.1) could be stated in more economical forms, for instance introducing only a left identity and left inverses, and then showing that the more general properties in (1.1) do hold.

One can immediately think of various qualitatively different possibilities. The number of (distinct) elements in G may be finite. Then this number, denoted by $|G|$, is called the order of G. Some other possibilities are that the number of elements may be a discrete infinity, or else a continuous infinity with G being a manifold of some dimension.

1.2 Some Examples

(i) The symmetric group, the group of permutations on n objects, is finite, of order $n!$, and is denoted by S_n. We mention only a few pertinent properties now, and go into some more detail in Chapter 2. Each $p \in S_n$ can be written in several convenient ways:

$$\begin{aligned} p &= \begin{pmatrix} 1 & 2 & \cdots & n \\ p(1) & p(2) & \cdots & p(n) \end{pmatrix} = \begin{pmatrix} j \\ p(j) \end{pmatrix}_{j=1\cdots n} \\ &= (1\ p(1)\ p(p(1))\cdots)\ (j\ p(j)\ p(p(j))\cdots)\cdots(k\ p(k)\ p(p(k))\cdots). \end{aligned} \tag{1.2}$$

In the first form the columns can be rearranged at will, while in the second form the factors and their entries are distinct. The composition law can be developed as follows:

$$qp = \begin{pmatrix} k \\ q(k) \end{pmatrix}\begin{pmatrix} j \\ p(j) \end{pmatrix} = \begin{pmatrix} p(j) \\ q(p(j)) \end{pmatrix}\begin{pmatrix} j \\ p(j) \end{pmatrix} = \begin{pmatrix} j \\ q(p(j)) \end{pmatrix},$$

i.e.,

$$(qp)(j) = q(p(j)), \quad j = 1, 2, \cdots, n. \tag{1.3}$$

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We have followed here the rule of 'reading from right to left'. The identity and inverses are:

$$e = \begin{pmatrix} 1\ 2 \cdots n \\ 1\ 2 \cdots n \end{pmatrix} = \begin{pmatrix} j \\ j \end{pmatrix}_{j=1\cdots n} = (1)(2)\cdots(n);$$

$$p^{-1} = \begin{pmatrix} p(j) \\ j \end{pmatrix}_{j=1\cdots n}, \text{ i.e., } p^{-1}(p(j)) = j. \tag{1.4}$$

All the conditions in (1.1) can and should be checked.

(ii) All integers with respect to addition. Here the group 'multiplication' law is arithmetic addition. The identity is 0, and inverses are negatives.

(iii) All positive real (or rational) numbers with respect to multiplication. Now the identity is 1, and inverses are reciprocals.

(iv) All vectors in any (real or complex) linear vector space with respect to vector addition. Similar to (ii) above, the identity is the zero vector $\mathbf{0}$, and inverses are negatives.

(v) $SO(2)$ and $O(2)$, $SO(3)$ and $O(3)$: these are the proper and full groups of rotations in a plane or in three dimensions, respectively. These are continuous groups with infinitely many elements. In the $O(2)$ and $O(3)$ cases, the group is made up of two disjoint components, each of which is continuous and connected in an obvious intuitive sense. We study these in Chapter 3.
We can also consider discrete subsets of these, leaving invariant some given regular plane figure or solid. These are the point groups.

(vi) The rotation groups $SO(3), O(3)$ generalise to any dimension n and to the complex case. Thus we arrive at the groups $SO(n)$ and $O(n)$, $SU(n)$ and $U(n)$ for various integers n, which we will study in some detail later. Yet others are the complex orthogonal groups $SO(n, \mathbb{C})$.

(vii) Groups related to spacetime. Here we have the Euclidean, Galilean, Lorentz, and Poincaré groups. All of these are continuous groups with infinitely many elements. They have more than one connected component if discrete operations like space and/or time reflections are included. We will study some of these groups in Chapter 10.

A group G is said to be abelian if for any pair of elements a and b, $ab = ba$. Otherwise it is nonabelian. In the examples listed above, (ii), (iii) and (iv) are abelian.

Among the symmetric groups, S_2 is abelian while S_n for $n \geq 3$ is nonabelian. $SO(2)$ is abelian, and $O(2)$, $SO(3)$ and $O(3)$ are all nonabelian.

The existence and properties of a unique identity element and unique inverse a^{-1} for each $a \in G$ lead to the useful cancellation rules:

$$ab = ac \Leftrightarrow b = c; \quad ba = ca \Leftrightarrow b = c. \tag{1.5}$$

For a finite group G the composition law or entire structure can be displayed in a *multiplication table* with $|G|$ 'rows' and 'columns' labelled by the group elements:

$$\begin{array}{c|cccc} & e & \cdots & b & \cdots \\ \hline e & e & \cdots & b & \cdots \\ \vdots & \vdots & & \vdots & \\ a & a & \cdots & ab & \cdots \\ \vdots & \vdots & & \vdots & \end{array} \tag{1.6}$$

At the intersection of row a and column b stands the product ab. These entries must be consistent with the group laws (1.1). So by (1.5) each row (column) contains every element of G exactly once.

1.3 Operations within a Group

For the present we continue to have in mind a finite group, though many concepts we will go on to introduce are more generally meaningful. For any elements $a, b \in G$, conjugation of a by b leads to another element $a' \in G$:

$$a' = bab^{-1}, \; a = b^{-1}a'b. \tag{1.7}$$

We then say a and a' are conjugate to one another. For each $a \in G$, its *equivalence class or conjugacy class* $\mathcal{C}(a)$ consists of all a' conjugate to a:

$$\mathcal{C}(a) = \text{equivalence class of } a = \{bab^{-1} |\, b \in G, \text{ omit repetitions}\} \subset G. \tag{1.8}$$

Different classes are generally of different 'sizes'. For instance, $\mathcal{C}(e) = \{e\}$ consists of one element alone. It is easy and important to check:

$$\begin{aligned} &(i)\ \mathcal{C}(a) = \mathcal{C}(bab^{-1}), \text{ any } b, \text{so } \mathcal{C}(a) \text{ is determined by any one of its elements;} \\ &(ii)\ \mathcal{C}(a) \cap \mathcal{C}(b) = \mathcal{C}(a) \text{ if } b \in \mathcal{C}(a), \text{ null otherwise, so two classes cannot} \\ &\qquad \text{overlap partially;} \\ &(iii)\ G \text{ is the union of } \textit{disjoint} \text{ equivalence classes.} \end{aligned} \tag{1.9}$$

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In S_n, for example, all elements in one class have common cycle structure and vice versa. Thus if we write any $p \in S_n$ in the form

$$\begin{aligned} p &= \underbrace{(i_1 \; p(i_1) \; p(p(i_1)) \cdots)}_{m_1} \; \underbrace{(i_2 \; p(i_2) \cdots)}_{m_2} \cdots , \\ n &= m_1 + m_2 \cdots , \end{aligned} \tag{1.10}$$

where i_2 is not one of the earlier m_1 entries, i_3 is not one of the earlier $m_1 + m_2$ entries, $\cdots$, then $m_1, m_2, \cdots$ is some partition of n. The cycle structure of p is denoted by $(m_1, m_2, \cdots)$, and cycle structures determine equivalence classes and conversely. All $p' \in S_n$ conjugate to p in (1.10) have the same cycle structure as p, and conversely.

