J. Aust. Math. Soc. **96** (2014), 338–353
doi:10.1017/S1446788713000657

GROEBNER BASES AND NONEMBEDDINGS OF SOME FLAG MANIFOLDS

ZORAN Z. PETROVIĆ✉ and BRANISLAV I. PRVULOVIĆ

(Received 21 April 2012; accepted 21 November 2013; first published online 31 March 2014)

Communicated by D. Chan

Abstract

Groebner bases for the ideals determining mod 2 cohomology of the real flag manifolds $F(1, 1, n)$ and $F(1, 2, n)$ are obtained. These are used to compute appropriate Stiefel–Whitney classes in order to establish some new nonembedding and nonimmersion results for the manifolds $F(1, 2, n)$.

2010 *Mathematics subject classification*: primary 13P10; secondary 57R40, 57R42.

Keywords and phrases: flag manifold, Groebner basis, immersion, embedding, cuplength.

1. Introduction

The real flag manifold $F(n_1, n_2, \ldots, n_r)$ is defined as the set of flags of type $(n_1, n_2, \ldots, n_r)$ (r-tuples $(V_1, V_2, \ldots, V_r)$ of mutually orthogonal subspaces in $\mathbb{R}^m$, where $m = n_1 + \cdots + n_r$ and $\dim(V_i) = n_i$) with the manifold structure coming from the natural identification $F(n_1, n_2, \ldots, n_r) = O(n_1 + \cdots + n_r)/O(n_1) \times \cdots \times O(n_r)$. Obviously, there is no loss of generality in assuming that $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_r$. There are r canonical vector bundles $\gamma_1, \gamma_2, \ldots, \gamma_r$ ($\dim(\gamma_i) = n_i$) over $F(n_1, n_2, \ldots, n_r)$. By Borel's description [4], the mod 2 cohomology algebra of $F(n_1, n_2, \ldots, n_r)$ is isomorphic to the polynomial algebra on the Stiefel–Whitney classes of bundles $\gamma_1, \gamma_2, \ldots, \gamma_{r-1}$ modulo the ideal $I_{n_1,\ldots,n_r}$ generated by the dual classes.

The theory of Groebner bases is a natural choice when it comes to calculating in the quotient of the polynomial algebra by an ideal. This is our approach to $\mathbb{Z}_2$-cohomology of flag manifolds in this paper.

An algebraic background is given in Section 2. In this brief opening section we recall some basic notions from the theory of Groebner bases and point out a few elementary facts which will be used in proving our results.

The first author was partially supported by Ministry of Education, Science and Technological Development of Republic of Serbia Project #174032. The second author was partially supported by Ministry of Education, Science and Technological Development of Republic of Serbia Project #174034.

In Section 3, we construct reduced Groebner bases for ideals $I_{1,1,n}$ (for all $n \geqslant 1$) and present an additive basis for $H^*(F(1,1,n);\mathbb{Z}_2)$. These results are stated in Theorem 3.1 and Corollary 3.2. We apply these Groebner bases and give a simple proof of the theorem of Ajayi and Ilori [2] concerning nonembeddings and nonimmersions of the manifolds $F(1,1,n)$. Here, we also point out a mistake that the authors made in [2] while showing that a certain Stiefel–Whitney class is nonzero. However, this oversight is not essential, since the Stiefel–Whitney class in question is really nonzero. It just has a slightly different form than the one specified in [2]. We specify the mistake in a remark in Section 3.2.

The main results are presented in Section 4. Reduced Groebner bases for ideals $I_{1,2,n}$ (for all $n \geqslant 2$) are given in Theorem 4.2. As a consequence, an additive basis for cohomology algebra $H^*(F(1,2,n);\mathbb{Z}_2)$ is obtained in Corollary 4.3. We apply these Groebner bases to establish the following nonembedding and nonimmersion results for the flag manifolds $F(1,2,n)$. In the theorem, $\mathrm{em}(F(1,2,n)) = \min\{d \mid F(1,2,n) \text{ embeds into } \mathbb{R}^d\}$ and $\mathrm{imm}(F(1,2,n)) = \min\{d \mid F(1,2,n) \text{ immerses into } \mathbb{R}^d\}$.

THEOREM 1.1. *Let $n \geqslant 2$ and $s \geqslant 3$ be such that $2^{s-1} < n+3 \leqslant 2^s$.*

(a) *If $2^{s-1} \leqslant n \leqslant 2^s - 3$, then*

$$\mathrm{em}(F(1,2,n)) \geqslant 3 \cdot 2^s - 2 \quad \textit{and} \quad \mathrm{imm}(F(1,2,n)) \geqslant 3 \cdot 2^s - 3.$$

(b) *For $s \geqslant 4$,*

$$\mathrm{em}(F(1,2,2^{s-1}-2)) \geqslant 4 \cdot 2^{s-1} - 2 \quad \textit{and} \quad \mathrm{imm}(F(1,2,2^{s-1}-2)) \geqslant 4 \cdot 2^{s-1} - 3;$$
$$\mathrm{em}(F(1,2,2)) \geqslant 11 \quad \textit{and} \quad \mathrm{imm}(F(1,2,2)) \geqslant 10.$$

(c) *For $s \geqslant 4$,*

$$\mathrm{em}(F(1,2,2^{s-1}-1)) \geqslant 5 \cdot 2^{s-1} - 3 \quad \textit{and} \quad \mathrm{imm}(F(1,2,2^{s-1}-1)) \geqslant 5 \cdot 2^{s-1} - 4;$$
$$\mathrm{em}(F(1,2,3)) \geqslant 16 \quad \textit{and} \quad \mathrm{imm}(F(1,2,3)) \geqslant 15.$$

Nonimmersions in the cases $n = 2, 3, 4$ are known, due to Stong [13], and these are the cases where these lower bounds for the immersion dimension coincide with the upper bounds obtained by Lam in [8]. Also, when n is a power of two, Theorem 1.1 gives pretty high lower bounds for $\mathrm{imm}(F(1,2,n))$. Namely, we show that $3 \cdot 2^s - 3 \leqslant \mathrm{imm}(F(1,2,2^{s-1})) \leqslant 3 \cdot 2^s - 1$ for $s \geqslant 4$.

As another illustration of usage of Groebner bases, at the end of Section 4, we give an alternative proof of the result of Korbaš and Lörinc [7] concerning the $\mathbb{Z}_2$-cup-length of the manifolds $F(1,2,n)$.

2. Reduction of polynomials, Groebner bases

Let $\mathbb{F}$ be a field and $\mathbb{F}[x_1, x_2, \ldots, x_k]$ the polynomial algebra over $\mathbb{F}$ on k variables. A *term* on variables $x_1, x_2, \ldots, x_k$ is a product of powers $x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k} \in \mathbb{F}[x_1, x_2, \ldots, x_k]$, where $a_1, a_2, \ldots, a_k \geqslant 0$. The set of all terms in $\mathbb{F}[x_1, x_2, \ldots, x_k]$ will be denoted by T.

A *term ordering* (*monomial ordering*) in $\mathbb{F}[x_1, x_2, \ldots, x_k]$ is a linear ordering $\preccurlyeq$ on T such that $1 \in T$ is the minimum and such that $t \preccurlyeq s$ implies $t \cdot r \preccurlyeq s \cdot r$ for all $r, s, t \in T$. It is immediate from the definition that $t \mid s$ implies $t \preccurlyeq s$ for all $s, t \in T$ and any term ordering $\preccurlyeq$.

In the rest of this section we assume that a term ordering $\preccurlyeq$ is fixed. For a nonzero polynomial $f = \sum_{i=1}^{m} \alpha_i t_i \in \mathbb{F}[x_1, x_2, \ldots, x_k]$, where t_i are pairwise different terms and $\alpha_i \in \mathbb{F} \setminus \{0\}$, let $T(f) := \{t_i \mid 1 \leqslant i \leqslant m\}$. We define the *leading term* of f, denoted by $\mathrm{LT}(f)$, as $\max T(f)$ with respect to $\preccurlyeq$. The *leading coefficient* of f, denoted by $\mathrm{LC}(f)$, is the coefficient of $\mathrm{LT}(f)$ in f.

We are now able to define the notion of reduction (see [3, page 195]).

Definition 2.1. Let $f, g, p \in \mathbb{F}[x_1, x_2, \ldots, x_k]$, where $p \neq 0$, and let $P \subseteq \mathbb{F}[x_1, x_2, \ldots, x_k]$.

