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RIESZ DECOMPOSITIONS

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1. Introduction. All functions mentioned in this paper will be real-valued. If f_1, f_2, g are nonnegative functions on a set S that satisfy $g \leqq f_1 + f_2$, the *Riesz decomposition problem* associated with these data is to find functions g_i on S such that

$$(1) \qquad g = g_1 + g_2 \quad \text{and} \quad 0 \leqq g_i \leqq f_i \qquad (i = 1, 2).$$

The formula

$$(2) \qquad g_i = \begin{cases} f_i g/(f_1 + f_2) & \text{where } f_1 + f_2 > 0 \\ 0 & \text{where } f_1 + f_2 = 0 \end{cases} \qquad (i = 1, 2)$$

always furnishes a solution. The problem becomes more interesting if one asks under what conditions one can find solutions that are, roughly speaking, as smooth as the data.

To state the resulting problems concisely, we introduce some definitions; the abbreviation RD will stand for *Riesz decomposition.*

Definition 1. [**1**; **3**, p. 27] A collection Y of real functions on a set S will be called an *RD-space* if every RD-problem with data in Y has solutions in Y.

For example, (2) shows that the following are RD-spaces:

(i) The space of all bounded functions on any set S.

(ii) The space of all real-analytic functions on the line $\mathbf{R}^1$ [**5**, p. 122].

(iii) The space of all continuous functions on any topological space S. (This fact is useful in the proof of the Riesz representation theorem; see [**4**, Theorem 6.19].)

Definition 2. For $n = 1, 2, 3, \ldots$, $C = C(\mathbf{R}^n)$ is the space of all continuous real functions on the Euclidean space $\mathbf{R}^n$.

For $k = 1, 2, 3, \ldots$, $C^k = C^k(\mathbf{R}^n)$ is the space of all f whose kth-order partial derivatives are in C. Also, $C^0 = C$.

$D^k = D^k(\mathbf{R}^n)$ consists of all $f \in C^{k-1}$ whose kth-order partial derivatives exist at all points of $\mathbf{R}^n$.

$B^k = B^k(\mathbf{R}^n)$ consists of all $f \in D^k$ whose kth-order partial derivatives are locally bounded (i.e., are bounded on every compact subset of $\mathbf{R}^n$).

On every $\mathbf{R}^n$ we thus have the following chain of spaces:

$$D^1 \supset B^1 \supset C^1 \supset D^2 \supset B^2 \supset C^2 \supset D^3 \supset B^3 \supset \ldots \supset C^\infty.$$

Which of these are RD-*spaces*?

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THEOREM 1. *If $n = 1$, then D^1, B^1, C^1, D^2, B^2, C^2, and D^3 are* RD-*spaces.*

THEOREM 2. *If $n > 1$, then D^1, B^1, C^1, and B^2 are RD-spaces.*

Note that D^2 is skipped in Theorem 2. The following examples show that our theorems are sharp:

Example 1. *There is an RD-problem on $\mathbf{R}^1$, with C^∞-data, which has no B^3-solution.*

Example 2. *There is an RD-problem on $\mathbf{R}^2$, whose data are quadratic polynomials, but which has no C^2-solution.*

Example 3. *There is an RD-problem on $\mathbf{R}^2$, with D^2-data, which has no D^2-solution.*

Theorems 1 and 2 will be proved by showing that the formula (2) gives a solution in Y if the data are in Y and if Y is any one of the spaces mentioned in the theorems.

In an appendix we show that the RD-problem can be restated in lattice-theoretic terms. This equivalence was pointed out to us by Creighton Buck.

2. Some Lemmas. The following lemmas describe the behavior of non-negative differentiable functions on the line at points where they are 0.

LEMMA 1. *Suppose $f \geqq 0$ on $\mathbf{R}^1$, $x_0 \in \mathbf{R}^1$, and $f(x_0) = 0$.*

(a) *If $f \in D^1$ then $f'(x_0) = 0$.*

(b) *If $f \in D^2$ and $f''(x_0) \neq 0$ then $f''(x_0) > 0$ and x_0 is an isolated point of the zero-set of f.*

(c) *If $f \in D^2$, $x \in \mathbf{R}^1$, and*

$$M(x) = M(f, x_0, x) = \sup\{|f''(\xi)| : |\xi - x_0| \leqq 2|x - x_0|\}$$

then $|f'(x)|^2 \leqq 2f(x)M(x)$.