In $SO(3)$, as we will recall later, each rotation is by some right handed angle about some axis. Then each class consists of all rotations by a given angle about all possible axes.

Subgroups

A subset $H \subset G$ is a *subgroup* if its elements obey all the conditions to be a group, given the composition law in G. $H = \{e\}$ or G are trivial cases. An elegant and compact criterion for a subset to be a subgroup is this:

$$H \subset G \text{ is a subgroup} \Leftrightarrow \text{for all } h_1, h_2 \in H,\; h_1^{-1} h_2 \in H. \tag{1.11}$$

As examples of subgroups in familiar groups we have: all even integers in the additive group of all integers; all even permutations in S_n making up the alternating subgroup A_n; S_{n-1} within S_n, i.e., permutations p with $p(n) = n$; all rotations in $SO(3)$ about a given axis.

In a finite group, each element leads via its 'powers' to a subgroup called its *cycle*:

$$\begin{aligned} &a \in G \to \{e, a, a^2, \cdots, a^{q-1}\} = \text{cycle of } a = \text{subgroup in } G, \\ &q = \text{least positive integer such that } a^q = e, q = \text{order of } a. \end{aligned} \tag{1.12}$$

We will see that q divides $|G|$; it is clear that $a^{-1} = a^{q-1}$, etc.

Given a subgroup $H \subset G$ and any $g \in G$, conjugation leads to another *conjugate subgroup*:

$$H_g = gHg^{-1} = \{ghg^{-1} \mid h \in H \text{ varying, } g \text{ fixed}\} \subset G. \tag{1.13}$$

Of course, $g \in H$ implies $H_g = H$; and if $g \notin H$, H_g could be different from H.

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Cosets

Given a subgroup $H \subset G$, there are two generally distinct and useful ways to break G up into subsets, based on two kinds of cosets:

$$\begin{aligned} aH &= \{ab|\ a \text{ fixed, all } b \in H\} = \text{ right coset containing } a \in G; \\ Ha &= \{ba|\ a \text{ fixed, all } b \in H\} = \text{ left coset containing } a \in G. \end{aligned} \tag{1.14}$$

As we saw with equivalence classes, here too it is easy to see the following: each coset is determined by any one of its elements; two cosets (both right or both left) either coincide fully or are disjoint; G is a union of (right or left) cosets. However, unlike classes, each coset is of the same 'size', namely as 'big' as H. It follows that $|G|/|H|$ is an integer, the number of disjoint right (or left) cosets. This is *Lagrange's theorem*. The claim made after (1.12) is now understood.

For a general subgroup $H \subset G$, right and left cosets are different, leading to different ways of expressing G as a union of disjoint subsets. However, if $H_g = H$ for all $g \in G$, i.e., H is self conjugate, then every right coset is also a left coset and vice versa. Then we say H is a *normal* or an *invariant* subgroup:

$$\begin{aligned} H \text{ is an invariant subgroup } &\Leftrightarrow H_g = gHg^{-1} = H \text{ for every } g \in G \\ &\Leftrightarrow aH = Ha \text{ for every } a \in G \\ &\Leftrightarrow \text{ the two kinds of cosets coincide.} \end{aligned} \tag{1.15}$$

Then these (common) cosets can themselves be regarded as elements of a group, the *quotient group* or *factor group* G/H: the identity is the coset containing the identity element, $eH = H$, ie, H itself; the composition of two cosets is given by $aH \cdot bH = ab \cdot H$; and for inverses we take $(aH)^{-1} = a^{-1}H$. It is easy to check that all the group laws are obeyed. The orders obey $|G/H| = |G|/|H|$.

For any group G, there are two natural invariant subgroups, the *centre* and the *commutator* subgroup. The former consists of all those elements which 'commute' with all elements,

$$Z = \text{centre of } G = \{a \in G|\ ab = ba \text{ for all } b \in G\}, \tag{1.16}$$

so obviously it is abelian. The latter is more intricate and in fact is a way of 'measuring' the extent to which G is nonabelian. If G is abelian, then always $ab = ba$. In general,

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we define the *commutator* of any two elements a and b as the element

$$q(a,b) = aba^{-1}b^{-1} = \text{measure of noncommutativity of } a \text{ and } b,$$
$$q(a,b) = e \Leftrightarrow ab = ba. \tag{1.17}$$

It is easy to see that under inversion and conjugation,

$$q(a,b)^{-1} = q(b,a),\ \ cq(a,b)c^{-1} = q(cac^{-1}, cbc^{-1}). \tag{1.18}$$

The commutator subgroup $Q \subset G$ is now defined as consisting of products of any numbers of commutator factors:

$$Q = \{q(a_1,b_1)q(a_2,b_2)\cdots q(a_m,b_m)|\text{ any } m, a\text{'s},\ \ b\text{'s}\}. \tag{1.19}$$

It is easy to see from (1.18) that Q is an invariant subgroup of G. Further, since $ab = q(a,b)ba = baq(a^{-1}, b^{-1})$, we see that G/Q is abelian. Thus in the quotient all noncommutativity in G has been removed.

There is a converse to this statement: if H is an invariant subgroup in G such that G/H is abelian, then H contains Q.

For the symmetric group S_n, Q is the alternating group A_n of all even permutations, and S_n/A_n is the two element abelian group.

We mention that the concept of the commutator subgroup is basic to the definitions of so-called solvable and nilpotent groups, but we do not go into them here.

Returning to the general concept of invariant subgroups, we have two important definitions:

$$(a)\ \ G \text{ is simple } \Leftrightarrow G \text{ has no (proper) invariant subgroup;}$$
$$(b)\ \ G \text{ is semisimple } \Leftrightarrow G \text{ has no invariant abelian subgroup.} \tag{1.20}$$

This leads to a one-way relationship: G is simple $\Rightarrow$ G is semisimple.

Among the symmetric groups S_n we always have the alternating subgroup A_n which is invariant, so all S_n are nonsimple. For $n = 3$ or $n \geq 5$, A_n is the only invariant subgroup in S_n. In the case of S_4, apart from A_4 there is one other invariant subgroup K of 4 elements, and the quotient $S_4/K \sim S_3$. In contrast, the rotation group $SO(3)$ is simple.

The last operation within a given group we consider is that of an *automorphism*. Given G, an automorphism of G is a map $\tau : G \to G$ which is one-to-one, onto,

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invertible and preserves products:

$$a \in G \to \tau(a) \in G\colon \tau(a)\tau(b) = \tau(ab),\ \tau^{-1} \text{ exists}, \tau(G) = G. \tag{1.21}$$

It is easy to see that $\tau(a)^{-1} = \tau(a^{-1}),\ \tau(e) = e$. Conjugation by a fixed $g \in G$ is an automorphism:

$$g \in G,\ \text{fixed:}\ \ \tau_g(a) = gag^{-1},\ \ a \in G. \tag{1.22}$$

The conditions in (1.21) are immediately verified. These are called *inner* automorphisms, all others are *outer*. Clearly if G is abelian, every nontrivial automorphism is outer. We will later come across physically important examples of automorphisms.