(i) We say that f *reduces to* g *modulo* p (and write $f \longrightarrow_p g$) if there exists $t \in T(f)$ such that $\mathrm{LT}(p) \mid t$ and $g = f - (\alpha/\mathrm{LC}(p)) \cdot s \cdot p$, where $\alpha \in \mathbb{F} \setminus \{0\}$ is the coefficient of t in f and $s \in T$ is such that $t = s \cdot \mathrm{LT}(p)$.

(ii) We say that f *reduces to* g *modulo* P (and write $f \longrightarrow_P g$) if there exists $p \in P$ such that $f \longrightarrow_p g$.

(iii) The relation $\stackrel{*}{\longrightarrow}_P$ is defined as the reflexive–transitive closure of $\longrightarrow_P$ in $\mathbb{F}[x_1, x_2, \ldots, x_k]$. In other words, $f \stackrel{*}{\longrightarrow}_P g$ means that either $f = g$ or there are polynomials $f_0, f_1, \ldots, f_n \in \mathbb{F}[x_1, x_2, \ldots, x_k]$ ($n \geqslant 1$) such that $f = f_0 \longrightarrow_P f_1 \longrightarrow_P \cdots \longrightarrow_P f_n = g$.

The statement of the following lemma is obvious from the definition.

Lemma 2.2. *If* $P \subseteq F \subseteq \mathbb{F}[x_1, x_2, \ldots, x_k]$ *and* $f \stackrel{*}{\longrightarrow}_P g$, *then* $f \stackrel{*}{\longrightarrow}_F g$.

The proof of the next lemma can be found in [3, Lemma 5.24(i)].

Lemma 2.3. *If* $g \in P \subseteq \mathbb{F}[x_1, x_2, \ldots, x_k]$, *then* $g \cdot h \stackrel{*}{\longrightarrow}_P 0$ *for every* $h \in \mathbb{F}[x_1, x_2, \ldots, x_k]$.

Recall that for $f, g \in \mathbb{F}[x_1, x_2, \ldots, x_k]$, the *S-polynomial* of f and g is defined as

$$S(f, g) := \mathrm{LC}(g) \cdot \frac{u}{\mathrm{LT}(f)} \cdot f - \mathrm{LC}(f) \cdot \frac{u}{\mathrm{LT}(g)} \cdot g,$$

where $u = \mathrm{lcm}(\mathrm{LT}(f), \mathrm{LT}(g))$ is the least common multiple of $\mathrm{LT}(f)$ and $\mathrm{LT}(g)$.

Let us now prove an important fact concerning the S-polynomials. The greatest common divisor of terms s and t is denoted by $\gcd(s, t)$.

Proposition 2.4. *Let* $f, g \in \mathbb{F}[x_1, x_2, \ldots, x_k]$ *be nonzero polynomials and* $P = \{f, g\}$. *If* $\gcd(\mathrm{LT}(f), t) = 1$ *for all* $t \in T(g)$, *then* $S(f, g) \stackrel{*}{\longrightarrow}_P 0$.

Proof. Let $f = \alpha_1 s_1 + \alpha_2 s_2 + \cdots + \alpha_m s_m$ and $g = \beta_1 t_1 + \beta_2 t_2 + \cdots + \beta_n t_n$, where s_i, $i = \overline{1, m}$, (and likewise t_j, $j = \overline{1, n}$) are pairwise different terms and α_i (β_j) are nonzero scalars. We may assume that $s_1 > s_2 > \cdots > s_m$ and $t_1 > t_2 > \cdots > t_n$,

https://doi.org/10.1017/S1446788713000657 Published online by Cambridge University Press

so $\mathrm{LT}(f) = s_1$, $\mathrm{LC}(f) = \alpha_1$, $\mathrm{LT}(g) = t_1$ and $\mathrm{LC}(g) = \beta_1$. From $\gcd(s_1, t_1) = 1$, we conclude that $\mathrm{lcm}(s_1, t_1) = s_1 \cdot t_1$, so

$$\begin{aligned} S(f, g) &= \beta_1 t_1 \cdot f - \alpha_1 s_1 \cdot g \\ &= \beta_1 t_1 \cdot (\alpha_1 s_1 + \alpha_2 s_2 + \cdots + \alpha_m s_m) - \alpha_1 s_1 \cdot (\beta_1 t_1 + \beta_2 t_2 + \cdots + \beta_n t_n) \\ &= \beta_1 t_1 \cdot (\alpha_2 s_2 + \cdots + \alpha_m s_m) - \alpha_1 s_1 \cdot (\beta_2 t_2 + \cdots + \beta_n t_n), \end{aligned}$$

where the expressions in the brackets are understood to be zero if $m = 1$ ($n = 1$).

Consider now the polynomials

$$h_r := (\beta_1 t_1 + \cdots + \beta_r t_r) \cdot (\alpha_2 s_2 + \cdots + \alpha_m s_m) - \alpha_1 s_1 \cdot (\beta_{r+1} t_{r+1} + \cdots + \beta_n t_n),$$

$r = \overline{1, n-1}$, and

$$h_n := (\beta_1 t_1 + \cdots + \beta_n t_n) \cdot (\alpha_2 s_2 + \cdots + \alpha_m s_m) = g \cdot (\alpha_2 s_2 + \cdots + \alpha_m s_m).$$

Obviously, $h_1 = S(f, g)$, and now we prove the following claim:

$$h_r \longrightarrow_f h_{r+1} \quad \text{for all } r = \overline{1, n-1}.$$

Note that $s_1 \cdot t_{r+1} \in T(h_r)$ (that is, the coefficient of $s_1 \cdot t_{r+1}$ in h_r is nonzero). Namely, if this was false, then the term $s_1 \cdot t_{r+1}$ would have to be canceled in the upper expression for h_r by some other term. Hence, it would be equal either to some $s_1 \cdot t_j$ ($r + 2 \leqslant j \leqslant n$) or to some $s_i \cdot t_j$ ($2 \leqslant i \leqslant m$, $1 \leqslant j \leqslant r$). The first option is impossible since $t_j \neq t_{r+1}$ for $j \neq r + 1$. The second one implies $s_1 \mid s_i \cdot t_j$, but since $\gcd(s_1, t_j) = 1$, we would have $s_1 \mid s_i$, and consequently $s_1 \preccurlyeq s_i$, which contradicts the fact that $s_1 \succ s_i$.

So, $s_1 \cdot t_{r+1} \in T(h_r)$, and we conclude that $h_r \longrightarrow_f h_r - ((-\alpha_1 \beta_{r+1})/\alpha_1) \cdot t_{r+1} \cdot f$. It is now routine to check that $h_r + ((\alpha_1 \beta_{r+1})/\alpha_1) \cdot t_{r+1} \cdot f = h_{r+1}$. This proves the claim.

Finally, since $h_1 \xrightarrow{*}_P h_n$, that is, $S(f, g) \xrightarrow{*}_P g \cdot (\alpha_2 s_2 + \cdots + \alpha_m s_m)$, Lemma 2.3 finishes the proof of the proposition. □

There is a number of equivalent ways of defining a Groebner basis for an ideal in $\mathbb{F}[x_1, x_2, \ldots, x_k]$ (see [3, page 207]). The most appropriate for our context is the following one.

Definition 2.5. Let $G \subseteq \mathbb{F}[x_1, x_2, \ldots, x_k]$, $0 \notin G$, be a finite set of polynomials and let $I = (G)$ be the ideal in $\mathbb{F}[x_1, x_2, \ldots, x_k]$ generated by the set G. We say that G is a *Groebner basis* for I if $f \xrightarrow{*}_G 0$ for all $f \in I$.

If G is a Groebner basis for $I = (G)$ and $g_1, g_2 \in G$, then clearly, $S(g_1, g_2) \in I$ and so, $S(g_1, g_2) \xrightarrow{*}_G 0$. The Buchberger criterion ([5], see also [3, Theorem 5.48]) states that the converse is also true, that is, in order to prove that G is a Groebner basis, it suffices to check that $S(g_1, g_2) \xrightarrow{*}_G 0$ for all $g_1, g_2 \in G$. In the following theorem, along with this characterization of Groebner bases, we outline another one which we will use in the upcoming sections (see [3, Theorem 5.35(x)], [1, Proposition 2.1.6]).

https://doi.org/10.1017/S1446788713000657 Published online by Cambridge University Press

Theorem 2.6. *Let $G \subseteq \mathbb{F}[x_1, x_2, \ldots, x_k]$, $0 \notin G$, be a finite set of polynomials and let $I = (G)$ be the ideal in $\mathbb{F}[x_1, x_2, \ldots, x_k]$ generated by G. Then the following three conditions are equivalent.*

(i) *G is a Groebner basis for I.*

(ii) *For all $g_1, g_2 \in G$, $S(g_1, g_2) \xrightarrow{*}_G 0$.*

(iii) *The set of classes (cosets) of all terms in $\mathbb{F}[x_1, x_2, \ldots, x_k]$ that are not divisible by any of the leading terms $\mathrm{LT}(g)$, $g \in G$, forms an additive basis for the quotient algebra $\mathbb{F}[x_1, x_2, \ldots, x_k]/I$.*

The Groebner basis G is *reduced* if all $g \in G$ are monic ($\mathrm{LC}(g) = 1$) and if for all $g, g_1 \in G$, $g \neq g_1$, and all $t \in T(g)$, $\mathrm{LT}(g_1) \nmid t$, that is, if no term of $g \in G$ is divisible by some leading term in $G \setminus \{g\}$. It is a theorem (see [3, Theorem 5.43]) that the reduced Groebner basis for a given ideal is unique.