(d) *If $f \in D^3$ and $f''(x_0) = 0$ then also $f'''(x_0) = 0$.*

Part (c) is in [**2**]; we include its proof for the sake of completeness.

Proof. (a) This is a triviality.

(b) To every $x \neq x_0$ corresponds a ξ between x and x_0 such that

$$(3) \qquad f(x) = f'(\xi)(x - x_0) = \frac{f'(\xi)}{\xi - x_0} \cdot (\xi - x_0)(x - x_0).$$

Since $(\xi - x_0)(x - x_0) > 0$ and $f'(\xi)/(\xi - x_0) \to f''(x_0)$ as $x \to x_0$, (b) follows from (3).

(c) Fix $x \neq x_0$, and assume $f'(x) \neq 0$ (otherwise there is nothing to prove). By (a) and the mean value theorem, $|f'(x)| \leqq M(x)|x - x_0|$. Hence there exists t, $|t| \leqq |x - x_0|$, so that $f'(x) = tM(x)$. To this t corresponds a ξ

between x and $x - t$ such that

$$0 \leqq f(x-t) = f(x) - f'(x)t + \tfrac{1}{2}f''(\xi)t^2$$
$$\leqq f(x) - f'(x)t + \tfrac{1}{2}M(x)t^2 = f(x) - [f'(x)]^2/2M(x).$$

This proves (c).

(d) To every $x \neq x_0$ corresponds a ξ between x and x_0 such that $0 \leqq f(x) = \frac{1}{2}f''(\xi)(x-x_0)^2$. Hence $f''(\xi) \geqq 0$ for values of ξ that are arbitrarily close to x_0, on either side. Since $f'''(x_0)$ is assumed to exist, it follows that $f'''(x_0) = 0$.

LEMMA 2. *Suppose* $0 \leqq \alpha \leqq f$ *on* $\mathbf{R}^1$, Ω *is the set on which* $f(x) > 0$, *and* x_0 *is a boundary point of* Ω. *Define*

$$(4) \qquad u(x) = \frac{\alpha(x)}{f(x)} \quad (x \in \Omega).$$

(a) *If* $f \in D^2$, $\alpha \in D^2$, *then* $\alpha''(x_0) \leqq f''(x_0)$.
(b) *If* $f \in D^2$, $\alpha \in D^2$, *and* $f''(x_0) > 0$ *then*

$$(5) \qquad \lim_{x \to x_0} u(x) = \frac{\alpha''(x_0)}{f''(x_0)}$$

and

$$(6) \qquad \lim_{x \to x_0} [u'(x)]^2 f(x) = 0.$$

(c) *If* $f \in B^2$, $\alpha \in B^2$, *then* $u'^2 f$ *is bounded on every bounded subset of* Ω.
(d) *If* $f \in C^2$, $\alpha \in C^2$, *then* (6) *holds.*
(e) *If* $f \in D^3$, $\alpha \in D^3$, *then*

$$(7) \qquad \lim_{x \to x_0} \frac{[u'(x)]^2 f(x)}{x - x_0} = 0.$$

(In (5), (6), and (7), x is of course confined to Ω.)

Proof. If $f \in D^2$ and $x = x_0 + \delta$, the Taylor formulas

$$(8) \qquad f'(x) = f''(x_0)\delta + o(\delta), \qquad f(x) = \tfrac{1}{2}f''(x_0)\delta^2 + o(\delta^2)$$

hold. Analogous formulas hold for α' and α. They yield (a) and (b).

(c) In Ω, $|fu'| = |\alpha' - f'u| \leqq |\alpha'| + |f'|$. If this is combined with Lemma 1(c) one obtains

$$(9) \qquad [u'(x)]^2 f(x) \leqq 4M(\alpha, x_0, x) + 4M(f, x_0, x) \qquad (x \in \Omega).$$

This proves (c).

(d) Because of (b) we may assume that $f''(x_0) = 0$. Then $\alpha''(x_0) = 0$, by (a). Since f'' and α'' are assumed to be continuous, the right side of (9) tends to 0 as $x \to x_0$. This proves (d).