1.4 Operations with and Relations between Groups

We consider four of these:

(A) *Homomorphism* Given two groups G' and G, a homomorphism is a map $\Phi : G' \to G$ such that images of products are products of images:

$$\Phi(a'b') = \Phi(a')\Phi(b'),\ \ \text{all } a', b' \in G'. \tag{1.23}$$

It is easy to see that $\Phi(e') = e \in G$, $\Phi(a'^{-1}) = \Phi(a')^{-1}$, $\Phi(G') \subset G$ is a subgroup in G. We may generally assume $\Phi(G') = G$ or else limit ourselves to $\Phi(G')$ in G. The kernel of a homomorphism is

$$K = \{g' \in G' |\ \Phi(g') = e\} = \text{invariant subgroup in } G', \tag{1.24}$$

so we can form the quotient G'/K.

(B) *Isomorphism* This is a particular case of homomorphism when Φ is one to one, onto and invertible, so $\Phi(G') = G$. Thus as groups G' and G are 'identical' or essentially the same. Returning to a general homomorphism, case (A), we see that, assuming $\Phi(G') = G$,

$$G'/K \text{ is isomorphic to } G. \tag{1.25}$$

In an isomorphism G' and G play symmetric roles, but in a homomorphism this is not so.

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(C) *Direct product* Given two groups G_1 and G_2, their direct (or Cartesian) product $G_1 \times G_2$ is another group. Its elements are ordered pairs (a_1, a_2) with $a_1 \in G_1$, $a_2 \in G_2$. The group law is trivial, each entry 'minding its own business':

$$\begin{aligned} (a_1, a_2)(b_1, b_2) &= (a_1 b_1, a_2 b_2); \\ \text{identity} &= (e_1, e_2); \\ (a_1, a_2)^{-1} &= (a_1^{-1}, a_2^{-1}). \end{aligned} \tag{1.26}$$

The orders multiply: $|G_1 \times G_2| = |G_1||G_2|$. In a natural sense, G_1 and G_2 are invariant subgroups in $G_1 \times G_2$ and are recoverable as factor groups with respect to the appropriate kernels.

(D) *Semidirect product* This is a more intricate way of combining G_1 and G_2, a direct product with a 'twist', in which G_1 and G_2 are not treated symmetrically. For each $a_1 \in G_1$, we need an automorphism τ_{a_1} of G_2 obeying certain conditions:

$$\begin{aligned} \tau_{a_1'}(\tau_{a_1}(a_2)) &= \tau_{a_1' a_1}(a_2), \\ \tau_{a_1^{-1}} &= \tau_{a_1}^{-1}. \end{aligned} \tag{1.27}$$

Then the group law for ordered pairs is:

$$G_1 \rtimes G_2 : (a_1, a_2)(b_1, b_2) = (a_1 b_1, a_2 \tau_{a_1}(b_2)). \tag{1.28}$$

It is instructive to verify the laws of group structure here; it is not as trivial as with the direct product. In a natural manner we do find that G_1 is a subgroup, G_2 is an invariant one, and the quotient $G_1 \rtimes G_2 / G_2 \simeq G_1$.

We will see several physically important examples of semidirect products, especially in Chapter 10.

1.5 Realisations and Representations of Groups

A realisation of a group G arises in the following way. We have a set X, and for each $g \in G$, a map $\phi_g : X \to X$ obeying the group and other laws:

$$\begin{aligned} &\text{(i)} \quad \text{each } \phi_g \text{ one-to-one, onto, invertible;} \\ &\text{(ii)} \quad \phi_e = \mathrm{Id}_X, \text{ identity map;} \\ &\text{(iii)} \quad \phi_{g'} \circ \phi_g = \phi_{g'g}, \text{ all } g', g \in G; \\ \text{so } &\text{(iv)} \quad \phi_{g^{-1}} = (\phi_g)^{-1}. \end{aligned} \tag{1.29}$$

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(For composition of maps, associativity is automatic; in a group in the abstract, it is explicitly postulated). Then we say we have a realisation of G by maps on X.

In the context of realisation of a group G on a set X as defined above, the following notions naturally arise:

$$\text{Orbit of } x \in X : \vartheta(x) = \{x' \in X \mid x' = \phi_g(x), \text{ some } g \in G\}$$

$$\text{Equivalence relation on } X : x \sim y \Leftrightarrow \phi_g(x) = y \text{ for some } g \in G$$

$$\text{Stability (Isotropy) group } H(x) \text{ of } x \in X : H(x) = \{g \in G \mid \phi_g(x) = x\}$$

$$\text{Fixed points } X_g \text{ of } g \in G : X_g = \{x \in X \mid \phi_g(x) = x \text{ for a fixed } g \in G\}$$

In classical Hamiltonian mechanics, groups usually act as canonical transformations on phase space. In quantum mechanics, the state space is a linear vector space, a Hilbert space, and we have the Superposition Principle. In this context, linear representations of groups become relevant. We will hereafter study only these.

1.6 Group Representations

These are particular cases of realisations, with added features. Given a group G and a (real or complex) linear vector space V, we have a representation D of G on V if the following hold:

$$\begin{aligned}
&\text{(i)} \quad \text{For each } g \in G,\ D(g) = \text{ invertible linear operator on } V;\\
&\text{(ii)} \quad D(e) = 1\!\!1 = \text{ identity or unit operator on } V;\\
&\text{(iii)} \quad D(g')D(g) = D(g'g),\ \text{all } g', g \in G;\\
\text{so }&\text{(iv)} \quad D(g^{-1}) = D(g)^{-1}.
\end{aligned} \tag{1.30}$$

The $D(g)$ are also called linear transformations on V. The representation is faithful if

$$g' \neq g \Rightarrow D(g') \neq D(g), \tag{1.31}$$

otherwise it is nonfaithful. The dimension of the representation D is that of V; it may be finite or infinite.

A group G generally has many representations, on various V, of various dimensions.

Given a realisation of a group G on a set X, not naturally endowed with a vector space structure, one can elevate it to a representation of G by defining a vector space $\mathbb{F}[X]$ over a field $\mathbb{F}$ consisting of all formal linear combinations $\sum_i c_i x_i$ of elements

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x_i of X with the coefficients c_i drawn from the field $\mathbb{F}$. A particularly interesting and important case arises when the set X is chosen to be G itself, which would be analysed in greater detail later in this chapter. Other instances where this device can be profitably employed are discussed in Chapter 2 in the context of the representations of the group S_n.

1.7 Equivalent Representations

Let D on V, D' on V' be two representations of G. (It may happen that $V' = V$ but $D' \neq D$.) We say they are *equivalent* if there is a linear invertible map $S : V \to V'$ such that

$$D'(g) = SD(g)S^{-1}, \text{ all } g \in G. \tag{1.32}$$

If no such S exists, they are *inequivalent*. We are often interested in representations determined up to equivalence.

For clarity we may assume that both V and V' are real vector spaces, or both complex ones. Cases where one is real and the other complex can be handled suitably.

1.8 Unitary/Orthogonal Cases – UR's

Let the complex (or real) space V of a representation of G carry a hermitian (or symmetric) inner product of vectors, written as (x, y) or $\langle x|y\rangle$ for $x, y \in V$. The representation $D(\,\cdot\,)$ of G on V is said to be unitary (or real orthogonal) if

$$(D(g)x, D(g)y) = (x, y), \text{ all } g \in G,\ x, y \in V. \tag{1.33}$$

Unitary representations are usually denoted as UR's. For the most part we will work with unitary group representations. Given a representation $D(\,\cdot\,)$ of G on V, it may happen that there is no choice of inner product on V such that (1.33) holds. Then $D(\,\cdot\,)$ is essentially nonunitary (or essentially nonorthogonal). For all finite groups as well as so-called continuous compact groups, every representation is equivalent to a unitary one. The Lorentz group $SO(3, 1)$ is an important case where this does not happen.