3. The real flag manifolds $F(1, 1, n)$

The real flag manifold $F(1, 1, n)$, $n \geqslant 1$, is a manifold of dimension $2n + 1$ which consists of triples (l_1, l_2, V), where l_1 and l_2 are mutually orthogonal lines through the origin in $\mathbb{R}^{n+2}$ and V is the n-dimensional subspace of $\mathbb{R}^{n+2}$ orthogonal to both l_1 and l_2. Since V is uniquely determined by l_1 and l_2, note that the map $(l_1, l_2, V) \mapsto (l_1, l_2)$ is a natural embedding of $F(1, 1, n)$ into $\mathbb{R}P^{n+1} \times \mathbb{R}P^{n+1}$.

3.1. Groebner basis for cohomology of $F(1, 1, n)$. Let $n \geqslant 1$ be a fixed integer. It is well known that the mod 2 cohomology algebra of $F(1, 1, n)$ is isomorphic to the quotient algebra $\mathbb{Z}_2[x, y]/I_{1,1,n}$, where $x, y \in H^1(F(1, 1, n); \mathbb{Z}_2)$ are Stiefel–Whitney classes of two canonical line bundles over $F(1, 1, n)$ and $I_{1,1,n} = (z_{n+1}, z_{n+2})$ is the ideal in $\mathbb{Z}_2[x, y]$ generated by the dual classes z_{n+1} and z_{n+2}. These dual classes are actually dual to Stiefel–Whitney classes of the Whitney sum of two canonical line bundles, so they are obtained from the equation

$$1 + z_1 + z_2 + \cdots = (1 + x)^{-1}(1 + y)^{-1} = \sum_{s \geqslant 0} x^s \cdot \sum_{t \geqslant 0} y^t.$$

In cohomological dimension $r \geqslant 1$ we have the equality $z_r = \sum_{t=0}^{r} x^{r-t} y^t$. Observe that

$$x z_{n+1} + z_{n+2} = \sum_{t=0}^{n+1} x^{n+2-t} y^t + \sum_{t=0}^{n+2} x^{n+2-t} y^t = y^{n+2}. \tag{3.1}$$

This means that the ideal generated by z_{n+1} and y^{n+2} coincides with the ideal $I_{1,1,n} = (z_{n+1}, z_{n+2})$, that is, the set $\{z_{n+1}, y^{n+2}\}$ is a basis for $I_{1,1,n}$.

Denote by $\preccurlyeq$ the lexicographic term ordering (lex ordering) in $\mathbb{Z}_2[x, y]$ with $x > y$. Hence, $x^a y^b \preccurlyeq x^c y^d$ if and only if $a < c$ or else $a = c$ and $b \leqslant d$.

Theorem 3.1. *The set $\{z_{n+1}, y^{n+2}\}$ is the reduced Groebner basis for $I_{1,1,n}$ with respect to the lex ordering $\preccurlyeq$.*

https://doi.org/10.1017/S1446788713000657 Published online by Cambridge University Press

PROOF. It is clear that $\mathrm{LT}(z_{n+1}) = x^{n+1}$ and $\gcd(x^{n+1}, y^{n+2}) = 1$, so the conditions of Proposition 2.4 are satisfied. By that proposition and Theorem 2.6 we conclude that $\{z_{n+1}, y^{n+2}\}$ is a Groebner basis for the ideal $(z_{n+1}, y^{n+2}) = I_{1,1,n}$.

It is pretty obvious that $\mathrm{LT}(y^{n+2}) = y^{n+2}$ does not divide any of the terms in z_{n+1} and *vice versa*, so this Groebner basis is the reduced one. □

As we have noticed in the proof of the theorem, $\mathrm{LT}(z_{n+1}) = x^{n+1}$ and $\mathrm{LT}(y^{n+2}) = y^{n+2}$. Now, as a consequence of Theorems 3.1 and 2.6, we obtain the following corollary (compare to [4]).

COROLLARY 3.2. *If $x, y \in H^1(F(1, 1, n); \mathbb{Z}_2)$ are Stiefel–Whitney classes of two canonical line bundles over $F(1, 1, n)$, then $\{x^a y^b \mid a \leqslant n,\ b \leqslant n + 1\}$ is a vector space basis for $H^*(F(1, 1, n); \mathbb{Z}_2)$.*

The height of the class y in $H^*(F(1, 1, n); \mathbb{Z}_2)$, $\mathrm{ht}(y) = \max\{i \mid y^i \neq 0\}$, is equal to $n + 1$ since y^{n+1} is a basis element in $H^*(F(1, 1, n); \mathbb{Z}_2)$ and $y^{n+2} \in I_{1,1,n}$, so $y^{n+2} = 0$ in $H^*(F(1, 1, n); \mathbb{Z}_2)$.

Likewise, $\mathrm{ht}(x) = n + 1$. This is because $x^{n+1} + \sum_{t=1}^{n+1} x^{n+1-t} y^t = z_{n+1} \in I_{1,1,n}$, thus $x^{n+1} = \sum_{t=1}^{n+1} x^{n+1-t} y^t \neq 0$ in $H^*(F(1, 1, n); \mathbb{Z}_2)$ since this is a (nonempty) sum of distinct basis elements. On the other hand, in a similar way as for (3.1), one obtains that $x^{n+2} = y z_{n+1} + z_{n+2} \in I_{1,1,n}$, so $x^{n+2} = 0$.

3.2. Nonembeddings and nonimmersions of $F(1, 1, n)$. Now, we are going to give another proof of nonembedding and nonimmersion results for $F(1, 1, n)$, $n \geqslant 2$, obtained by Ajayi and Ilori in [2].

Let ν be the stable normal bundle of $F(1, 1, n)$. It is well known (see [9, pages 120 and 49]) that nontriviality of the class $w_k(\nu)$ implies $\mathrm{em}(F(1, 1, n)) \geqslant \dim(F(1, 1, n)) + k + 1 = 2n + k + 2$ and $\mathrm{imm}(F(1, 1, n)) \geqslant 2n + k + 1$.

It is also known that the following formula holds for the total Stiefel–Whitney class of ν (see for example, [2, page 52]):

$$w(\nu) = (1 + x + y)(1 + x)^{-n-2}(1 + y)^{-n-2}.$$

If $r \geqslant 2$ is the integer such that $2^{r-1} \leqslant n + 1 < 2^r$, then, since $\mathrm{ht}(x) = \mathrm{ht}(y) = n + 1$ and $2^r \geqslant n + 2$, we may multiply the right-hand side of the formula by $(1 + x)^{2^r}(1 + y)^{2^r} = (1 + x^{2^r})(1 + y^{2^r}) = 1$ and thus obtain that

$$w(\nu) = (1 + x + y)(1 + x)^{2^r-n-2}(1 + y)^{2^r-n-2}. \tag{3.2}$$

THEOREM 3.3 [2]. *Let $n \geqslant 2$ and $r \geqslant 2$ be such that $2^{r-1} \leqslant n + 1 < 2^r$. Then:*

(a) *for $2^{r-1} \leqslant n \leqslant 2^r - 2$, $\mathrm{em}(F(1, 1, n)) \geqslant 2^{r+1} - 1$ and $\mathrm{imm}(F(1, 1, n)) \geqslant 2^{r+1} - 2$;*

(b) *$\mathrm{em}(F(1, 1, 2^{r-1} - 1)) \geqslant 3 \cdot 2^{r-1} - 1$ and $\mathrm{imm}(F(1, 1, 2^{r-1} - 1)) \geqslant 3 \cdot 2^{r-1} - 2$.*

PROOF. The top class in (3.2) is in dimension $2^{r+1} - 2n - 3$ and moreover

$$w_{2^{r+1}-2n-3}(\nu) = (x + y)x^{2^r-n-2}y^{2^r-n-2} = x^{2^r-n-1}y^{2^r-n-2} + x^{2^r-n-2}y^{2^r-n-1}.$$

https://doi.org/10.1017/S1446788713000657 Published online by Cambridge University Press

If $2^r - n - 1 \leqslant n$, that is, $n \geqslant 2^{r-1}$, this is the sum of two distinct basis elements (Corollary 3.2), so $w_{2^{r+1}-2n-3}(\nu) \neq 0$ in this case. This proves (a).