(e) If $x = x_0 + \delta$, we now have the Taylor formulas

$$(10) \qquad \begin{aligned} f''(x) &= f''(x_0) + f'''(x_0)\delta + o(\delta) \\ f'(x) &= f''(x_0)\delta + \tfrac{1}{2}f'''(x_0)\delta^2 + o(\delta^2) \\ f(x) &= \tfrac{1}{2}f''(x_0)\delta^2 + \tfrac{1}{6}f'''(x_0)\delta^3 + o(\delta^3) \end{aligned}$$

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and their analogues for α'', α', α. If $f''(x_0) > 0$, these formulas establish (7). If $f''(x_0) = 0$, then also $\alpha''(x_0) = 0$, by (a), and Lemma 1(d) implies

$$M(f, x_0, x) = o(\delta), \qquad M(\alpha, x_0, x) = o(\delta),$$

so that (7) follows from (9).

3. Proof of Theorems 1 and 2. As was mentioned earlier, Theorems 1 and 2 will be proved by means of formula (2). To simplify the notation, replace f_1 by α, g by β, $f_1 + f_2$ by f. Let Ω be the set on which $f(x) > 0$, let E be the set on which $f(x) = 0$. Then the following has to be proved.

THEOREM 3. *Let Y be any one of the spaces listed in Theorems* 1 *and* 2. *Assume $\alpha, \beta, f \in Y$, $0 \leqq \alpha \leqq f$, $0 \leqq \beta \leqq f$. Define*

$$(11) \quad h(x) = \begin{cases} \dfrac{\alpha(x)\beta(x)}{f(x)} & \text{if } x \in \Omega, \\ 0 & \text{if } x \in E. \end{cases}$$

Then $h \in Y$.

Proof. Put $u(x) = \alpha(x)/f(x)$, $v(x) = \beta(x)/f(x)$, for $x \in \Omega$. Note that $0 \leqq u \leqq 1$, $0 \leqq v \leqq 1$.

We begin with the case $n = 1$.

If $Y = D^1(\mathbf{R}^1)$ and $x_0 \in E$, then $\beta'(x_0) = 0$; since $h = u\beta$ in Ω, it follows that $h'(x_0) = 0$. Hence

$$(12) \quad h' = \begin{cases} u\beta' + v\alpha' - uvf' & \text{in } \Omega, \\ 0 & \text{in } E. \end{cases}$$

Thus $h \in D^1$.

Formula (12) also proves the theorem if $Y = B^1(\mathbf{R}^1)$ and if $Y = C^1(\mathbf{R}^1)$, since $f' = \alpha' = \beta' = 0$ in E.

If $Y = D^2(\mathbf{R}^1)$ we split E into two sets, E_0 and E_1: $f'' = 0$ on E_0, $f'' > 0$ on E_1. Lemma 1(b) shows that every point of E_1 is an *isolated* point of E.

We claim that

$$(13) \quad h'' = \begin{cases} u\beta'' + v\alpha'' - uvf'' + 2fu'v' & \text{in } \Omega, \\ \alpha''\beta''/f'' & \text{in } E_1, \\ 0 & \text{in } E_0. \end{cases}$$

The first line in (13) follows from (12) by a simple computation. If $x_0 \in E_1$ and $x \to x_0$ $(x \neq x_0)$, then (5) holds, for β as well as for α; hence each term in the first line of (12), divided by $x - x_0$, tends to $(\alpha''\beta''/f'')(x_0)$. Lemmas 2(a) and 1(b) show that

$$(14) \quad \alpha''(x_0) = \beta''(x_0) = f''(x_0) = 0 \qquad (x_0 \in E_0),$$

so that $h''(x_0) = 0$ on E_0, by (12). This proves (13), and settles the case $Y = D^2(\mathbf{R}^1)$.

Assume $Y = B^2(\mathbf{R}^1)$. Lemma 2(c) shows that $fu'v'$ is bounded on every bounded portion of Ω; Lemma 2(a) implies that $0 \leqq h'' \leqq \beta''$ on E_1. Hence (13) shows that h'' is locally bounded.

If $Y = C^2(\mathbf{R}^1)$, Lemma 2(d) shows that $fu'v'$ has a continuous extension to $\mathbf{R}^1$ which is 0 on E. The continuity of h'' follows therefore from (5), (13), and (14).

