1. Introduction and main results. We start by recalling some of the notations and definitions used throughout this paper. Let \( C[a, b] \) and \( C^k[a, b] \) be, respectively, the sets of all continuous and \( k \)-times continuously differentiable functions on \([a, b]\), and let \( L_p[a, b], 0 < p < \infty \), be the set of measurable functions on \([a, b]\) such that \( \|f\|_{L_p[a, b]} < \infty \), where

\[
\|f\|_{L_p[a, b]} := \left( \int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}}.
\]

Throughout this paper \( L_\infty[a, b] \) is understood as \( C[a, b] \) with the usual uniform norm, to simplify the notation. We also denote by \( W^k_p[a, b], p \geq 1 \), the set of all functions \( f \) on \([a, b]\) such that \( f^{(k-1)} \) are absolutely continuous and \( f^{(k)} \in L_p \), and by \( P_n \) the set of all polynomials of degree \( \leq n \). The \( m \)-th symmetric difference of \( f \) is given by

\[
\Delta^m_n(f, x, [a, b]) := \sum_{i=0}^m \left( \frac{m!}{i!} \right)(-1)^{m-i}f(x - \frac{m}{2} + ih), \quad \text{if } x \pm \frac{m}{2} \in [a, b],
\]


\[
\text{otherwise.}
\]

Then the \( m \)-th (usual) modulus of smoothness of \( f \in L_p[a, b] \) is defined by

\[
\omega^m(f, \cdot, [a, b])_p := \sup_{0 < h \leq 1} \|\Delta^m_n(f, \cdot, [a, b])\|_{L_p[a, b]}.
\]

We will also use the so-called \( \tau \)-modulus, an averaged modulus of smoothness, defined for all bounded measurable functions on \([a, b]\) by

\[
\tau^m(f, \cdot, [a, b])_p := \|\omega^m(f, \cdot, \cdot)\|_{L_p[a, b]},
\]

where

\[
\omega^m(f, x, t) := \sup\{\|\Delta^m_n(f, y)\| : y \pm mh/2 \in [x - mt/2, x + mt/2] \cap [a, b]\}
\]

is the \( m \)-th local modulus of smoothness of \( f \). (We set \( \tau^m(f, \cdot, [a, b])_p := \infty \) if the function \( f \) is unbounded.) From the definition one can easily see that

\[
(1.1) \quad \tau^m(f, \cdot, [a, b])_\infty = \omega^m(f, t, [a, b])_\infty.
\]
The following relationship between the $\omega$- and $\tau$-moduli holds for any $f \in W_p^1[a, b]$ and $1 \leq p \leq \infty$ (Sendov and Popov, [29, Theorem 1.5])

$$\tau^m(f, t, [a, b])_p \leq C_m \omega^{m-1}(f', t, [a, b])_p, \quad t \geq 0.$$  

(1.2)

If the interval $[-1, 1]$ is used in any of the above notations, it will be omitted for the sake of simplicity, for example,

$$\|f\|_p := \|f\|_{L_p[-1, 1]}, \quad \omega^m(f, t)_p := \omega^m(f, t, [-1, 1])_p.$$

The moduli $\omega$ and $\tau$ measure the smoothness of $f$ over the interval uniformly. It is well known that polynomials approximate better near the endpoints of the interval than in the middle, and this leads to either pointwise estimates (if $p = \infty$), or the introduction of “non-uniform” moduli of smoothness. The pointwise estimates for constrained approximation that we obtain in this paper are given in terms of $\omega^m(f, \Delta_n(x))_\infty$, where $\Delta_n(x) := n^{-1/2} \sqrt{1 - x^2}$. The “non-uniform” modulus that we use is the $m$-th Ditzian-Totik modulus of smoothness, defined for $f \in L_p[-1, 1]$ by

$$\omega^m_\varphi(f, t)_p := \sup_{0 < j < t} \|\Delta^j_{\varphi(n)}(f, \cdot, [-1, 1])\|_p,$$

with $\varphi(x) := \sqrt{1 - x^2}$. We have

$$\omega_\varphi^m(f, t)_p \leq \omega^m(f, t)_p \leq \tau^m(f, t)_p \leq 2^\tau \omega^m(f, t)_\infty, \quad 1 \leq p \leq \infty$$

and

$$\omega_\varphi^m(f, t)_p \leq \omega^m(f, t)_p \leq 2^\tau \omega^m(f, t)_\infty, \quad 0 < p < 1.$$

Let $Y_s := \{y_1, \ldots, y_s \mid y_0 := -1 < y_1 < y_2 < \cdots < y_s < 1 =: y_{s+1}\}, s \geq 0$. We denote by $\Delta^0(Y_s)$ the set of all functions $f$ such that $(-1)^{s-k}f(x) \geq 0$ for $x \in [y_k, y_{k+1}]$, $k = 0, \ldots, s$, i.e., those that have $0 \leq s < \infty$ sign changes at the points in $Y_s$ and are nonnegative near 1. In particular, $\Delta^0 := \Delta^0(Y_0)$ denotes the set of all nonnegative functions on $[-1, 1]$. Functions $f$ and $g$ which belong to the same class $\Delta^0(Y_s)$ are said to be copositive.