For (b), it suffices to show that $w_n(\nu) \neq 0$ for $n = 2^{r-1} - 1 > 1$. In this case, by (3.2)

$$\begin{aligned} w(\nu) &= (1+x+y)(1+x)^{2^{r-1}-1}(1+y)^{2^{r-1}-1} \\ &= (1+x+y)(1+x+x^2+\cdots+x^{2^{r-1}-1})(1+y+y^2+\cdots+y^{2^{r-1}-1}) \\ &= (1+x+y)(1+x+x^2+\cdots+x^n)(1+y+y^2+\cdots+y^n). \end{aligned}$$

In dimension n, we obtain that

$$\begin{aligned} w_n(\nu) &= \sum_{t=0}^{n} x^{n-t}y^t + (x+y)\sum_{t=0}^{n-1} x^{n-1-t}y^t \\ &= \sum_{t=0}^{n} x^{n-t}y^t + \sum_{t=0}^{n-1} x^{n-t}y^t + \sum_{t=0}^{n-1} x^{n-1-t}y^{t+1} \\ &= y^n + \sum_{t=0}^{n-1} x^{n-1-t}y^{t+1} = \sum_{t=0}^{n-2} x^{n-1-t}y^{t+1}, \end{aligned}$$

and this is nonzero since $n > 1$ and all summands of the last sum are (distinct) basis elements (Corollary 3.2). □

Remark. In [2], the authors also show that $w_{2^{r-1}-1}(\nu) \neq 0$ for the manifold $F(1, 1, 2^{r-1} - 1)$, but, unlike here, there the terms (basis elements) $x^{2^{r-1}-1}$ and $y^{2^{r-1}-1}$ have nonzero coefficients in $w_{2^{r-1}-1}(\nu)$. This is because, in that paper, authors made a mistake on page 53. The sum of binomial coefficients, appearing at the very beginning of their calculation of $\overline{w}_{n-2}$ (that is, $w_{2^{r-1}-1}(\nu)$), should be $\binom{2^{r-1}-1-i}{i} + \binom{2^{r-1}-1-i}{i+1}$. This implies that later, instead of even powers, only odd powers of σ_2 will remain. Also, the statement $\binom{2^{r-1}-2i+1}{2i} \equiv 1 \pmod 2$ for all $i = \overline{0, 2^{r-3} - 1}$ is false.

At the end of this section, we give a few comments on the number $\operatorname{imm}(F(1, 1, n))$. As it is stated in [2], the previous theorem gives the highest lower bound for $\operatorname{imm}(F(1, 1, n))$ when n is a power of two. Also, it gives the best possible result when $n = 2, 3, 4$. Furthermore, if $n = 2^{r-1}$, then, as we have already noticed, $F(1, 1, n)$ embeds in $\mathbb{R}P^{2^{r-1}+1} \times \mathbb{R}P^{2^{r-1}+1}$. On the other hand, by the result of Sanderson [11], $\mathbb{R}P^{2^{r-1}+1}$ immerses into $\mathbb{R}^{2^r-1}$ for $r \geqslant 4$, so we obtain an immersion of $F(1, 1, 2^{r-1})$ into $\mathbb{R}^{2^r-1} \times \mathbb{R}^{2^r-1} = \mathbb{R}^{2^{r+1}-2}$. Hence, Theorem 3.3 gives the best possible result for all powers of two and we have that

$$\operatorname{imm}(F(1, 1, 2^{r-1})) = 2^{r+1} - 2, \quad r \geqslant 2.$$

4. The real flag manifolds $F(1, 2, n)$

The real flag manifold $F(1, 2, n)$, $n \geqslant 2$, is a manifold of dimension $3n + 2$ which consists of triples (l, α, V), where l is a line through the origin in $\mathbb{R}^{n+3}$, α is a plane

https://doi.org/10.1017/S1446788713000657 Published online by Cambridge University Press

through the origin in $\mathbb{R}^{n+3}$ orthogonal to l, and V is the n-dimensional subspace of $\mathbb{R}^{n+3}$ orthogonal to both l and α. Obviously, V is uniquely determined by l and α, so the map $(l, \alpha, V) \mapsto (l, \alpha)$ is a natural embedding of $F(1, 2, n)$ into $\mathbb{R}P^{n+2} \times G_{2,n+1}$, where $G_{2,n+1} = G_2(\mathbb{R}^{n+3})$ is the Grassmann manifold.

4.1. Groebner basis for cohomology of $F(1, 2, n)$. Let $n \geqslant 2$ be a fixed integer. As in the case of manifolds $F(1, 1, n)$, the mod 2 cohomology algebra of $F(1, 2, n)$ is known to be isomorphic to the quotient algebra $\mathbb{Z}_2[x, y_1, y_2]/I_{1,2,n}$, where $x \in H^1(F(1, 2, n); \mathbb{Z}_2)$ is the Stiefel–Whitney class of the canonical line bundle over $F(1, 2, n)$, $y_i \in H^i(F(1, 2, n); \mathbb{Z}_2)$, $i = 1, 2$, are Stiefel–Whitney classes of the canonical two-dimensional bundle over $F(1, 2, n)$ and $I_{1,2,n} = (z_{n+1}, z_{n+2}, z_{n+3})$ is the ideal in $\mathbb{Z}_2[x, y_1, y_2]$ generated by the dual classes z_{n+1}, z_{n+2} and z_{n+3}. In a similar fashion as in the previous section, the dual classes are calculated from the equation

$$1 + z_1 + z_2 + \cdots = (1 + x)^{-1}(1 + y_1 + y_2)^{-1}. \tag{4.1}$$

It is clear that the classes

$$f_t := \sum_{a+2b=t} \binom{a+b}{a} y_1^a y_2^b, \quad t \geqslant 0,$$

are dual to y_i ($i = 1, 2$), that is, that $(1 + y_1 + y_2)^{-1} = \sum_{t\geqslant 0} f_t$, so from (4.1) we now obtain that

$$1 + z_1 + z_2 + \cdots = (1 + x)^{-1} \sum_{t\geqslant 0} f_t = \sum_{s\geqslant 0} x^s \cdot \sum_{t\geqslant 0} f_t.$$

Finally, we have that for all $r \geqslant 1$

$$z_r = \sum_{t=0}^{r} x^{r-t} f_t.$$

Let $G := \{g_0, g_1, \ldots, g_{n+2}\}$ be the set of polynomials

$$g_m := \sum_{a+2b=n+2+m} \binom{a+b-m}{a} y_1^a y_2^b, \quad 0 \leqslant m \leqslant n+2. \tag{4.2}$$

It is understood that the sum is over nonnegative integers a and b. Obviously, $g_0 = f_{n+2}$, and also, it is not hard to verify the relations $y_1 g_0 + g_1 = f_{n+3}$ and $y_2 g_m + y_1 g_{m+1} = g_{m+2}$, $m = \overline{0, n}$ [10, page 115]. Note that G corresponds to the Groebner basis for the ideal determining the $\mathbb{Z}_2$-cohomology of Grassmann manifold $G_{2,n+1}$ obtained in [10], while the set G defined in [10, page 115] is the Groebner basis for the corresponding ideal for the Grassmannian $G_{2,n}$.