We turn to the case $Y = D^3(\mathbf{R}^1)$. By Lemma 2(e), the continuous extension of $fu'v'$ that we mentioned in the preceding paragraph has derivative 0 on E. At points of Ω at which $f'' \neq 0$, (13) can be rewritten in the form

$$(15)\quad h'' - \frac{\alpha''\beta''}{f''} = 2fu'v' - \frac{(f\alpha'' - f''\alpha)(f\beta'' - f''\beta)}{f^2 f''}.$$

Insert the Taylor formulas (10) (for f, α, β) into the right side of (15). If $x_0 \in E_1$, one sees that the right side of (15) is $O((x - x_0)^2)$. Thus

$$(16)\quad h'''(x_0) = (\alpha''\beta''/f'')'(x_0) \qquad (x_0 \in E_1).$$

If $x_0 \in E_0$, Lemma 1(d) and (14) show that $\alpha'''(x_0) = \beta'''(x_0) = f'''(x_0) = 0$. Since $0 \leqq h'' \leqq \beta''$ on E_1, it follows from (13) that $h'''(x_0) = 0$. Since h''' obviously exists at every point of Ω, we have proved that $h \in D^3(\mathbf{R}^1)$.

This completes the case $n = 1$.

When $n > 1$, we shall denote partial differentiation with respect to the ith variable by D_i.

If $Y = D^1(\mathbf{R}^n)$, the analogue of (12) is now

$$(17)\quad D_i h = \begin{cases} uD_i\beta + vD_i\alpha - uvD_if & \text{in } \Omega, \\ 0 & \text{in } E, \end{cases}$$

for $i = 1, \ldots, n$. This proves the theorem for $Y = D^1(\mathbf{R}^n)$, for $Y = B^1(\mathbf{R}^n)$, and for $Y = C^1(\mathbf{R}^n)$, as in the case $n = 1$.

There remains the case $Y = B^2(\mathbf{R}^n)$, $n > 1$. Let V be a bounded open set in $\mathbf{R}^n$. Choose $\varphi \in C^2(\mathbf{R}^n)$, with compact support, so that $\varphi = 1$ in V. Replace α, β, f by $\alpha\varphi, \beta\varphi, f\varphi$. These functions have bounded (not just locally bounded) second-order derivatives and h is not affected in V by the introduction of φ. Hence we may assume, without loss of generality, that f, α, β have compact support, and that $|D_iD_jf| \leqq M$, $|D_iD_j\alpha| \leqq M$, $|D_iD_j\beta| \leqq M$ in all of $\mathbf{R}^n$, for all i and j, and for some $M < \infty$.

Fix $x \in \mathbf{R}^n$. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbf{R}^n$, consisting of orthogonal unit vectors. Lemma 1(c), applied to the function $t \to f(x + te_j)$, shows that

$$(18)\quad |(D_jf)(x)|^2 \leqq 2Mf(x) \qquad (x \in \mathbf{R}^n, j = 1, \ldots, n).$$

The same inequality holds with α and β in place of f.

Since $f D_ju = D_j\alpha - uD_jf$ in Ω, it follows from (18) that

$$(19)\quad |(D_ju)(x)|^2 \leqq 8M/f(x) \qquad (x \in \Omega, j = 1, \ldots, n);$$

the same holds with v in place of u.

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By (18) and (19), differentiation of (17) yields

(20) $|(D_jD_ih)(x)| \leqq 19M \qquad (x \in \Omega, i, j = 1, \ldots, n).$

If $x \in E$ and $(D_j^2f)(x) = 0$, then $f(x + te_j) = o(t^2)$; by (18),

$$(D_if)(x + te_j) = o(t);$$

the same holds with α and β in place of f; hence (17) shows that

(21) $(D_jD_ih)(x) = 0$ if $x \in E$ and $(D_j^2f)(x) = 0$.

If $x \in E$ and $(D_j^2f)(x) > 0$, then

$$\lim_{t\to 0} u(x + te_j) = (D_j^2\alpha/D_j^2f)(x)$$

by Lemma 2(b); the same holds with v, β in place of u, α. It now follows from (17) that $(D_jD_ih)(x)$ exists, and that

(22) $|(D_jD_ih)(x)| \leqq 3M$ if $x \in E$ and $(D_j^2f)(x) > 0$.