Copheritive approximation is the approximation of functions $f$ from $\Delta^0(Y_s)$ class by polynomials and splines that are copositive with $f$. For $f \in L_p[-1, 1]$ let

$$E_n(f)_p := \inf_{P_n \in P_n} \|f - P_n\|_p$$

denote the degree of unconstrained approximation, and let

$$E_n^{(0)}(f, Y_s)_p := \inf_{P_n \in P_n \cap \Delta^0(Y_s)} \|f - P_n\|_p$$

be the degree of copositive polynomial approximation of $f$. (In particular, $E_n^{(0)}(f)_p := E_n^{(0)}(f, Y_0)_p := \inf_{P_n \in P_n \cap \Delta^0} \|f - P_n\|_p$ is the degree of positive approximation.)
The best onesided approximation of $f$ by means of algebraic polynomials $P_n \in P_n$ in $L_p$-metric is given by

$$\tilde{E}_n(f)_p := \inf \{\|P - Q\|_p; P, Q \in P_n \text{ and } P(x) \geq f(x) \geq Q(x), -1 \leq x \leq 1\}.$$ 

A natural extension of (co)positive and onesided approximations is the concept of so-called intertwining (or co-onesided) approximation.

**Definition.** For the set $Y_s := \{y_1, \ldots, y_s \mid y_0 := -1 < y_1 < y_2 < \cdots < y_s < 1; y_{s+1}\}$ the best intertwining polynomial approximation of a function $f \in L_p[-1, 1]$ is given by

$$\tilde{E}_n(f, Y_s)_p := \inf \{\|P - Q\|_p; P, Q \in P_n, P - f \in \Delta^0(Y_s) \text{ and } f - Q \in \Delta^0(Y_s)\}.$$ 

We call $\{P, Q\}$ an intertwining pair of polynomials for $f$ with respect to $Y_s$ if $P - f, f - Q \in \Delta^0(Y_s)$.

Clearly, in the case $s = 0$ the above definition becomes the definition of the best onesided polynomial approximation: $\tilde{E}_n(f, Y_0)_p = E_n(f)_p$.

We have the following relationships among the above quantities:

- If $f(x) \geq 0, x \in [-1, 1]$, then $E_n^{(0)}(f)_p \leq \tilde{E}_n(f)_p$.
- If $f \in \Delta^0(Y_s)$, then $E_n^{(0)}(f, Y_s)_p \leq \tilde{E}_n(f, Y_s)_p$.

1.1. **Positive and onesided approximations.** First of all, if $p = \infty$ (i.e., in the uniform metric) the uniform estimates for positive and onesided approximations are not of much interest since

$$E_n(f)_\infty \leq \tilde{E}_n(f)_\infty \leq 2E_n(f)_\infty$$

for any $f \in C[-1, 1]$, and

$$E_n(f)_\infty \leq E_n^{(0)}(f)_\infty \leq 2E_n(f)_\infty$$

for $f \in C[-1, 1] \cap \Delta^0$.

At the same time, if $1 \leq p < \infty$, then the situation is quite different. It was shown by Stojanova [31] (see also Hristov and Ivanov [9], [10], and [11]) that for any bounded and measurable function on $[-1, 1]$, and $m \in N$,

$$\tilde{E}_n(f)_p \leq C(m)\tau^m(f, n^{-1})_p, \quad 1 \leq p < \infty.$$ 

In fact, the estimates obtained in [31] were given in terms of $\tau^m(f, \Delta_n(x))_p$, which is smaller than $\tau^m(f, n^{-1})_p$, and which is, in a sense, “the right” quantity for estimation of degree of onesided approximation.

We also remark that $\tau$ is the “correct” modulus in (1.5) (i.e., it can not be replaced by $\omega$ or $\omega_*$), since the estimate $E_n(f)_p \leq C\|f\|_p$, certainly, can not be correct for all $f \in L_p[-1, 1], p < \infty$. To see this it is sufficient to consider the function $f$ such that $f(0) = 1$ and $f(x) = 0, x \neq 0$. Then $\|f\|_p = 0$ and $\tilde{E}_n(f)_p > 0$. 