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1.9 Matrices of a Representation

Assume for simplicity that we are dealing with finite dimensional representations. Given $D(\cdot)$ of G on V, choose a basis $\{e_j\}$ in V, and for each $g \in G$ write

$$D(g)e_j = D^k{}_j(g)e_k, \text{ sum on } k. \tag{1.34}$$

The $n \times n$ matrices $(D^j{}_k(g))$, where n = dimension of V, j is a row index and k a column index, are the matrices of the representation in this basis. We simply write them too as $D(g)$. They obviously obey (1.30) in matrix form:

$$\begin{aligned} &\text{(i)}\ \ D^j{}_k(g_1 g_2) = D^j{}_l(g_1)D^l{}_k(g_2); \\ &\text{(ii)}\ \ D^j{}_k(e) = \delta^j{}_k; \\ &\text{(iii)}\ \ D^j{}_k(g^{-1}) = \text{elements of inverse of matrix } (D^j{}_k(g)). \end{aligned} \tag{1.35}$$

Any change of basis $\{e_j\} \to \{e'_j\}$ in V according to

$$e'_j = (S^{-1})^k{}_j e_k, \tag{1.36}$$

where S is a (real or complex as the nature of V) nonsingular $n \times n$ matrix leads to the relation (compare (1.32))

$$\text{matrix } D'(g) = S \left(\text{matrix } D(g)\right) S^{-1}, \tag{1.37}$$

so the representation matrices experience a similarity transformation.

If the representation $D(\cdot)$ on V is unitary or real orthogonal, and we use an orthonormal basis in V, then the representation matrices are themselves unitary or real orthogonal in the familiar senses.

1.10 Some Operations with Group Representations

We consider a few of these here. Picture a representation via its matrices $D(g)$, and do not assume unitarity or real orthogonality. We define three operations starting with $D(g)$:

$$\begin{aligned} \text{Contragredient representation: } & g \to (D(g)^T)^{-1} = D(g^{-1})^T; \\ \text{Adjoint representation: } & g \to (D(g)^\dagger)^{-1} = D(g^{-1})^\dagger; \\ \text{Complex conjugate representation: } & g \to D(g)^*. \end{aligned} \tag{1.38}$$

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In each case we clearly see that from the initial representation we have created a new one. The adjoint is the complex conjugate of the contragredient.

In case $D(g)$ is a UR, it is self adjoint, and the contragredient equals the complex conjugate. If $D(g)$ is real orthogonal, it is self contragredient.

Direct Sums and Products

Let D_1 on V_1 and D_2 on V_2 be two representations of G. Assume V_1 and V_2 are both complex or both real, but in general of different dimensions. Then the *direct sum* representation $D = D_1 \oplus D_2$ is a representation on $V = V_1 \oplus V_2$ defined in the natural way:

$$x_1 \in V_1,\ x_2 \in V_2 \to x = x_1 + x_2 \in V,$$
$$D(g)x = D_1(g)x_1 + D_2(g)x_2 \in V, \quad \text{all } g \in G. \tag{1.39}$$

The representation property is immediate, and the dimension is the sum of the individual ones.

The *direct product* representation $D = D_1 \times D_2$ acts on the tensor product space $V_1 \times V_2$: for product vectors,

$$x \in V_1,\ y \in V_2,\ g \in G:\ D(g)(x \times y) = D_1(g)x \times D_2(g)y. \tag{1.40}$$

This is then extended by linearity to general vectors in $V_1 \times V_2$.

1.11 Character of a Representation

This is an important concept which we will often come back to. It is most simply defined using the matrices of a representation in any basis. The character $\chi^{(D)}$ of a representation D of G on V, a function on G, is the trace of the representation matrices:

$$\chi^{(D)}(g) = D^j{}_j(g) = \mathrm{Tr}\ D(g). \tag{1.41}$$

By Eq. (1.37) this is actually basis independent. It is in general a complex valued function on the group, and constant over each conjugacy class:

$$\chi^{(D)}(g' g g'^{-1}) = \chi^{(D)}(g). \tag{1.42}$$

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So we say $\chi^{(D)}(g)$ is a class function. For a UR, $\chi(g^{-1}) = \chi(g)^*$. For sums and products of representations, the character behaves very simply:

$$\chi^{(D_1 \oplus D_2)}(g) = \chi^{(D_1)}(g) + \chi^{(D_2)}(g),$$
$$\chi^{(D_1 \times D_2)}(g) = \chi^{(D_1)}(g)\chi^{(D_2)}(g). \qquad (1.43)$$

We will soon see that the character of a representation determines the representation completely up to equivalence.

1.12 Invariant Subspaces, Reducibility, Irreducibility – UIR's

Given a representation D of G on V, it is *reducible* if there is a nontrivial subspace $V_1 \subset V$ *invariant* under G action:

$$x \in V_1,\ g \in G \Rightarrow D(g)x \in V_1. \qquad (1.44)$$

If no such V_1 exists, $D(\,\cdot\,)$ is an *irreducible* representation, or an irrep. If it is also unitary, we say it is an UIR – unitary irreducible representation.

The convenient abbreviations 'irreps', 'UR' and 'UIR' will be used throughout. In general we will keep in mind the complex case.

In the reducible case we can go further and ask: can we supplement V_1 with another (disjoint) subspace $V_2 \subset V$ which is also invariant and such that $V = V_1 \oplus V_2$? If we can, then $D(\,\cdot\,)$ is *decomposable*, otherwise it is *indecomposable*. So the arrangement of these ideas can be indicated in this way:

Irrep No nontrivial invariant subspace V_1

↗

Rep. D of G on V

↘

Reducible $V_1 \subset V$, nontrivial, invariant under D

↙ ↘

Decomposable *Indecomposable*

$V = V_1 \oplus V_2$, V_2 also invariant No such V_2

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In matrix form, in suitable bases, these cases mean:

$$D(g) \text{ reducible } \Leftrightarrow D(g) \text{ can be brought to the form } \begin{pmatrix} D_1(g) & \vdots & B(g) \\ \dots & \vdots & \dots \\ 0 & \vdots & D_2(g) \end{pmatrix};$$
$$D(g) \text{ reducible, decomposable } \Leftrightarrow \text{can choose basis so that } B(g) = 0;$$
$$D(g) \text{ reducible, indecomposable } \Leftrightarrow \text{always } B(g) \neq 0. \tag{1.45}$$

In the decomposable case we can 'look inside' V_1 and V_2 and try to repeat the process. For UR's, reducibility always implies decomposability, as we can simply take $V_2 = V_1^{\perp}$, the orthogonal complement of V_1. So by repeating this analysis we finally arrive at UIR's and can say: every UR is the direct sum of UIR's. This holds for all finite and all continuous compact groups. Thus in these cases, the basic building blocks for representation theory are the UIR's, which for these types of groups are also finite dimensional. Anticipating a little, we have: for a finite group, the number of inequivalent UIR's is finite; for a continuous compact group they are a denumerable infinity.