By (20), (21), and (22), $h \in B^2(\mathbf{R}^n)$. This completes the proof.

4. Construction of examples.

Example 1. Let $\{\psi_k\}$ be a sequence of nonnegative C^∞-functions on $\mathbf{R}^1$, with pairwise disjoint compact supports, all lying in $[0, 1]$, such that $\psi_k(x) = 1$ in a segment $(t_k - \delta_k, t_k + \delta_k)$. Choose constants $c_k > 0$, so small that the kth derivative of $c_k\psi_k$ is everywhere less than $1/k^2$. Choose ϵ_k, $0 < \epsilon_k < \delta_k$, so that $\epsilon_k/c_k \to 0$ as $k \to \infty$. Define

$$f_1(x) = \sum_{k=1}^{\infty} c_k(x - t_k)^2\psi_k(x),$$

$$f_2(x) = \sum_{k=1}^{\infty} c_k\epsilon_k^2\psi_k(x),$$

$$g(x) = \frac{1}{2}\sum_{k=1}^{\infty} c_k(x - t_k + \epsilon_k)^2\psi_k(x).$$

Our choice of $\{c_k\}$ and $\{\epsilon_k\}$ ensures that $f_1, f_2, g \in C^\infty(\mathbf{R}^1)$. Since

$$\tfrac{1}{2}(x - t + \epsilon)^2 \leqq (x - t)^2 + \epsilon^2,$$

we see that f_1, f_2, g are RD-data.

Suppose g_1, g_2 solve the corresponding RD-problem, and that they are in D^3. On $(t_k - \delta_k, t_k + \delta_k)$ we have

$$f_1(x) = c_k(x - t_k)^2,$$
$$f_2(x) = c_k\epsilon_k^2,$$
$$g(x) = \tfrac{1}{2}c_k(x - t_k + \epsilon_k)^2.$$

Since $g(t_k - \epsilon_k) = 0$, we have $g_1(t_k - \epsilon_k) = 0$.

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Since $f_1(t_k) = 0$, we have $g_1(t_k) = 0$; also, $g_1 \geqq 0$, so that $g_1'(t_k) = 0$. Since $g = f_1 + f_2$ when $x = t_k + \epsilon_k$, we have

$$g_1(t_k + \epsilon_k) = f_1(t_k + \epsilon_k) = c_k \epsilon_k^2.$$

Taylor's formula shows that there are real numbers η_k and ξ_k such that

$$t_k - \epsilon_k < \eta_k < t_k < \xi_k < t_k + \epsilon_k$$

and

$$c_k \epsilon_k^2 = g_1(t_k + \epsilon_k) = \tfrac{1}{2} g_1''(t_k)\epsilon_k^2 + \tfrac{1}{6} g_1'''(\xi_k)\epsilon_k^3,$$
$$0 = g_1(t_k - \epsilon_k) = \tfrac{1}{2} g_1''(t_k)\epsilon_k^2 - \tfrac{1}{6} g_1'''(\eta_k)\epsilon_k^3.$$

If we subtract the last equation from the preceding one, we obtain

$$g_1'''(\xi_k) + g_1'''(\eta_k) = 6c_k/\epsilon_k.$$

Since $c_k/\epsilon_k \to \infty$ as $k \to \infty$, g_1''' is not bounded on $[0, 1]$.

This RD-*problem with C^∞-data has therefore no B^3-solution.*

Example 2. Define

$$f_1(x, y) = (x + y)^2, \qquad f_2(x, y) = (x - y)^2, \qquad g(x, y) = 2x^2.$$

These polynomials obviously form RD-data. Assume that g_1, g_2 is a twice-differentiable solution.

Since $0 \leqq g_1 \leqq g$ and $g(0, y) = 0$, g_1 has a local minimum at each point $(0, y)$, so that $(D_1 g_1)(0, y) = 0$. Hence $(D_2 D_1 g_1)(0, 0) = 0$.

When $y = 0$, then $g = f_1 + f_2$, hence $g_1 = f_1$. Thus $g_1 - f_1$ has a local maximum at each point $(x, 0)$, so that $(D_2 g_1)(x, 0) = (D_2 f_1)(x, 0) = 2x$. Hence $(D_1 D_2 g_1)(0, 0) = 2$.