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An immediate consequence of (1.5) and (1.2) is the fact that if \( f \in W^1_p[-1, 1] \), then
\[
E_n(f)_p \leq C(m)n^{-1} \omega^m(f', n^{-1})_p, \quad 1 \leq p < \infty.
\]
Moreover, it was shown in [31] that for any \( f \in W^1_p[-1, 1] \)
\begin{equation}
E_n(f)_p \leq Cn^{-1}E_{n-1}(f')_p, \quad 1 \leq p < \infty.
\end{equation}
(Though, the estimate (1.6) was not explicitly stated in [31], it immediately follows from the proof of Corollary 1 in that paper.)

As for positive approximation, it was shown in [13] (see also Ivanov [17]) that for any \( f \in L_p[-1, 1] \cap \Delta^0 \) and \( 0 < p < \infty \)
\[
E_n^0(f)_p \leq C\omega_p(f, n^{-1})_p.
\]
At the same time, for every \( n \in \mathbb{N} \), \( 0 < p < \infty \) and \( A > 0 \) there exists a function \( f \in L_p[-1, 1] \cap \Delta^0 \) such that
\[
E_n^0(f)_p > A\omega(f, 1)_p.
\]
In this paper we show, in particular, that pointwise estimates in terms of \( \omega^m(f, \Delta_n(x))_\infty \) are true for onesided (and, therefore, for positive) approximation in \( C[-1, 1] \), thus, in a sense, completing the investigation of these types of approximation. (Of course, some improvements are possible if measures of smoothness different from those considered here are used.)

**Theorem 1 (Onesided Approximation).** Let \( f \in C[-1, 1] \) and \( m \in \mathbb{N} \). Then for every \( n \geq m - 1 \) there exist polynomials \( P, Q \in P_n \) such that \( P(x) \geq f(x) \geq Q(x) \), \( -1 \leq x \leq 1 \), and
\begin{equation}
|P(x) - Q(x)| \leq C(m)\omega^m(f, \Delta_n(x))_\infty.
\end{equation}

**Corollary 2 (Positive Approximation).** Let \( m \in \mathbb{N} \) and \( f \in C[-1, 1] \) be such that \( f(x) \geq 0 \), \( -1 \leq x \leq 1 \). Then for every \( n \geq m - 1 \) there exist a polynomial \( P \in P_n \), \( P(x) \geq 0 \), \( -1 \leq x \leq 1 \) satisfying
\begin{equation}
|f(x) - P(x)| \leq C(m)\omega^m(f, \Delta_n(x))_\infty.
\end{equation}
While preparing this paper for publication, the authors learned that Corollary 2 was also recently proved by G. Dzyubenko [8].

The above results can be summarized as follows.

<table>
<thead>
<tr>
<th>Onesided approximation</th>
<th>p = \infty</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f \in C )</td>
<td>( \exists P_n, Q_n: P_n(x) \geq f(x) \geq Q_n(x), ) such that (</td>
</tr>
<tr>
<td>( 1 \leq p &lt; \infty )</td>
<td></td>
</tr>
<tr>
<td>( f \in L_p )</td>
<td>( \tilde{E}_n(f)_p \leq Cr^m(f, n^{-1})_p )</td>
</tr>
<tr>
<td>( f \in W^1_p )</td>
<td>( \tilde{E}_n(f)_p \leq C</td>
</tr>
<tr>
<td>( f \in W^1_p )</td>
<td>( \tilde{E}_n(f)<em>p \leq Cn^{-1}E</em>{n-1}(f')_p )</td>
</tr>
</tbody>
</table>

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Positive approximation

| $p = \infty$ | $f \in \mathbb{C}$ | $\exists P_n, P_n(x) \geq 0, \text{ such that } |f(x) - P_n(x)| \leq C\omega^m(f, \Delta_n(x))_{\infty}$ | Corollary 2, see also Dzyubenko [8] |
|--------------|-----------------|---------------------------------|---------------------------------|
| $1 \leq p < \infty$ | $f \in L_p$ | $E_{n}^{(0)}(f)_p \leq C\omega^m(f, n^{-1})_p$ | Stojanova [31] |
|               |                 | $E_{n}^{(0)}(f)_p \leq C\omega^p(f, n^{-1})_p$ | [13], see also Ivanov [17] |
|               |                 | $E_{n}^{(0)}(f)_p \leq C\omega^2(f, 1)_p$ | [13] |
|               | $f \in W^1_p$ | $E_{n}^{(0)}(f)_p \leq Cn^{-1}E_{n-1}(f^t)_p$ | Stojanova [31] |