In what follows, we are going to prove that the set $F := \{z_{n+1}\} \cup G$ is the reduced Groebner basis for the ideal $I_{1,2,n} = (z_{n+1}, z_{n+2}, z_{n+3})$ with respect to a term ordering which will be defined later. Now we prove that F is a basis for $I_{1,2,n}$.

https://doi.org/10.1017/S1446788713000657 Published online by Cambridge University Press

LEMMA 4.1. *If (F) is the ideal in $\mathbb{Z}_2[x, y_1, y_2]$ generated by F, then*

$$(F) = I_{1,2,n}.$$

PROOF. The following relations are crucial in proving the lemma:

$$xz_{n+1} + g_0 = \sum_{t=0}^{n+1} x^{n+2-t} f_t + f_{n+2} = \sum_{t=0}^{n+2} x^{n+2-t} f_t = z_{n+2};$$

$$\begin{aligned} x^2 z_{n+1} + (x + y_1)g_0 + g_1 &= x(xz_{n+1} + g_0) + y_1 g_0 + g_1 = xz_{n+2} + f_{n+3} \\ &= \sum_{t=0}^{n+2} x^{n+3-t} f_t + f_{n+3} = \sum_{t=0}^{n+3} x^{n+3-t} f_t = z_{n+3}. \end{aligned}$$

It is now clear that $z_{n+2}, z_{n+3} \in (F)$, so $I_{1,2,n} \subseteq (F)$. Conversely, $g_0 = xz_{n+1} + z_{n+2} \in I_{1,2,n}$ by the first relation and $g_1 = x^2 z_{n+1} + (x + y_1)g_0 + z_{n+3} \in I_{1,2,n}$ by the second. Now, using induction and the relations $g_{m+2} = y_2 g_m + y_1 g_{m+1}$ ($m = \overline{0, n}$), one can show that $G \subseteq I_{1,2,n}$, and consequently $(F) \subseteq I_{1,2,n}$. □

In [10] it is shown that the set G is the reduced Groebner basis for the ideal $(G) \triangleleft \mathbb{Z}_2[y_1, y_2]$ (which determines the $\mathbb{Z}_2$ cohomology of the Grassmannian $G_{2,n+1}$) with respect to the grlex ordering in $\mathbb{Z}_2[y_1, y_2]$ ($y_1^a y_2^b \preccurlyeq_{\text{grlex}} y_1^c y_2^d$ if and only if $a + b < c + d$ or else $a + b = c + d$ and $a \leqslant c$). We wish to define a term ordering in $\mathbb{Z}_2[x, y_1, y_2]$ which restricts to this grlex ordering in $\mathbb{Z}_2[y_1, y_2]$.

In order to do so, for two terms $x^k y_1^a y_2^b$ and $x^l y_1^c y_2^d$ in $\mathbb{Z}_2[x, y_1, y_2]$, we write $x^k y_1^a y_2^b \preccurlyeq x^l y_1^c y_2^d$ if and only if $k < l$ or else $k = l$ and $y_1^a y_2^b \preccurlyeq_{\text{grlex}} y_1^c y_2^d$. It is routine to check that $\preccurlyeq$ is a term ordering in $\mathbb{Z}_2[x, y_1, y_2]$, and obviously, it has the desired property: $y_1^a y_2^b \preccurlyeq y_1^c y_2^d$ if and only if $y_1^a y_2^b \preccurlyeq_{\text{grlex}} y_1^c y_2^d$.

THEOREM 4.2. *The set $F = \{z_{n+1}\} \cup G$ is the reduced Groebner basis for the ideal $I_{1,2,n}$ with respect to the ordering $\preccurlyeq$.*

PROOF. By Lemma 4.1, $(F) = I_{1,2,n}$. In order to prove that F is a Groebner basis for $I_{1,2,n}$, by Theorem 2.6 it is enough to show that $S(f, g) \xrightarrow{*}_F 0$ for all $f, g \in F$.

If $g_l, g_m \in G \subseteq F$, then $S(g_l, g_m) \xrightarrow{*}_G 0$ since G is a Groebner basis [10, Theorem 2.7]. According to Lemma 2.2, $S(g_l, g_m) \xrightarrow{*}_F 0$.

Also, since $z_{n+1} = \sum_{t=0}^{n+1} x^{n+1-t} f_t$ and f_t are polynomials in variables y_1 and y_2, we conclude that the leading term of z_{n+1} is obtained for $t = 0$. Since $f_0 = 1$, we have that $\mathrm{LT}(z_{n+1}) = x^{n+1}$. Furthermore, the polynomials $g_m \in G$ are also polynomials in variables y_1 and y_2, so $\gcd(x^{n+1}, t) = 1$ for all $t \in T(g_m)$. By Proposition 2.4 and Lemma 2.2, $S(z_{n+1}, g_m) \xrightarrow{*}_F 0$ for all $g_m \in G$.

We are left to prove that F is reduced. Since $\gcd(\mathrm{LT}(z_{n+1}), t) = 1$ for all $t \in T(g_m)$ and all $g_m \in G$, no term of $g_m \in G$ is divisible by $\mathrm{LT}(z_{n+1})$. Also, no term of $g_m \in G$ is divisible by the leading term of some $g_l \in G \setminus \{g_m\}$ since G is reduced [10, Theorem 2.7]. It remains to show that no term of z_{n+1} is divisible by some $\mathrm{LT}(g_m)$,

$m = \overline{0, n+2}$. This is due to the fact that $\mathrm{LT}(g_m) = y_1^{n+2-m} y_2^m$ [10, page 115] and the fact that for all terms $x^{n+1-t} y_1^a y_2^b$ appearing in $z_{n+1} = \sum_{t=0}^{n+1} x^{n+1-t} f_t$, the sum of the exponents $a + b \leqslant a + 2b = t \leqslant n + 1$. □

As we have outlined in the proof of the theorem, $\mathrm{LT}(z_{n+1}) = x^{n+1}$ and $\mathrm{LT}(g_m) = y_1^{n+2-m} y_2^m$, $m = \overline{0, n+2}$. This means that a term $x^k y_1^a y_2^b$ is not divisible by any of the leading terms in F if and only if $k \leqslant n$ and $a + b \leqslant n + 1$. The following corollary is immediate from this observation, Theorems 4.2 and 2.6.

Corollary 4.3. *If $x \in H^1(F(1, 2, n); \mathbb{Z}_2)$ is the Stiefel–Whitney class of the canonical line bundle over $F(1, 2, n)$ and $y_i \in H^i(F(1, 2, n); \mathbb{Z}_2)$, $i = 1, 2$, are Stiefel–Whitney classes of the canonical two-dimensional bundle over $F(1, 2, n)$, then the set $\{x^k y_1^a y_2^b \mid k \leqslant n,\ a + b \leqslant n + 1\}$ is a vector space basis for $H^*(F(1, 2, n); \mathbb{Z}_2)$.*

Remark. Note that, if p is a polynomial in variables y_1 and y_2 and if $p \longrightarrow_f q$ for some $f \in F$ and $q \in \mathbb{Z}_2[x, y_1, y_2]$, then f must belong to G and q must also be a polynomial in variables y_1 and y_2 only. This is because $p \longrightarrow_f q$ implies that $\mathrm{LT}(f)$ divides some $t \in T(p)$ and $\mathrm{LT}(z_{n+1}) = x^{n+1}$. So $f \in F \setminus \{z_{n+1}\} = G$ and now since both p and f are polynomials in variables y_1 and y_2 only, the same is true for q (see Definition 2.1(i)).

The quotient algebra $\mathbb{Z}_2[y_1, y_2]/(G)$ is isomorphic to $H^*(G_{2,n+1}; \mathbb{Z}_2)$ [10]. It is known that the projection map $F(1, 2, n) \to G_{2,n+1}$ $((l, \alpha, V) \mapsto \alpha)$ induces a monomorphism in $\mathbb{Z}_2$-cohomology [7, page 146], so this algebra is (algebraically) embedded in $H^*(F(1, 2, n); \mathbb{Z}_2)$. This fact is also easily seen from Theorem 4.2, as we show in the following proposition and corollary.

Proposition 4.4. *Let $p = p(y_1, y_2) \in \mathbb{Z}_2[y_1, y_2] \subset \mathbb{Z}_2[x, y_1, y_2]$. Then $p \in (G)$ if and only if $p \in (F) = I_{1,2,n}$. In other words, $p(y_1, y_2)$ is trivial in the quotient algebra $\mathbb{Z}_2[y_1, y_2]/(G)$ if and only if it is trivial in $\mathbb{Z}_2[x, y_1, y_2]/I_{1,2,n} \cong H^*(F(1, 2, n); \mathbb{Z}_2)$.*

Proof. Since $G \subset F$, one direction is trivial. Conversely, if $p \in (F)$, then $p \overset{*}{\longrightarrow}_F 0$ since F is a Groebner basis. By the remark above, we conclude that $p \overset{*}{\longrightarrow}_G 0$ and so, $p \in (G)$. □

Corollary 4.5. *The subalgebra of $H^*(F(1, 2, n); \mathbb{Z}_2)$ generated by the classes y_1 and y_2 is isomorphic to $\mathbb{Z}_2[y_1, y_2]/(G) \cong H^*(G_{2,n+1}; \mathbb{Z}_2)$.*

Via the isomorphism $\mathbb{Z}_2[y_1, y_2]/(G) \cong H^*(G_{2,n+1}; \mathbb{Z}_2)$, the classes y_1 and y_2 correspond to the Stiefel–Whitney classes w_1 and w_2 of the canonical bundle γ_2 over the Grassmannian $G_{2,n+1}$. The heights of these Stiefel–Whitney classes are well known [12], so the following corollary is straightforward (compare to [7, page 147]).