Since $D_2 D_1 g_1 \neq D_1 D_2 g_1$, we conclude that $g_1 \notin C^2(\mathbf{R}^2)$.

Example 3. Choose $\psi \in C^\infty(\mathbf{R}^1)$, with support in $[-1, 1]$, so that $0 \leqq \psi \leqq 1$, $\psi(0) > 0$, $\psi'(0) > 0$. Put $f_1(0, y) = f_2(0, y) = g(0, y) = 0$; when $x \neq 0$, define

$$f_1(x, y) = x^4 \sin^2(\log|x|),$$
$$f_2(x, y) = x^4 \cos^2(\log|x|),$$
$$g(x, y) = x^4 \psi(y/x^3).$$

It is clear that these functions form RD-data, and that $f_1 \in D^2(\mathbf{R}^2)$, $f_2 \in D^2(\mathbf{R}^2)$. As regards g, note that $g(x, y) = 0$ if $|y| \geqq |x^3|$. The origin is therefore the only point where the existence of derivatives is in question. Since

$$(D_1 g)(x, y) = \begin{cases} 4x^3\psi(y/x^3) - 3y\psi'(y/x^3) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

and

$$(D_2 g)(x, y) = \begin{cases} x\psi'(y/x^3) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

we see that $g \in C^1(\mathbf{R}^2)$. Also, at $(0, 0)$ we have

$$D_1^2 g = D_2^2 g = D_2 D_1 g = 0, \qquad D_1 D_2 g = \psi'(0).$$

Thus $g \in D^2(\mathbf{R}^2)$.

Assume g_1, g_2 is a differentiable solution of this RD-problem. Choose $s_1 > t_1 > s_2 > t_2 > \ldots$ so that $s_m \to 0$ and $f_1(s_m, 0) = 0$, $f_2(t_m, 0) = 0$. Then g_1 has a local minimum at $(s_m, 0)$, so that

$$(D_2 g_1)(s_m, 0) = 0 \qquad (m = 1, 2, 3, \ldots).$$

Also, $g_2(t_m, 0) = 0$, so that $g_1(t_m, 0) = g(t_m, 0)$. Hence $g_1 - g$ has a local maximum at $(t_m, 0)$, so that

$$(D_2 g_1)(t_m, 0) = (D_2 g)(t_m, 0) = \psi'(0) t_m \qquad (m = 1, 2, 3, \ldots).$$

The quotients $(D_2 g_1)(x, 0)/x$ oscillate therefore between 0 and $\psi'(0)$ as $x \to 0$. Consequently, $(D_1 D_2 g_1)(0, 0)$ does not exist.

Thus $g_1 \notin D^2(\mathbf{R}^2)$.

(We point out that $D_2^2 g$ is not bounded in any neighborhood of the origin, so that $g \notin B^2(\mathbf{R}^2)$; this agrees with Theorem 2.)

Appendix. A collection Y of real functions on a set S is called a *weak lattice* if it has the following property.

If $h_i \in Y$, $H_i \in Y$, *and* $h_i \leqq H_j$ *for* $i, j = 1, 2$, *then there exists* $\varphi \in Y$ *such that* $h_i \leqq \varphi \leqq H_j$ *for* $i, j = 1, 2$.

Within the class of additive groups, every RD-space is a weak lattice, and vice versa; this is due to Buck [**1**]:

Suppose Y is an RD-space and h_i, H_i are as above. Then $f_1 = H_2 - h_1$, $f_2 = H_1 - h_2$, $g = H_1 - h_1$ are RD-data; if g_1, g_2 is a solution in Y, then

$$h_1 + g_1 = h_2 + (f_2 - g_2) = H_1 - g_2 = H_2 - (f_1 - g_1).$$

Hence $\varphi = h_1 + g_1$ shows that Y is a weak lattice.

Conversely, suppose Y is a weak lattice, and f_1, f_2, g are RD-data in Y. Put $h_1 = 0$, $h_2 = g - f_2$, $H_1 = g$, $\mathrm{H}_2 = f_1$. There exists $\varphi \in Y$ so that $h_i \leqq \varphi \leqq H_j$ $(i, j = 1, 2)$. Hence $g_1 = \varphi$, $g_2 = g - \varphi$ provides an RD-solution in Y.

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