1.2. Copositive and intertwining approximations. Copositive approximation was extensively studied in recent years. A number of results were obtained (see [8], [12], [13], [14], [15], [16], [20], [24], [25], [26], [28], [32], [33], [34], for example). Recently, Kopotun [20] showed that if $f \in \mathbb{C}[-1, 1] \cap \Delta^0(Y_s)$, then

$$E_{n}^{(0)}(f, Y_s)_\infty \leq C(Y_s)\omega^3(f, n^{-1})_{\infty}, \quad n \geq 2.$$  

(1.9)

(See Hu and Yu [16], Hu, Leviatan and Yu [14, 15] for weaker but earlier results.) This is the best possible estimate in the sense that $\omega^3$ in (1.9) can not be replaced by $\omega^4$ (Zhou [33]). If $f$ is continuously differentiable, Hu, Leviatan and Yu [15] gave an estimate in terms of higher order modulus of $f$, i.e., for any function $f \in \mathbb{C}^1[-1, 1] \cap \Delta^0(Y_s)$

$$E_{n}^{(0)}(f, Y_s)_\infty \leq Cn^{-1}E_{n-1}(f^t)_p\omega^m(f^t, n^{-1})_p, \quad n \geq C_1,$$

(1.10)

where the constants $C_1$ and $C_2$ depend only on $m$ and $Y_s$. In fact, using a slight modification of the proof in [15], one can show that $\omega^m$ in (1.10) can be replaced by $\omega^m_2$. In this paper we use a different method to show that, and also obtain pointwise estimates improving (1.10) (see Corollary 6).

As for $f \in L_p \cap \Delta^0(Y_s)$, the authors [12, 13] have shown that the copositive approximation is quite different from other kinds of constrained approximation such as monotone or convex approximation, with which we have seen similarities between approximations in $L_p$ and in $\mathbb{C}$. For example, DeVore, Hu and Leviatan [3] recently proved that the degree of convex polynomial approximation in $L_p$, $0 < p < \infty$, has order $\omega^3(f, n^{-1})_p$, which is a natural extension of Kopotun’s result in [19] for the space $\mathbb{C}$. By contrast, if $0 < p < \infty$, the degree of copositive polynomial approximation is merely given by (see [13])

$$E_{n}^{(0)}(f, Y_s)_p \leq C\omega_p(f, n^{-1})_p,$$

(1.11)

which is significantly lower than (1.9). We will show in this paper that $\omega_p$ in (1.11) is the best possible in the sense that it can not be replaced even by $\omega^2(f, 1)_p$ (see also Zhou [34], where a similar result was proved for $1 < p < \infty$), and that an analogue of this holds true for splines. (As was shown in [13] and mentioned above, this is also the case for positive approximation.)
Also, we extend our investigation of copositive approximation to the Sobolev spaces $W^k_p$, $p \geq 1$, and consider intertwining approximation in $W^k_p$ as well, obtaining the estimates which are exact in the sense of the orders of moduli of smoothness. We prove that if $f \in W^2_p$, then the degree of intertwining (and, hence, copositive) polynomial approximation has the order $n^{-2} \omega^m(f'', n^{-1})_p$ for any positive integer $m$. If $f$ is merely in $W^1_p$, then it deteriorates to $n^{-1} \omega^m(f', n^{-1})_p$, which is the best in the sense that one can not replace $n^{-1} \omega^m(f', n^{-1})_p$ by $\|f\|_p$. At the same time, the degree of copositive approximation does not deteriorate that bad. In particular, the estimate $E_n^0(f, Y_p) \leq Cn^{-1} \omega^2(f', n^{-1})_p$ holds true. (This estimate is exact in the sense that $n^{-1} \omega^2(f', n^{-1})_p$ cannot be replaced by $\omega^2(f', 1)_p$.) Analogues of these again are true for splines.

The investigation of constrained approximation in $L_p$, $0 < p < 1$ quasi-norm is not our goal in this paper. (We prove some of our results in the case $p < 1$ as well. However, it is done only if the proof is similar to that for $p \geq 1$, and no extra effort or discussions are required.) It is known that for unconstrained approximation the usual Jackson type estimates, involving the first derivatives of functions, are no longer valid if $p < 1$ (see Kopotun [21], for example). However, it does not guarantee that the same is true in constrained case, since the functions satisfying some shape preserving constraint form a proper subset of $W^k_p$. In fact, it was shown in [21] that for convex polynomial approximation one can get estimates which are not true in the general (unconstrained) case. At the same time, the restriction $f \in \Delta^0(Y_p)$, for example, is not as “strong” as $f \in \Delta^2$ (i.e., $f$ is convex), and does not eliminate those functions $f$ which “bring anomalous properties” into $L_p$ for $p < 1$. (See [21] for further discussions. We only mention that the proof of Theorem 3 of [21] can be used to show that for every $A > 0$, $B > 0$, $0 < p < 1$, $n \in \mathbb{N}$ and a set $Y_p$, there exists a function $f \in AC[-1, 1] \cap \Delta^0(Y_p)$ such that $E_n^0(f, Y_p) > An^p \|f\|_p$.)