1.13 Schur's Lemma: Proof and Applications

This is a key result, a very powerful tool to deal with irreps, equivalences, etc. Let D on V, D' on V' be two given irreps of G, both assumed complex for definiteness. Suppose there exists, or we are able to somehow construct, a linear operator $T : V \to V'$ such that it 'intertwines' D and D' in the sense:

$$T\ D(g) = D'(g)T, \text{all } g \in G. \tag{1.46}$$

Then we can prove that either

$$\text{(i)} \quad D \text{ and } D' \text{ are inequivalent, and } T = 0;$$
$$\text{or (ii)} \quad T \text{ is nonsingular, so } D \text{ and } D' \text{ are equivalent.} \tag{1.47}$$

(We can leave aside the case where D and D' are equivalent, yet $T = 0$.)
We must appreciate that what *cannot happen* is $T \neq 0$, but T is singular, i.e., non invertible.

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Proof. Define subspaces in V and in V' as follows:

$$\mathcal{N}(T) = \text{null space of } T = \text{subspace of } V \text{ on which } T \text{ vanishes;}$$
$$\mathcal{R}(T) = \text{range of } T = \text{subspace of } V' = \text{image of } V \text{ under } T; \tag{1.48}$$

i.e.,

$$T\mathcal{N}(T) = 0, \;\; \mathcal{N}(T) \subset V; \;\; \mathcal{R}(T) = T(V) \subset V'. \tag{1.49}$$

(Again we can leave aside the case $T = 0, \mathcal{N}(T) = V$.) Then from Eq. (1.46) we easily find: $\mathcal{N}(T)$ is invariant under $D(G)$, $\mathcal{R}(T)$ under $D'(g)$. Therefore the irreducibility of both D and D' implies: $\mathcal{N}(T) = 0$, so T is one-to-one; $\mathcal{R}(T) = V'$, so T is onto. (Again we can discard $\mathcal{R}(T) = 0$ as then $T = 0$.) Then if $T \neq 0$, T^{-1} exists; and the two representations, both irreducible, are equivalent.

For a first application of the Lemma, we consider a single irrep, and take $V' = V$, $D' = D$. Then suppose for some T, a map $V \to V$, we find:

$$TD(g) = D(g)T, \;\; \text{all } g \in G. \tag{1.50}$$

Then form $T - \lambda \cdot 1\!\!1$ where λ is (any) eigenvalue of T. Since $T - \lambda \cdot 1\!\!1$ is singular and also obeys (1.50), it must vanish, so we conclude:

$$D(g) \text{ an irrep of } G, \;\; TD(g) = D(g)T, \text{ all } g \Rightarrow T = \lambda \cdot \, 1\!\!1, \text{ some } \lambda. \tag{1.51}$$

The converse can also be easily shown: if only multiples of $1\!\!1$ commute with $D(g)$ for all g, $D(\,\cdot\,)$ must be an irrep.

The next application of the Lemma reveals the inner structure of a reducible representation, and the sense in which the reduction is unique. □

'Uniqueness' of Complete reduction

Let a representation $D(\cdot)$ of G on V of finite dimension be fully reducible into irreps, so $D(\cdot)$ is a direct sum of these. Imagine two reductions of V into irreducible invariant subspaces with corresponding irreps, namely,

$$V = \sum_{\rho=1,2,\cdots} \oplus \; \mathcal{L}_\rho, \quad \mathcal{L}_\rho \text{ carrying irrep } D^{(\rho)},$$
$$= \sum_{\sigma=1,2,\cdots} \oplus \; \mathcal{M}_\sigma, \quad \mathcal{M}_\sigma \text{ carrying irrep } D^{(\sigma)}. \tag{1.52}$$

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We want to examine the extent to which these two decompositions can differ, and what they must hold in common. Any vector $x_\rho \in \mathcal{L}_\rho$ can be expanded uniquely in terms of components in $\mathcal{M}_\sigma$:

$$x_\rho \in \mathcal{L}_\rho\colon x_\rho = \sum_\sigma y_\sigma, \quad y_\sigma \in \mathcal{M}_\sigma. \tag{1.53}$$

For each pair $\sigma\rho$, define the linear map $T_{\sigma\rho} : \mathcal{L}_\rho \to \mathcal{M}_\sigma$ by

$$T_{\sigma\rho} x_\rho = y_\sigma \tag{1.54}$$

as determined by (1.53). (Here there is no sum on ρ.) Apply $D(g)$ to both sides to get:

$$\begin{aligned} D(g)x_\rho \equiv D^{(\rho)}(g)x_\rho &= D(g)\sum_\sigma y_\sigma \\ &= \sum_\sigma D(g)y_\sigma = \sum_\sigma D^{(\sigma)}(g)y_\sigma, \\ \text{i.e.,}\ \ D^{(\sigma)}(g)y_\sigma &= T_{\sigma\rho}D^{(\rho)}(g)x_\rho, \\ \text{i.e.,}\ \ D^{(\sigma)}(g)T_{\sigma\rho}x_\rho &= T_{\sigma\rho}D^{(\rho)}(g)x_\rho, \text{ any } x_\rho \in \mathcal{L}_\rho. \end{aligned} \tag{1.55}$$

Therefore for each pair $\sigma\rho$ we have an intertwining relation

$$T_{\sigma\rho}D^{(\rho)}(g) = D^{(\sigma)}(g)T_{\sigma\rho}, \tag{1.56}$$

as in Eq. (1.46). We can apply Schur's Lemma to conclude:

$$\begin{aligned} &\text{(i)}\ \ T_{\sigma\rho} = 0 \ \text{ if } \ D^{(\rho)}, D^{(\sigma)} \text{ are not equivalent;} \\ &\text{(ii)}\ \ T_{\sigma\rho} \neq 0 \ \text{ only if they are equivalent.} \end{aligned} \tag{1.57}$$

We can now draw out the consequences. For each $\mathcal{L}_\rho$ carrying the irrep $D^{(\rho)}$, pick out all those $\mathcal{M}_\sigma$ such that $D^{(\sigma)}$ is equivalent to $D^{(\rho)}$. Then $x_\rho \in \mathcal{L}_\rho \Rightarrow y_\sigma \neq 0$ only in these $\mathcal{M}_\sigma; x_\rho \neq 0 \Rightarrow$ at least one such $y_\sigma \neq 0$, and at least one $T_{\sigma\rho} \neq 0$. Then by the Lemma, this $T_{\sigma\rho}$ is nonsingular.