Corollary 4.6. *Let $y_i \in H^i(F(1, 2, n); \mathbb{Z}_2)$, $i = 1, 2$, be Stiefel–Whitney classes of the canonical two-dimensional bundle over $F(1, 2, n)$. Then $\mathrm{ht}(y_2) = n + 1$ and if $s \geqslant 3$ is the integer such that $2^{s-1} < n + 3 \leqslant 2^s$, then $\mathrm{ht}(y_1) = 2^s - 2$.*

As a direct consequence of Corollaries 4.3 and 4.5, we obtain the following proposition.

PROPOSITION 4.7. *Every cohomology class $\sigma \in H^*(F(1,2,n);\mathbb{Z}_2)$ can be written in the form $\sigma = \sum_{i=0}^{n} x^i p_i(y_1, y_2)$, where $p_i(y_1, y_2)$ are polynomials in variables y_1 and y_2 only. Moreover, $\sigma = 0$ if and only if $p_i(y_1, y_2) = 0$ in $\mathbb{Z}_2[y_1, y_2]/(G) \cong H^*(G_{2,n+1};\mathbb{Z}_2)$ for all $i = \overline{0, n}$.*

By looking at this proposition, one might think that the cohomology algebra $H^*(F(1,2,n);\mathbb{Z}_2)$ is isomorphic to the tensor product of $H^*(G_{2,n+1};\mathbb{Z}_2)$ with truncated polynomial algebra $\mathbb{Z}_2[x]/(x^{n+1})$. However, this is not the case, since $x^{n+1} \neq 0$ in $H^*(F(1,2,n);\mathbb{Z}_2)$. Namely, $x^{n+1} + \sum_{t=1}^{n+1} x^{n+1-t} f_t = z_{n+1} = 0$ in $H^*(F(1,2,n);\mathbb{Z}_2)$, and so

$$x^{n+1} = \sum_{t=1}^{n+1} x^{n+1-t} f_t = \sum_{t=1}^{n+1} \sum_{a+2b=t} \binom{a+b}{a} x^{n+1-t} y_1^a y_2^b = \sum_{t=0}^{n} \sum_{a+2b=t+1} \binom{a+b}{a} x^{n-t} y_1^a y_2^b. \tag{4.3}$$

The summand for $t = 0$ in the last sum is $x^n \sum_{a+2b=1} \binom{a+b}{a} y_1^a y_2^b = x^n y_1$, and since $y_1 \neq 0$, by Proposition 4.7 we conclude that $x^{n+1} \neq 0$.

As for x^{n+1}, it will be convenient for our purposes to have x^{n+2} expressed as a sum of basis elements from Corollary 4.3. By (4.3), we have that

$$\begin{aligned} x^{n+2} &= \sum_{t=0}^{n} \sum_{a+2b=t+1} \binom{a+b}{a} x^{n+1-t} y_1^a y_2^b \\ &= \sum_{a+2b=1} \binom{a+b}{a} x^{n+1} y_1^a y_2^b + \sum_{t=1}^{n} \sum_{a+2b=t+1} \binom{a+b}{a} x^{n+1-t} y_1^a y_2^b \\ &= x^{n+1} y_1 + \sum_{t=0}^{n-1} \sum_{a+2b=t+2} \binom{a+b}{a} x^{n-t} y_1^a y_2^b. \end{aligned}$$

Observe that the (missing) summand for $t = n$ in the sum is actually equal to $g_0 \in G$ (see (4.2)) which vanishes in cohomology, so we may allow that $t = \overline{0, n}$ in this sum. Having this in mind and applying formula (4.3) again, we obtain that

$$\begin{aligned} x^{n+2} &= \sum_{t=0}^{n} \sum_{a+2b=t+1} \binom{a+b}{a} x^{n-t} y_1^{a+1} y_2^b + \sum_{t=0}^{n} \sum_{a+2b=t+2} \binom{a+b}{a} x^{n-t} y_1^a y_2^b \\ &= \sum_{t=0}^{n} \sum_{a+2b=t+2} \binom{a+b-1}{a-1} x^{n-t} y_1^a y_2^b + \sum_{t=0}^{n} \sum_{a+2b=t+2} \binom{a+b}{a} x^{n-t} y_1^a y_2^b. \end{aligned}$$

Finally, since $\binom{a+b-1}{a-1} + \binom{a+b}{a} \equiv \binom{a+b-1}{a} \pmod 2$, we have the desired relation

$$x^{n+2} = \sum_{t=0}^{n} \sum_{a+2b=t+2} \binom{a+b-1}{a} x^{n-t} y_1^a y_2^b. \tag{4.4}$$

Obviously, the summand for $t = 0$ is $x^n y_2$, so $x^{n+2} \neq 0$ by Proposition 4.7. Actually, it is a result of Korbaš and Lörinc that $\mathrm{ht}(x) = n + 2$ [7, page 147]. Using (4.4), it is not hard to prove that $x^{n+3} = 0$ by our method, but in order to save some space, we omit this proof.

4.2. Nonembeddings and nonimmersions of $F(1, 2, n)$. Let γ_1, γ_2 and γ_3 be canonical vector bundles over $F(1, 2, n)$ ($\dim(\gamma_1) = 1$, $\dim(\gamma_2) = 2$, $\dim(\gamma_3) = n$). If τ is the tangent bundle over $F(1, 2, n)$, then, due to Lam [8], we know that

$$\tau \cong (\gamma_1 \otimes \gamma_2) \oplus (\gamma_1 \otimes \gamma_3) \oplus (\gamma_2 \otimes \gamma_3).$$

If we add the bundle $(\gamma_1 \otimes \gamma_1) \oplus (\gamma_1 \otimes \gamma_2) \oplus (\gamma_2 \otimes \gamma_2)$ to both sides of this isomorphism, then, using the fact that $\gamma_1 \oplus \gamma_2 \oplus \gamma_3$ is a trivial $(n + 3)$-dimensional bundle, we obtain that

$$\tau \oplus (\gamma_1 \otimes \gamma_1) \oplus (\gamma_1 \otimes \gamma_2) \oplus (\gamma_2 \otimes \gamma_2) \cong (n + 3)\gamma_1 \oplus (n + 3)\gamma_2.$$

Now, $\gamma_1 \otimes \gamma_1$ is a trivial line bundle, since it is the tensor product of a line bundle with itself. Total Stiefel–Whitney classes of tensor products $\gamma_1 \otimes \gamma_2$ and $\gamma_2 \otimes \gamma_2$ are calculated by the method described in [9, Problem 7-C], so for the total Stiefel–Whitney class $w(\tau)$ one obtains the relation

$$w(\tau) \cdot (1 + y_1 + x^2 + xy_1 + y_2)(1 + y_1^2) = (1 + x)^{n+3}(1 + y_1 + y_2)^{n+3},$$

where x, y_1 and y_2 are, as before, the Stiefel–Whitney classes of canonical bundles γ_1 and γ_2. Hence, for the total Stiefel–Whitney class of the stable normal bundle ν over $F(1, 2, n)$, we have that

$$w(\nu) = (1 + y_1 + x^2 + xy_1 + y_2)(1 + y_1^2)(1 + x)^{-n-3}(1 + y_1 + y_2)^{-n-3}.$$

If $s \geqslant 3$ is the integer such that $2^{s-1} < n + 3 \leqslant 2^s$, then in view of the heights of classes x, y_1 and y_2, we have that $(1 + x)^{2^s} = 1 + x^{2^s} = 1$ and likewise, $(1 + y_1 + y_2)^{2^s} = 1$. Finally, this means that we may multiply the right-hand side of the above equality by $(1 + x)^{2^s}(1 + y_1 + y_2)^{2^s}$, and thus we obtain the formula

$$w(\nu) = (1 + y_1 + x^2 + xy_1 + y_2)(1 + y_1^2)(1 + x)^{2^s-n-3}(1 + y_1 + y_2)^{2^s-n-3}. \tag{4.5}$$

Proof of Theorem 1.1. We know that $w_k(\nu) \neq 0$ implies $\mathrm{em}(F(1, 2, n)) \geqslant \dim(F(1, 2, n)) + k + 1 = 3n + k + 3$ and $\mathrm{imm}(F(1, 2, n)) \geqslant 3n + k + 2$ (see [9, pages 49 and 120]).