We now state our results on copositive and intertwining approximations, and begin with a theorem on splines. We give local estimates in the theorems because they are stronger than the corresponding global estimates (also, this is the form needed in the proof of theorems on polynomial approximation). The global estimates follow immediately from the inequality (which can be shown directly from the definition of the $\tau$-modulus)

$$
\sum I \tau^m(f, t, I, I_p^p) \leq k \tau^m(f, t, I_p^p),
$$

where $\cup I_i = I$, and each $x$ in the interval $I$ is contained in at most $k$ subintervals $I_i$. Moreover, if $f$ is smooth, and the partition $I = \cup I_i$ is (close to) the one formed by zeros of Chebyshev’s polynomial $cos(n \arccos x)$, then the global estimates can be further improved since

$$
\sum I \tau^m(f, |I|, I, I_p^p) \leq C \sum |I| \omega^{m-1}(f', |I|, I_p^p) \leq C n^{-p} \omega^{m-1}(f', n^{-1})_p.
$$

Let $T_n := \{z_0, \ldots, z_n \mid -1 := z_0 < z_1 < \cdots < z_{n-1} < z_n := 1\}$, $n \geq 1$, be a given knot sequence on $[-1, 1]$, and set $z_i := -1, i < 0$, and $z_i := 1, i > n$. For $i = -1, \ldots, n$, let $J_i := [z_i, z_{i+1}]$. With this notation we have
where $I_j$ is an interval such that $J_i$ contains $I_j$ for some $i$. Consequently, if in addition $f \in W_p^m$, then

$$
\|\tilde{S} - S\|_{L_p(J_i)} \leq C |J_i|^{r-1}(q', |J_i|, J_i)_p,
$$

where $C$ is a constant depending on the maximum ratio of the knot sequence and direct estimates for constrained spline approximation.

For example, if $f$ is a polynomial on $[0, 1]$, then, clearly, no polynomial $P(x)$ satisfies $P(x) \leq S(x) = 0$ for $x < 0$, and $P(x) \geq S(x) = 1$ for $x \geq 0$. (In other words, an intertwining pair of polynomials for $S$ with respect to $\{y_i\}$ simply does not exist in this case.)

As shown in Section 3 the following results are almost straightforward consequences of the above theorem and direct estimates for constrained spline approximation.
THEOREM 5 (INTERTWining POLYNOMIAL APPROXIMATION IN $C^1[−1, 1]$). Let $f \in C^1[−1, 1]$, $m \in \mathcal{N}$, and $Y_s := \{y_1, \ldots, y_s \mid y_0 := -1 < y_1 < y_2 < \cdots < y_s < 1 =: y_{s+1}\}$, $s \geq 0$. Then
\begin{equation}
E_n(f, Y_s)_{\infty} \leq C(m, s)n^{-1}\omega_m^s(f', n^{-1})_{\infty}, \quad n \geq C(Y_s).
\end{equation}
Also, there exists an intertwining pair of polynomials $\{P_n, Q_n\} \subset P_n$ such that
\begin{equation}
|P_n(x) - Q_n(x)| \leq C(m, s)^{n-1}\omega_m^s(f', n^{-1})_{\infty}, \quad n \geq C(Y_s).
\end{equation}

COROLLARY 6 (COPOSITIVE POLYNOMIAL APPROXIMATION IN $C^1[−1, 1]$). Let $f \in C^1[−1, 1] \cap \Delta^0(Y_s)$, $Y_s := \{y_1, \ldots, y_s \mid y_0 := -1 < y_1 < y_2 < \cdots < y_s < 1 =: y_{s+1}\}$, $s \geq 0$, $m \in \mathcal{N}$. Then
\begin{equation}
E_n^0(f, Y_s)_{\infty} \leq C(m, s)n^{-1}\omega_m^s(f', n^{-1})_{\infty}, \quad n \geq C(Y_s).
\end{equation}
Also, there exists a polynomial $P_n \in P_n \cap \Delta^0(Y_s)$ such that
\begin{equation}
|f(x) - P_n(x)| \leq C(m, s)^{n-1}\omega_m^s(f', n^{-1})_{\infty}, \quad n \geq C(Y_s).
\end{equation}

The following result does not hold for intertwining approximation (see Theorem 13), and, therefore, copositive case is considered separately.