So for each irrep $D^{(\rho)}$ in the first reduction, there is at least one equivalent irrep $D^{(\sigma)}$ in the second reduction, *and conversely*. Now we can examine the question of multiplicities, the number of times a given irrep. repeats itself in a complete reduction. Several subspaces $\mathcal{L}_\rho$ (similarly $\mathcal{M}_\sigma$) may carry equivalent irreps. Let us label irreps

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(up to equivalence) by $\alpha, \beta, \cdots$, and multiple appearances by $j, k, \cdots$; so we write

$$\mathcal{L}_\rho \equiv \mathcal{L}_{\alpha,j}, \quad \mathcal{M}_\sigma \equiv \mathcal{M}_{\beta,k}. \tag{1.58}$$

Now 'combine' subspaces according to 'representation type':

$$\begin{aligned} V &= \sum_\rho \oplus \mathcal{L}_\rho = \sum_\alpha \oplus \left(\sum_j \oplus \mathcal{L}_{\alpha,j} \right) = \sum_\alpha \oplus \mathcal{L}^{(\alpha)}, \\ \mathcal{L}^{(\alpha)} &= \sum_j \oplus \mathcal{L}_{\alpha,j}; \\ V &= \sum_\sigma \oplus \mathcal{M}_\sigma = \sum_\beta \oplus \left(\sum_k \oplus \mathcal{M}_{\beta,k} \right) = \sum_\beta \oplus \mathcal{M}^{(\beta)}, \\ \mathcal{M}^{(\beta)} &= \sum_k \oplus \mathcal{M}_{\beta,k}. \end{aligned} \tag{1.59}$$

Any $x \in \mathcal{L}_{\alpha,j}$ has components only in $\mathcal{M}_{\alpha,k}$ and vice versa. So

$$\begin{aligned} T_{\alpha j,\beta k} &= 0 \ \text{ if } \ \alpha \neq \beta; \\ x \in \mathcal{L}^{(\alpha)} &\Leftrightarrow x \in \mathcal{M}^{(\alpha)}, \quad \mathcal{L}^{(\alpha)} = \mathcal{M}^{(\alpha)}. \end{aligned} \tag{1.60}$$

We see that the irreps present in D on V and their multiplicities are unique, so are the subspaces $\mathcal{L}^{(\alpha)}$. Only the further break up of each $\mathcal{L}^{(\alpha)}$ into $\mathcal{L}_{\alpha,j}$ is arbitrary.

The Orthogonality Relations

This is our third and last application of Schur's Lemma. It leads to important properties of the matrix elements of the representation matrices in the various irreducible representations.

Consider a complex-valued function $f(g)$ on G, viewed as an element in a complex vector space of dimension $|G|$, which we assume is finite. (More generally we could consider $f(\cdot)$ belonging to some linear space of some dimension.) For any given $f(\cdot)$ we define its mean value by an averaging process over G:

$$\mathcal{M}(f) \equiv \mathcal{M}_g(f(g)) = \frac{1}{|G|} \sum_{g \in G} f(g). \tag{1.61}$$

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The main properties which are evident upon inspection are linearity, nonnegativity, normalisation and 'translation' invariances:

$$\mathcal{M}(f_1 + f_2) = \mathcal{M}(f_1) + \mathcal{M}(f_2),\ \mathcal{M}(\lambda f) = \lambda \mathcal{M}(f);$$
$$f \geq 0 \Rightarrow \mathcal{M}(f) \geq 0;\ f = 1 \Rightarrow \mathcal{M}(f) = 1,$$
$$\mathcal{M}_g(f(g)) = \mathcal{M}_g(f(g_0 g) \text{ or } f(g g_0) \text{ or } f(g^{-1})),\ g_0 \text{ fixed.} \quad (1.62)$$

Now take two irreps $D(g)$ on V, $D'(g)$ on V'. Let $S : V \to V'$ be any linear map, and consider

$$T_S = \mathcal{M}_g(D'(g^{-1})SD(g)) = \text{linear map } V \to V'. \quad (1.63)$$

(So here the quantity being averaged is not a complex valued function but something belonging to a linear space, namely, the space of all linear maps $V \to V'$.) Using Eq. (1.62) we see that it obeys:

$$D'(g'^{-1})T_S = \mathcal{M}_g\Big(D'(g'^{-1})D'(g^{-1})SD(g)\Big) = T_S D(g'^{-1}),$$
$$\text{i.e.,}\ \ D'(g)T_S = T_S D(g), \text{any}\ \ g \in G. \quad (1.64)$$

Thus T_S intertwines the two irreps, and we can use Schur's Lemma:

$$D' \text{ and } D \text{ inequivalent } \Rightarrow\ T_S = 0,\ \text{any } S;$$
$$D' \text{ and } D \text{ equivalent, the same } \Rightarrow\ T_S = \lambda(S)\mathbb{1}, \text{some } \lambda(S). \quad (1.65)$$

So for the case of inequivalent irreps, in terms of the matrices in any basis, as S is arbitrary, we get:

$$\mathcal{M}_g(D'_{j'k'}(g^{-1})D_{jk}(g)) = 0,\ \text{all } j\ k\ j'\ k'. \quad (1.66)$$

In case these matrices are unitary, this reads:

$$\mathcal{M}_g(D'_{j'k'}(g)^* D_{jk}(g)) = 0,\ \text{all } j\ k\ j'\ k'. \quad (1.67)$$

For equivalent irreps, we assume $D' = D$ as in the second line of Eq. (1.65), and get:

$$\mathcal{M}_g(D_{jk}(g^{-1})S_{kl}D_{lm}(g)) = \lambda(S)\delta_{jm},\ \text{sum on all } k, l. \quad (1.68)$$

Setting $j = m$ and summing on j gives $\lambda(S) = \text{Tr}\, S/\dim\, V$, so (1.68) becomes

$$\mathcal{M}_g(D_{jk}(g^{-1})D_{lm}(g)) = \delta_{jm}\delta_{kl}/\dim\, V. \quad (1.69)$$

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And in the unitary case we have

$$\mathcal{M}_g(D_{jk}(g)^* D_{lm}(g)) = \delta_{jl}\delta_{km}/\dim V. \tag{1.70}$$

Incidentally, the last two equations imply that in any irrep, in any chosen basis, any *fixed* matrix element cannot vanish for all g.

We can regard the space of complex functions $f(g)$ on G as a Hilbert space $L^2(G)$ of dimension $|G|$, and we use $\mathcal{M}$ to define the inner product:

$$(f_1, f_2) = \mathcal{M}_g(f_1(g)^* f_2(g)). \tag{1.71}$$

At the same time, let us label the various inequivalent UIR's by $\alpha, \beta, \cdots$ as in Eq. (1.58), with dimensions $N_\alpha, N_\beta, \cdots$. Then Eqs. (1.67, 1.70) lead to the orthogonality relations

$$\left(D^{(\alpha)}_{jk}(\,\cdot\,)\,,\; D^{(\beta)}_{lm}(\,\cdot\,)\right) = \delta_{\alpha\beta}\delta_{jl}\delta_{km}/N_\alpha. \tag{1.72}$$

Each fixed matrix element in each UIR thus gives one nonzero vector in $L^2(G)$, mutually orthogonal if the UIR's are inequivalent or the row or column indices differ. This leads to the inequality

$$\sum_\alpha N_\alpha^2 \leq |G|. \tag{1.73}$$

From Eq. (1.72) we get important relations for the characters $\chi^{(\alpha)}(g)$ of the various UIR's: summing over $j = k$ and $l = m$,

$$\left(\chi^{(\alpha)}(\,\cdot\,),\; \chi^{(\beta)}(\,\cdot\,)\right) = \delta_{\alpha\beta}. \tag{1.74}$$

Now the χ's are class functions, constant over each class. In general such functions on G form a linear subspace in $L^2(G)$, and within this Eq. (1.74) means the $\chi^{(\alpha)}(g)$ are an orthonormal set. This gives another inequality to accompany (1.73):

$$\text{Number of inequivalent irreps or UIR}'s \leq \text{ number of equivalence classes in } G \tag{1.75}$$

Let the character of a general UR $D(\,\cdot\,)$ be written as $\chi(g)$, and upon reduction let $D(\,\cdot\,)$ contain $D^{(\alpha)}(\,\cdot\,)$ with multiplicity ν_α. Then we easily find from Eq. (1.74) and

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the definition of $\chi(g)$:

$$\chi(g) = \sum_\alpha \nu_\alpha \chi^{(\alpha)}(g),$$
$$\nu_\alpha = \text{multiplicity of occurrence of } D^{(\alpha)} \text{ in } D = (\chi^{(\alpha)}, \chi),$$
$$(\chi, \chi) = \sum_\alpha \nu_\alpha^2 \geq 1. \tag{1.76}$$

Thus the UR $D(\cdot)$ with character $\chi(g)$ is irreducible or reducible according as the squared norm (χ, χ) of χ is or exceeds unity.