The top class in (4.5) is in dimension $3 \cdot 2^s - 3n - 5$ and

$$\begin{aligned} w_{3\cdot 2^s-3n-5}(\nu) &= (x^2 + xy_1 + y_2)y_1^2 x^{2^s-n-3} y_2^{2^s-n-3} \\ &= x^{2^s-n-1}y_1^2 y_2^{2^s-n-3} + x^{2^s-n-2}y_1^3 y_2^{2^s-n-3} + x^{2^s-n-3}y_1^2 y_2^{2^s-n-2}. \end{aligned}$$

According to Corollary 4.3, if $2^s - n \leqslant n + 1$, that is, $n \geqslant 2^{s-1}$, this is a sum of three distinct basis elements and so, $w_{3\cdot 2^s-3n-5}(\nu) \neq 0$ in this case. This proves (a).

If $n = 2^{s-1} - 2$, then (4.5) simplifies to

$$w(v) = (1 + y_1 + x^2 + xy_1 + y_2)(1 + y_1^2)(1 + x)^{2^{s-1}-1}(1 + y_1 + y_2)^{2^{s-1}-1}.$$

We now multiply this equality by $(1 + x)(1 + y_1 + y_2)$ and obtain that

$$\begin{aligned} w(v) \cdot (1 + x)(1 + y_1 + y_2) \\ = (1 + y_1 + x^2 + xy_1 + y_2)(1 + y_1^2)(1 + x^{2^{s-1}})(1 + y_1^{2^{s-1}} + y_2^{2^{s-1}}). \end{aligned}$$

For $s \geqslant 4$, the class in dimension $2^{s-1} + 4$ on the right-hand side is

$$\begin{aligned} \sigma &= (x^2 + xy_1 + y_2)y_1^2(x^{2^{s-1}} + y_1^{2^{s-1}}) = (x^2y_1^2 + xy_1^3 + y_1^2y_2)(x^{n+2} + y_1^{n+2}) \\ &= x^{n+2}y_1^2y_2 + x^2y_1^{n+4} + xy_1^{n+5} + y_1^{n+4}y_2, \end{aligned}$$

since $\mathrm{ht}(x) = n + 2$. Using (4.4), we can write the class σ in the form $\sum_{i=0}^{n} x^i p_i(y_1, y_2)$. When we do so, we obtain that the polynomial $p_n(y_1, y_2)$ is

$$y_1^2y_2 \sum_{a+2b=2} \binom{a+b-1}{a} y_1^a y_2^b = y_1^2y_2^2 \neq 0,$$

since $n > 2$. By Proposition 4.7, $\sigma \neq 0$. Finally, this means that for some $k \geqslant 2^{s-1} + 1$, the class $w_k(v)$ must be nonzero.

By direct calculation, one shows that $w_2(v) = xy_1 + y_1^2 \neq 0$ for $n = 2$. This proves (b).

In the case $n = 2^{s-1} - 1$, (4.5) gives us that

$$w(v) = (1 + y_1 + x^2 + xy_1 + y_2)(1 + y_1^2)(1 + x)^{2^{s-1}-2}(1 + y_1 + y_2)^{2^{s-1}-2}.$$

If we multiply this relation by $(1 + x)^2(1 + y_1 + y_2)^2 = (1 + x^2)(1 + y_1^2 + y_2^2)$, we obtain that the class $w(v) \cdot (1 + x^2)(1 + y_1^2 + y_2^2)$ is equal to

$$(1 + y_1 + x^2 + xy_1 + y_2)(1 + y_1^2)(1 + x^{2^{s-1}})(1 + y_1^{2^{s-1}} + y_2^{2^{s-1}}).$$

If $s \geqslant 4$, the summand in dimension $2^s + 3$ of this class is

$$y_1^3(x^{2^{s-1}}y_1^{2^{s-1}} + y_2^{2^{s-1}}) = y_1^3(x^{n+1}y_1^{n+1} + y_2^{n+1}) = x^{n+1}y_1^{n+4} + y_1^3y_2^{n+1}.$$

Now, we use (4.3) to represent this class in the form $\sum_{i=0}^{n} x^i p_i(y_1, y_2)$ and obtain that in this case

$$p_n(y_1, y_2) = y_1^{n+4} \sum_{a+2b=1} \binom{a+b}{a} y_1^a y_2^b = y_1^{n+5} \neq 0,$$

since $\mathrm{ht}(y_1) = 2n$ (Corollary 4.6) and $n \geqslant 7$. Arguing as in the previous case, we conclude that $w_k(v) \neq 0$ for some $k \geqslant 2^s - 3$.

Again, the direct calculation shows that $w_4(v) = x^2y_1^2 + xy_1^3 + y_1^2y_2 \neq 0$ when $n = 3$ (one could use (4.3) and Proposition 4.7). □

https://doi.org/10.1017/S1446788713000657 Published online by Cambridge University Press

By Lam's estimate [8, Corollary 5.2], $\operatorname{imm}(F(1,2,n)) \leqslant 3n+3+\binom{n}{2}$. By this result and Theorem 1.1, one obtains that $\operatorname{imm}(F(1,2,2)) = 10$, $\operatorname{imm}(F(1,2,3)) = 15$, $\operatorname{imm}(F(1,2,4)) = 21$ and these are exactly the cases which are covered by Stong's result [13].

Excluding these low-dimensional cases, the strongest results on immersion dimension provided by Theorem 1.1 are those when n is a power of two. Namely, $\operatorname{imm}(F(1,2,2^{s-1})) \geqslant 3\cdot 2^s-3$. On the other hand, as we have already noticed, $F(1,2,2^{s-1})$ embeds into $\mathbb{R}P^{2^{s-1}+2}\times G_{2,2^{s-1}+1}$. But, for $s \geqslant 4$, $\mathbb{R}P^{2^{s-1}+2}$ immerses into $\mathbb{R}^{2^s}$ (since $\mathbb{R}P^{2^{s-1}+3}$ immerses into $\mathbb{R}^{2^s}$ by [11]) and $G_{2,2^{s-1}+1}$ immerses into $\mathbb{R}^{2^{s+1}-1}$ [10], so we conclude that there is an immersion of the flag manifold $F(1,2,2^{s-1})$ in $\mathbb{R}^{2^s}\times\mathbb{R}^{2^{s+1}-1} = \mathbb{R}^{3\cdot 2^s-1}$. Finally, this means that

$$3\cdot 2^s-3 \leqslant \operatorname{imm}(F(1,2,2^{s-1})) \leqslant 3\cdot 2^s-1, \qquad s\geqslant 4.$$

4.3. Cup-length for $F(1,2,n)$. We can now determine the $\mathbb{Z}_2$-cup-length of the manifolds $F(1,2,n)$ $(n\geqslant 2)$ and thus give another proof of the result originally obtained by Korbaš and Lörinc [7, Proposition 3.2.4]. Recall that, for a commutative ring R, the R-cup-length of a path connected space X, denoted by $\operatorname{cup}_R(X)$, is the supremum of all integers d such that there exist classes $a_1,a_2,\ldots,a_d\in\widetilde{H}^*(X;R)$ with nonzero cup product $(a_1a_2\cdots a_d\neq 0)$. First, we prove one lemma. As before, x, y_1 and y_2 are Stiefel–Whitney classes of canonical bundles over $F(1,2,n)$.

LEMMA 4.8. *For all $m\in\{0,1,\ldots,n\}$,*

$$x^{n+1}y_2^m = \sum_{t=0}^{n-m}\sum_{a+2b=t+1+2m}\binom{a+b-m}{a}x^{n-t}y_1^a y_2^b.$$

PROOF. The proof is by induction on m. Formula (4.3) verifies the statement of the lemma for $m=0$. Now, let $1\leqslant m\leqslant n$ and suppose that the lemma is true for the integer $m-1$. Then

$$\begin{aligned}x^{n+1}y_2^m = y_2(x^{n+1}y_2^{m-1}) &= \sum_{t=0}^{n-m+1}\sum_{a+2b=t-1+2m}\binom{a+b-m+1}{a}x^{n-t}y_1^a y_2^{b+1}\\ &= \sum_{t=0}^{n-m+1}\sum_{a+2b=t+1+2m}\binom{a+b-m}{a}x^{n-t}y_1^a y_2^b.\end{aligned}$$

The latter equality is due to the change of variable $b\mapsto b-1$ that was made in the sum. We note that we may suppose that the 'new' b is also $\geqslant 0$ since for $b=0$ in the last sum, we obtain the coefficient $\binom{a-m}{a}$, where $a=t+1+2m>m$, and obviously, this binomial coefficient is zero. Hence, to complete the induction step, we are left to prove that the summand obtained for $t=n-m+1$ in the upper sum is zero. But, this summand is

$$\sum_{a+2b=n+2+m}\binom{a+b-m}{a}x^{m-1}y_1^a y_2^b = x^{m-1}g_m = 0,$$

and we are done. □

https://doi.org/10.1017/S1446788713000657 Published online by Cambridge University Press