THEOREM 7 (COPOSITIVE POLYNOMIAL APPROXIMATION IN $C[−1, 1]$). Let $f \in C[−1, 1] \cap \Delta^0(Y_s)$, $Y_s := \{y_1, \ldots, y_s \mid y_0 := -1 < y_1 < y_2 < \cdots < y_s < 1 =: y_{s+1}\}$, $s \geq 0$. Then there is a polynomial $P_n \in P_n$, copositive with $f$, such that
\begin{equation}
|f(x) - P_n(x)| \leq C(s)^{n-1}\Delta_n^0(f', n^{-1})_{\infty}, \quad n \geq C(Y_s).
\end{equation}

Now, we state the results on approximation of functions in $L_p[−1, 1]$, $1 \leq p < \infty$ norm.

THEOREM 8 (INTERTWining POLYNOMIAL APPROXIMATION, $1 \leq p < \infty$). Let $f \in W^1_p[−1, 1]$, $1 \leq p < \infty$, $m \in \mathcal{N}$, and $Y_s := \{y_1, \ldots, y_s \mid y_0 := -1 < y_1 < y_2 < \cdots < y_s < 1 =: y_{s+1}\}$, $s \geq 0$. Then
\begin{equation}
E_n(f, Y_s)_p \leq C(m, s)n^{-1}r_m^s(f', n^{-1})_p, \quad n \geq C(Y_s).
\end{equation}
Moreover, if $f$ is also in $W^2_p[−1, 1]$, then
\begin{equation}
\tilde{E}_n(f, Y_s)_p \leq C(m, s)n^{-2}r_m^{s-1}(f'', n^{-1})_p, \quad n \geq C(Y_s).
\end{equation}

COROLLARY 9 (COPOSITIVE POLYNOMIAL APPROXIMATION, $1 \leq p < \infty$). Let $f \in W^1_p[−1, 1] \cap \Delta^0(Y_s)$, $1 \leq p < \infty$. Then
\begin{equation}
E_n^0(f, Y_s)_p \leq C(m, s)n^{-1}r_m^s(f', n^{-1})_p, \quad n \geq C(Y_s).
\end{equation}
Moreover, if $f$ is also in $W^2_p[−1, 1]$, then
\begin{equation}
\tilde{E}_n^0(f, Y_s)_p \leq C(m, s)n^{-2}r_m^{s-1}(f'', n^{-1})_p, \quad n \geq C(Y_s).
\end{equation}
Again, the following estimates are true for copositive approximation and false in the intertwining case.

**Theorem 10 (Copositive Polynomial Approximation, 1 ≤ p < ∞).** Let $f ∈ L_p[-1, 1] \cap Δ^0(Y_s)$, 1 ≤ p < ∞. Then
\[
E_n^0(f, Y_s)_p ≤ C(s)n^3(f, n^{-1})_p, \quad n ≥ C(Y_s).
\]
Moreover, if $f ∈ W_p^1[-1, 1]$, then
\[
E_n^0(f, Y_s)_p ≤ C(s)n^{-1}(f', n^{-1})_p, \quad n ≥ C(Y_s).
\]

As mentioned earlier, (1.28) is the best one can get for $f ∈ W_p^1$. We back this assertion by proving the following counterexample.

**Theorem 11.** For every $n ∈ N$, 0 < p < ∞, 0 < ε ≤ 1 and $A > 0$, there exists a monotone increasing function $f ∈ C^∞[-1, 1]$ with $f(0) = 0$ such that for every polynomial $P_n ∈ P_n$ with $P_n(0) = 0$ and $P_n(x) ≥ 0$ for $x ∈ [0, ε]$, the following inequality holds:
\[
\|f - P_n\|_{L_p[0, ε]} > A\omega^3(f', 1)_p.
\]

It also follows from Theorem 11 that the estimate
\[
\|f - S_n\|_p ≤ C\omega^2(f', δ)_p,
\]
where $δ$ is the mesh size of the knot sequence $T_n$, is the best possible for copositive spline approximation in the $L_p$ metric. Also, (1.29) and (1.2) imply that $τ^3$ in (1.27) cannot be replaced by $τ^4$.

In the second counterexample, we show that (1.11) and its spline analogue proved in [12, 13] are best possible.

**Theorem 12.** For every $n ∈ N$, 0 < p < ∞, 0 < ε ≤ 1 and $A > 0$, there exists a function $f ∈ C^∞[-1, 1]$ satisfying $sf(x) ≥ 0, x ∈ [-1, 1]$, and such that for every polynomial $P_n ∈ P_n$ with $P_n(0) ≥ 0$, the following inequality holds:
\[
\|f - P_n\|_{L_p[0, ε]} > A\omega^2(f, 1)_p.
\]

The following theorem shows that the estimates in terms of $ω(f, 1)_p$ or $τ(f, 1)_p$ are impossible for intertwining approximation. (Note, that $ω(f, 1)_p ≤ C∥f∥_p$ and $τ(f, 1)_p ≤ C∥f∥_p$.)