It was mentioned earlier that for any finite group (as well as for any continuous compact group) every representation may be assumed without loss of generality to be unitary. We leave it as an exercise to prove this (in the finite group case) using the idea of averaging over G as introduced in Eq. (1.61).

The Regular Representations of G

As the last but one item in this chapter, we explicitly construct two special UR's of G, each of which contains all UIR's, and which show that both inequalities (1.73, 1.75) are equalities. These are the two regular representations, both acting on $L^2(G)$. For finite G, the space $L^2(G)$ is finite dimensional, with inner product (1.71). For now we consider this case. Later for continuous G we will use an integral version of this inner product, possessing invariances as in the last line of (1.62).

We write $L(g), R(g)$ or L_g, R_g for the operators on $L^2(G)$ corresponding to the left and right regular representations of G. On a general complex function $f(g)$ on G, their actions are:

$$(L_{g'}f)(g) = f(g'^{-1}g), \ (R_{g'}f)(g) = f(gg'). \tag{1.77}$$

It is easy to check that these are unitary operators, and they give two mutually commuting UR's of G:

$$L_{g_1} L_{g_2} = L_{g_1 g_2}, \ R_{g_1} R_{g_2} = R_{g_1 g_2}, \ L_{g_1} R_{g_2} = R_{g_2} L_{g_1}. \tag{1.78}$$

If we introduce Dirac notation with kets and bras according to

$$f \in L^2(G) \ : \ f(g) = \langle g|f\rangle, \ \ \langle g'|g\rangle = \delta_{g',g}, \tag{1.79}$$

then the actions of $L_{g'}$, $R_{g'}$ on these basis kets are:

$$L_{g'}|g\rangle = |g'g\rangle, \ \ R_{g'}|g\rangle = |gg'^{-1}\rangle. \tag{1.80}$$

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As these are UR's, they are fully reducible, and their characters determine the contents. From Eq. (1.80) we obtain:

$$\chi^{(\text{left reg})}(g) = \text{Tr}\, L_g = \chi^{(\text{right reg})}(g) = \text{Tr}\, R_g = |G|\delta_{g,e}. \tag{1.81}$$

Then we ask: how often does each UIR $D^{(\alpha)}(\,\cdot\,)$ occur in each of these UR's? From Eq. (1.76) we get the answer:

$$\nu_\alpha = \left(\chi^{(\alpha)}, \chi^{(\text{left or right reg})}\right) = \frac{1}{|G|}\sum_g \chi^{(\alpha)}(g)^* \chi^{(\cdots)}(g) = \chi^{(\alpha)}(e)^* = N_\alpha. \tag{1.82}$$

Thus every UIR α is present as often as its dimension N_α. This allows us to strengthen (1.73) to an equality,

$$\sum_\alpha N_\alpha^2 = |G|. \tag{1.83}$$

It is important to observe that even though at this point the only UR's that have been constructed are the regular ones, and the various UIR's are as yet 'unknown', we have the information that each of the latter is definitely contained in (each of) the former.

From Eqs. (1.72, 1.83) we have that the 'normalised' matrix elements $\{N_\alpha^{1/2} D_{jk}^{(\alpha)}(g)\}$ form an orthonormal basis for $L^2(G)$. Any $f(g)$ thus has a unique expansion:

$$\begin{aligned}
f(g) &= \sum_\alpha \sum_{jk} N_\alpha^{1/2} D_{jk}^{(\alpha)}(g) f_{jk}^{(\alpha)}, \\
f_{jk}^{(\alpha)} &= N_\alpha^{1/2} \left(D_{jk}^{(\alpha)}(\,\cdot\,)\,,\, f(\,\cdot\,)\right), \\
\|f\|^2 &= \frac{1}{|G|}\sum_g |f(g)|^2 = \sum_\alpha \sum_{jk} |f_{jk}^{(\alpha)}|^2.
\end{aligned} \tag{1.84}$$

In this basis, $L_{g'}$-action alters only the first index j, while $R_{g'}$-action alters only k: This is consistent with their mutually commuting. For the former action, k is a multiplicity index; for the latter, j plays this role.

Since the collection $\{N_\alpha^{1/2} D_{jk}^{(\alpha)}(g)\}$ is both orthonormal and complete, we see that any class function can be expanded in terms of the characters of the UIR's:

$$\begin{aligned}
&f(gg'g^{-1}) = f(g'), \text{all } g \text{ and } g' \Rightarrow \\
&\qquad f(g) = \sum_\alpha f_\alpha \chi^{(\alpha)}(g),\; f_\alpha = (\chi^{(\alpha)}(\,\cdot\,), f(\,\cdot\,)).
\end{aligned} \tag{1.85}$$

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As a result, (1.75) too can be strengthened to an equality:

$$\text{Number of inequivalent irreps or UIR's} = \text{number of equivalence classes in } G \tag{1.86}$$

Complex Conjugation, Direct Products, of UIR's

Given G, we know that in principle all the UIR's $D^{(\alpha)}(\,\cdot\,)$ can be 'extracted' from the regular representations. We know how many there are, and have some information, (1.83), on their dimensions.

Now suppose all the $D^{(\alpha)}(\cdot)$ have been constructed, each up to unitary equivalence. From (1.76) in the irreducible case, we see that $D^{(\alpha)^*}(\,\cdot\,)$, the complex conjugate of $D^{(\alpha)}(\,\cdot\,)$, is also a UIR. (This is one of the three operations listed in Eq. (1.38).) By judicious use of Schur's Lemma, we can easily find out the qualitatively different cases that can arise. We describe them briefly, omitting the details of derivations.