Note that $x^{n+1}y_2^{n+1} = 0$ since the dimension of this class is $3n + 3 > 3n + 2 = \dim(F(1, 2, n))$. Also, when m is close to n, from Lemma 4.8 it is not hard to explicitly express the class $x^{n+1}y_2^m$ as the sum of basis elements from Corollary 4.3. For example, it is an easy exercise to show that

$$x^{n+1}y_2^n = x^n y_1 y_2^n; \tag{4.6}$$
$$x^{n+1}y_2^{n-1} = x^n y_1 y_2^{n-1} + x^{n-1}y_1^2 y_2^{n-1} + x^{n-1}y_2^n. \tag{4.7}$$

Theorem 4.9 [7]. *Let $n \geqslant 2$ and $s \geqslant 3$ be such that $2^{s-1} < n + 3 \leqslant 2^s$. Then*

$$\mathrm{cup}_{\mathbb{Z}_2}(F(1, 2, n)) = \begin{cases} 3n + 2 & \text{if } 2^{s-1} - 2 \leqslant n \leqslant 2^{s-1} - 1, \\ 2n + 2^{s-1} + 1 & \text{if } 2^{s-1} - 1 \leqslant n \leqslant 2^s - 3. \end{cases}$$

Proof. According to [6], the class $w_1^{2^s-2}w_2^{n-2^{s-1}+2} \in H^{2n+2}(G_{2,n+1}; \mathbb{Z}_2)$ is nonzero, where w_1 and w_2 are Stiefel–Whitney classes of the canonical two-dimensional bundle over the Grassmann manifold $G_{2,n+1}$. By Corollary 4.5, we conclude that the class $y_1^{2^s-2}y_2^{n-2^{s-1}+2} \in H^{2n+2}(F(1, 2, n); \mathbb{Z}_2)$ is also nonzero.

In the case $n = 2^{s-1} - 2$, we have that $y_1^{2n+2} = y_1^{2^s-2} \neq 0$, so by Proposition 4.7, $x^n y_1^{2n+2} \neq 0$. This means that $\mathrm{cup}_{\mathbb{Z}_2}(F(1, 2, n)) \geqslant 3n + 2$, and since $\dim(F(1, 2, n)) = 3n + 2$, we certainly have the opposite inequality.

For $2^{s-1} - 1 \leqslant n \leqslant 2^s - 3$, let us prove that the class $x^{n+2}y_1^{2^s-2}y_2^{n-2^{s-1}+1}$ is nonzero. Denote the class $y_1^{2^s-2}y_2^{n-2^{s-1}+1} \in H^{2n}(F(1, 2, n); \mathbb{Z}_2)$ by σ. By Corollary 4.3 and Proposition 4.7, σ is a $\mathbb{Z}_2$-linear combination of $y_1^2 y_2^{n-1}$ and y_2^n, say $\sigma = \alpha y_1^2 y_2^{n-1} + \beta y_2^n$, $\alpha, \beta \in \{0, 1\}$.

Now, $g_n = y_1^2 y_2^n + y_2^{n+1}$ [10, page 118], so $y_1^2 y_2^n = y_2^{n+1}$ in $H^*(F(1, 2, n); \mathbb{Z}_2)$. So, $y_2\sigma = \alpha y_1^2 y_2^n + \beta y_2^{n+1} = (\alpha + \beta)y_2^{n+1}$, and since $y_2\sigma$ is nonzero ($y_2\sigma = y_1^{2^s-2}y_2^{n-2^{s-1}+2} \neq 0$), we conclude that σ is either equal to $y_1^2 y_2^{n-1}$ or to y_2^n.

If $\sigma = y_1^2 y_2^{n-1}$, then using the fact that $y_1^3 y_2^{n-1} = 0$ in $H^*(F(1, 2, n); \mathbb{Z}_2)$ ($g_{n-1} = y_1^3 y_2^{n-1}$; [10, page 118]) and (4.7), we have that

$$x^{n+2}\sigma = xy_1^2(x^n y_1 y_2^{n-1} + x^{n-1}y_1^2 y_2^{n-1} + x^{n-1}y_2^n) = x^n y_1^2 y_2^n = x^n y_2^{n+1} \neq 0,$$

by Corollary 4.3. If $\sigma = y_2^n$, then by (4.6), we obtain that

$$x^{n+2}\sigma = x^{n+1}y_1 y_2^n = x^n y_1^2 y_2^n = x^n y_2^{n+1} \neq 0.$$

So, in either case, $x^{n+2}y_1^{2^s-2}y_2^{n-2^{s-1}+1} = x^{n+2}\sigma \neq 0$, and consequently, $\mathrm{cup}_{\mathbb{Z}_2}(F(1, 2, n)) \geqslant 2n + 2^{s-1} + 1$.

In order to prove the opposite inequality, we note first that, since the cohomology algebra $H^*(F(1, 2, n); \mathbb{Z}_2)$ is a quotient of the polynomial algebra $\mathbb{Z}_2[x, y_1, y_2]$, there is a monomial $x^k y_1^a y_2^b$ which realizes the cup-length. Since $\mathrm{ht}(x) = n + 2$ and $\mathrm{ht}(y_1) = 2^s - 2$, we have that $k \leqslant n + 2$ and $a \leqslant 2^s - 2$. Also, the dimension of the class $x^k y_1^a y_2^b$ is $k + a + 2b$ and $\dim(F(1, 2, n)) = 3n + 2$, so the inequality $k + a + 2b \leqslant 3n + 2$ must hold as well. Summing the previous three inequalities leads to $2(k + a + b) \leqslant 4n + 2^s + 2$, that is, $k + a + b \leqslant 2n + 2^{s-1} + 1$. This concludes the proof of the theorem. □

https://doi.org/10.1017/S1446788713000657 Published online by Cambridge University Press

Acknowledgement

The authors would like to thank Marko Stošić for suggesting to determine Groebner bases for ideals $I_{1,2,n}$.

References

[1] W. W. Adams and P. Loustaunau, *An Introduction to Gröbner Bases*, Graduate Studies in Mathematics, 3 (American Mathematical Society, Providence, RI, 1994).
[2] D. O. Ajayi and S. A. Ilori, 'Nonembeddings of the real flag manifolds $RF(1, 1, n-2)$', *J. Aust. Math. Soc. (Ser. A)* **66** (1999), 51–55.
[3] T. Becker and V. Weispfenning, *Gröbner Bases: A Computational Approach to Commutative Algebra*, Graduate Texts in Mathematics (Springer, New York, 1993).
[4] A. Borel, 'La cohomologie mod 2 de certains espaces homogènes', *Comm. Math. Helv.* **27** (1953), 165–197.
[5] B. Buchberger, 'A theoretical basis for the reduction of polynomials to canonical forms', *ACM SIGSAM Bull.* **10/3** (1976), 19–29.
[6] H. Hiller, 'On the cohomology of real Grassmannians', *Trans. Amer. Math. Soc.* **257** (1980), 521–533.
[7] J. Korbaš and J. Lörinc, 'The $\mathbb{Z}_2$-cohomology cup-length of real flag manifolds', *Fund. Math.* **178** (2003), 143–158.
[8] K. Y. Lam, 'A formula for the tangent bundle of flag manifolds and related manifolds', *Trans. Amer. Math. Soc.* **213** (1975), 305–314.
[9] J. W. Milnor and J. D. Stasheff, *Characteristic Classes*, Annals of Mathematics Studies, 76 (Princeton University Press, New Jersey, 1974).
[10] Z. Z. Petrović and B. I. Prvulović, 'On Groebner bases and immersions of Grassmann manifolds $G_{2,n}$', *Homology Homotopy Appl.* **13**(2) (2011), 113–128.
[11] B. J. Sanderson, 'Immersions and embeddings of projective spaces', *Proc. Lond. Math. Soc.* **14** (1964), 137–153.
[12] R. E. Stong, 'Cup products in Grassmannians', *Topology Appl.* **13** (1982), 103–113.
[13] R. E. Stong, 'Immersions of real flag manifolds', *Proc. Amer. Math. Soc.* **88** (1983), 708–710.

ZORAN Z. PETROVIĆ, University of Belgrade, Faculty of Mathematics,
Studentski trg 16, Belgrade, Serbia
e-mail: zoranp@matf.bg.ac.rs

BRANISLAV I. PRVULOVIĆ, University of Belgrade, Faculty of Mathematics,
Studentski trg 16, Belgrade, Serbia
e-mail: bane@matf.bg.ac.rs