**Theorem 13.** For every $n ∈ N$, 0 < p ≤ ∞, 0 < ε ≤ 1 and $A > 0$, there exists a monotone increasing function $f ∈ C^∞[-1, 1]$ with $f(0) = 0$ such that for every polynomial $P_n ∈ P_n$ with $P_n(0) = 0$ and $P_n(x) ≥ f(x), 0 ≤ x ≤ ε$, the following inequality holds:
\[
\|f - P_n\|_{L_p[0, ε]} > A∥f∥_{L_p[-1, 1]}.
\]

Under the same conditions except that 0 < p < ∞, there also exists such a function f that
\[
\|f - P_n\|_{L_p[0, ε]} > A∥f∥_{L_p[-1, 1]}.
\]

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Finally, we summarize all the results discussed in this section in the form of the following two tables.

### Intertwining approximation

| $p = \infty$ | $f \in \mathbb{C}$ | $E_n(f, Y_s) \leq C ||f||_{\infty}$ | Theorem 13 |
|--------------|-------------------|---------------------------------|------------|
| $f \in \mathbb{C}$ | $E_n(f, Y_s) \leq C n^{-1} \omega_\infty(f', n^{-1})_{\infty}$ | Theorem 5 |
| $f \in \mathbb{C}'$ | $\exists$ an intertwining pair $\{P_n, Q_n\}$ for $f$ satisfying $|P_n(x) - Q_n(x)| \leq C \Delta_n(x) \omega(m)(f', \Delta_n(x))_{\infty}$ | Theorem 5 |
| $1 \leq p < \infty$ | $f \in L_p$ | $E_n(f, Y_s) \leq C ||f||_p$ | Theorem 13 |
| | $E_n(f, Y_s) \leq Cr(f, 1)_p$ | Theorem 13 and ineq. (1.2) |
| | $f \in W^1_p$ | $E_n(f, Y_s) \leq C ||f'||_p$ | Theorem 13 |
| | $E_n(f, Y_s) \leq C n^{-1} \omega_\infty(f', n^{-1})_p$ | Theorem 8 |
| | $f \in W^2_p$ | $E_n(f, Y_s) \leq C n^{-2} \omega_\infty(f'', n^{-1})_p$ | Theorem 8 |

### Copositive approximation

<table>
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<th>$p = \infty$</th>
<th>$f \in \mathbb{C}$</th>
<th>$E_n^0(f, Y_s) \leq C \omega^3(f, n^{-1})_{\infty}$</th>
<th>Kopotun [20]</th>
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<tr>
<td></td>
<td>$\exists P_n$, copositive with $f$, such that $</td>
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<td>\leq C \omega^3(f, \Delta_n(x))_{\infty}$</td>
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<td></td>
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<td>$E_n^0(f, Y_s) \leq C n^{-1} \omega^m(f', n^{-1})_{\infty}$</td>
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</tr>
<tr>
<td>$f \in \mathbb{C}'$</td>
<td>$\exists P_n$, copositive with $f$, such that $</td>
<td>f(x) - P_n(x)</td>
<td>\leq C \Delta_n(x) \omega^m(f', \Delta_n(x))_{\infty}$</td>
</tr>
<tr>
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<td></td>
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<tr>
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The rest of this paper is organized as follows. The affirmative results for splines, including the proof of Theorem 3, are given in Section 2, and those for polynomials in Section 3. The last section is devoted to proving Theorems 11–13.

2. Constrained spline approximation.

2.1. Intertwining spline approximation.

Proof of Theorem 3. We only consider the case for $1 \leq p < \infty$. The proof for $p = \infty$ is almost identical. We use some idea in the proof of [16, Theorem 3]. Let $d := 2(r - 1)^2$, $m := [(n + d - 1)/d]$ and $\bar{z}_i := z_d i$. Note that $\bar{z}_i = -1$ for $i \leq 0$ and $\bar{z}_i = 1$ for $i \geq m$. We first construct overlapping polynomial pieces of degree $< m$ on the coarser partition $\mathcal{T}_n := \{\bar{z}_i\}_{i=0}^m$. We call the interval $[\bar{z}_i, \bar{z}_{i+1}]$ contaminated if $\bar{z}_j < y_j \leq \bar{z}_{i+1}$ for some point $y_j \in Y$. By assumption, there exists exactly one $y_j$ in each of the contaminated intervals $I_m, j = 1, \ldots, s$, and there is at least one non-contaminated interval between $I_m$ and $\bar{I}_m$, that is,

$$m_j < m_j + 2 \leq m_{j+1}, \quad j = 1, \ldots, s - 1.$$

For convenience we also denote $m_0 := -1$, and $m_{s+1} := m$. Note that there are no $y_j$'s between $I_m$ and $\bar{I}_m$, $j = 0, \ldots, s$.