To begin with, and this is obvious,

$$\begin{aligned} \chi^{(\alpha)}(g)^* \neq \chi^{(\alpha)}(g) &\Leftrightarrow D^{(\alpha)}(\,\cdot\,)^*,\ D^{(\alpha)}(\,\cdot\,) \text{ are inequivalent UIR's;} \\ \chi^{(\alpha)}(g) \text{ real } &\Leftrightarrow D^{\alpha}(\,\cdot\,)^*,\ D^{(\alpha)}(\,\cdot\,) \text{ are equivalent UIR's.} \end{aligned} \tag{1.87}$$

In the latter case one can ask whether the matrices $D^{(\alpha)}(g)$ can all be made real by a suitable choice of orthonormal basis in the representation space. Here one finds the results:

$$\begin{aligned} &\chi^{(\alpha)}(g) \text{ real} \Leftrightarrow D^{(\alpha)}(g)^* = CD^{(\alpha)}(g)C^{-1}, \\ &C^\dagger C = 1\!\!1, C^T = \lambda C,\ \lambda = \pm 1; \\ &\quad \lambda = +1 \Leftrightarrow D^{(\alpha)}(g)^* = D^{(\alpha)}(g) \text{ in suitable basis – real case;} \\ &\quad \lambda = -1 \Leftrightarrow D^{(\alpha)}(g)^* \neq D^{(\alpha)}(g) \text{ in any basis – pseudoreal case.} \end{aligned} \tag{1.88}$$

(In the last case, of course, we mean that there are some g for which $D^{(\alpha)}(g)$ is complex.) Thus the three possibilities are – complex, pseudoreal, and real. There is a nice criterion in terms of $\chi^{(\alpha)}(\,\cdot\,)$ to directly distinguish between the last two, due to Wigner, but we omit the details.

As a final item in representation theory we consider direct products of UIR's, the Clebsch–Gordan problem. For any pair α, β we have a direct product UR (1.40)

$$\begin{aligned} D(g) &= D^{(\alpha)}(g) \times D^{(\beta)}(g), \quad \text{dimension} \quad N_\alpha N_\beta, \\ \text{Character} \quad \chi(g) &= \chi^{(\alpha)}(g)\chi^{(\beta)}(g). \end{aligned} \tag{1.89}$$

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In the complete reduction of this product into UIR's, each $D^{(\gamma)}$ occurs with some multiplicity:

$$D^{(\alpha)} \times D^{(\beta)} = \sum_{\gamma} \oplus \, \nu_{\alpha\beta,\gamma} D^{(\gamma)}, \quad \nu_{\alpha\beta,\gamma} = \text{integer} \geq 0, \tag{1.90}$$

where by Eq. (1.76)

$$\nu_{\alpha\beta,\gamma} = (\chi^{(\gamma)}, \ \chi^{(\alpha)}\chi^{(\beta)}). \tag{1.91}$$

The series on the right in (1.90) is called the Clebsch–Gordan series, and the change of basis in the space of the product representation $D^{(\alpha)} \times D^{(\beta)}$ to effect this block diagonalisation involves Clebsch–Gordan coefficients which are basis dependent.

For an abelian group G, every UIR is one dimensional, every element is a class by itself, and UIR's are usually called 'characters'. If G is nonabelian, at least some UIR's are multi dimensional.

1.14 Group Algebra

Given a finite group G and a field $\mathbb{F}$, we can construct out of them a new set, denoted by $\mathbb{F}[G]$, consisting of all formal linear combinations of the form

$$a = \sum_{i} a_i x_i, a_i \in F, x_i \in G. \tag{1.92}$$

The set $\mathbb{F}[G]$ can be viewed in several different ways:

1. As a vector space over $\mathbb{F}$: elements of $\mathbb{F}[G]$ can be added and multiplied by elements of $\mathbb{F}$ in a natural way. This is the vector space which underlies the regular representation of G as stated earlier.
2. As a unital ring: elements of $\mathbb{F}[G]$ can be added and (using the composition law of G) multiplied together to yield elements of $\mathbb{F}[G]$, with the identity element $1.e_G$ of G serving as the multiplicative unit of the ring. If G is abelian (nonabelian) $\mathbb{F}[G]$ is a commutative (noncommutative) ring.
   With the identification of $c \in \mathbb{F}$ with $ce_G \in \mathbb{F}[G]$ one can consider $\mathbb{F}$ as sitting in $\mathbb{F}[G]$.
3. As an algebra over $\mathbb{F}$: in addition to being a ring, $\mathbb{F}[G]$ is also closed under multiplication by elements of $\mathbb{F}$.

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1.15 Representations of G and Its Group Algebra $\mathbb{F}[G]$

Recall that by a linear representation of a group G on a vector space V over a field $\mathbb{F}$ we mean a collection of linear operators $\{D(g)\}$ on V respecting the composition law in G. Given a representation $\{D(g)\}$, we can define multiplication of elements v of the vector space by elements a of $\mathbb{F}[G]$ to obtain other elements v' of V as follows:

$$\begin{aligned} a.v &= (\sum_i a_i x_i).v \\ &\rightarrow \sum_i a_i D(x_i) v = \sum_i a_i v_i \in V. \end{aligned} \tag{1.93}$$

Here v_i are the vectors in V to which v is mapped by $D(x_i)$. We are thus led to a new mathematical structure – a module M over the ring $\mathbb{F}[G]$. (The notion of a module is same as that of a vector space with the field replaced by a ring.)

With this at hand, all aspects of the representation theory of finite groups can be couched in the language of modules – a subspace V_1 of the vector space V invariant under all of $\{D(g)\}$ corresponds to a submodule M_1 of M, an irreducible representation of G corresponds to a simple or irreducible submodule of M and so on. A crucial advantage of this language for studying group representations lies in the fact that constructing submodules of M (and hence subrepresentations of G) can be shown to reduce to the task of finding idempotent (essentially idempotent) elements in the ring $\mathbb{F}[G]$ – elements a in $\mathbb{F}[G]$ such that $a^2 = a$ ($a^2 = ca, c \in \mathbb{F}$). Further, constructing irreducible submodules (and hence irreducible representations of G) can be shown to reduce to the task of finding idempotents in $\mathbb{F}[G]$ which are primitive, i.e., those idempotents in $\mathbb{F}[G]$ which can not be expressed as a sum of two 'orthogonal' idempotents. (Two idempotents a and b in $\mathbb{F}[G]$ are said to be orthogonal provided $ab = ba = 0$.)

We shall put this machinery to use in Chapter 2 in the context of the irreducible representations of the symmetric group. Note also that the considerations here apply to any field $\mathbb{F}$. However, later in this book we will exclusively be concerned with the case when $\mathbb{F}$ is the complex field $\mathbb{C}$.

Problems

P1.1 Given an associative group composition law, and left inverses and left identity only, i.e.,

(i) unique e such that $ea = a$ for all a,
(ii) for each a, unique a^{-1} such that $a^{-1}a = e$;

derive all the results in Eq. $(1.1(iii), (iv))$.

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P1.2 For a finite group of prime order, show that there are no nontrivial subgroups.

P1.3 If H is an invariant subgroup of G such that G/H is abelian, show that the commutator subgroup Q of G is a subgroup of H as well.

P1.4 The group of translations in one real dimension, $x \to x + a$, is abelian. Show that

$$a \to \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

is a representation, and find out whether it is reducible or irreducible, decomposable or indecomposable.

P1.5 For a finite group G acting on a set X, show that

(i) If $x \in X$ and $y \in X$ lie on the same orbit then their stability subgroups are conjugate subgroups in G, i.e., $H(y) = gH(x)g^{-1}$, some $g \in G$,

(ii) For any $x \in X$, there is a natural bijective map from $\vartheta(x)$ to the coset space $G/H(x)$ and hence $|\vartheta(x)| = |G/H(x)|$,

(iii) If the set X is decomposed into disjoint orbits using the equivalence relation noted in Section 1.5 then one has

$$\text{Number of orbits} = \frac{1}{|G|} \sum_{g \in G} |X_g| \quad \text{(Burnside's Lemma)}.$$

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