If $m_{j+1} = m_j + 2$ (i.e., if there is only one non-contaminated interval between $I_m$ and $\bar{I}_m$), then the following construction is not needed, and the next two paragraphs can be skipped.

In the case $m_{j+1} > m_j + 2$, by Whitney’s Theorem for onesided polynomial approximation (see Theorem 2.6 of [12], but most of the credit goes to V. H. Hristov and K. G. Ivanov), on each of the intervals $[\bar{z}_i, \bar{z}_{i+2}], i = m_j + 1, \ldots, m_{j+1} - 2$, there exist two polynomials $P_i$ and $Q_i$ of degree $< r$ such that

$$P_i(x) \geq f(x) \geq Q_i(x), \quad \forall x \in [\bar{z}_i, \bar{z}_{i+2}]$$

$$\|P_i - Q_i\|_{L_p([\bar{z}_i, \bar{z}_{i+2}])} \leq C r^{0} f \left[ \bar{I}_i, \bar{I}_i, \bar{z}_{i+2} \right]_{p'}.$$

We define $p_i$ and $q_i$ on $[\bar{z}_i, \bar{z}_{i+2}]$ by $p_i := P_i$ and $q_i := Q_i$ if $(-1)^{r-i} > 0$, and $p_i := Q_i$ and $q_i := P_i$ if $(-1)^{r-i} < 0$. Hence, $(-1)^{r-i} (p_i(x) - f(x)) \geq 0$, $(-1)^{r-i} (q_i(x) - f(x)) \leq 0$, and

$$\|p_i - q_i\|_{L_p([\bar{z}_i, \bar{z}_{i+2}])} = \|P_i - Q_i\|_{L_p([\bar{z}_i, \bar{z}_{i+2}])} \leq C r^{0} f \left[ \bar{I}_i, \bar{I}_i, \bar{z}_{i+2} \right]_{p'} \leq C \left[ \bar{I}_i, \bar{I}_i, \bar{z}_{i+2} \right]_{p'}.$$

where, in the last step, we have used the inequality (see [29])

$$t^m(f, t)_{p} \leq C_{m} t^{m-1}(f', t)_{p}, \quad t \geq 0.$$

We should emphasize that when we speak of a polynomial on an interval, we mean the restriction of the polynomial to the interval, hence it is considered undefined outside.
Near each point \( y_j \), we construct local polynomials differently. More precisely, we approximate \( f' \) on \( [\tilde{z}_{m-1}, \tilde{z}_{m+2}] \), \( j = 1, \ldots, s \), from above and below by two polynomials \( \tilde{P}_{m_j} \) and \( \tilde{Q}_{m_j} \) of degree \( < r - 1 \). Then

\[
\tilde{P}_{m_j}(x) \geq f'(x) \geq \tilde{Q}_{m_j}(x), \quad \forall x \in [\tilde{z}_{m-1}, \tilde{z}_{m+2}]
\]

\[
\| \tilde{P}_{m_j} - \tilde{Q}_{m_j} \|_{\mathbb{L}^p[\tilde{z}_{m-1}, \tilde{z}_{m+2}]} \leq C r^{-1}(f', |\tilde{I}_{m_j}|, [\tilde{z}_{m-1}, \tilde{z}_{m+2}])_p.
\]

Define \( \tilde{P}_{m_j} := \tilde{P}_{m_j} \) and \( \tilde{Q}_{m_j} := \tilde{Q}_{m_j} \) if \((-1)^{r-j} > 0\), and \( \tilde{P}_{m_j} := \tilde{Q}_{m_j} \) and \( \tilde{Q}_{m_j} := \tilde{P}_{m_j} \) otherwise. It is easy to check that

\[
p_{m_j}(x) := \int_{y_j}^x \tilde{P}_{m_j}(t) \, dt + f(y_j)
\]

and

\[
q_{m_j}(x) := \int_{y_j}^x \tilde{Q}_{m_j}(t) \, dt + f(y_j)
\]

satisfy the inequalities

\[
(-1)^{r-j}(p_{m_j}(x) - f(x)) \text{ sgn}(x - y_j) \geq 0,
\]

\[
(-1)^{r-j}(q_{m_j}(x) - f(x)) \text{ sgn}(x - y_j) \leq 0,
\]

and

\[
|p_{m_j} - q_{m_j}|_p = \left\| \int_{y_j}^x (\tilde{P}_{m_j}(t) - \tilde{Q}_{m_j}(t)) \, dt \right\|_p \leq \left\| \tilde{I}_{m_j} \right\|_p C r^{-1}(f', |\tilde{I}_{m_j}|, [\tilde{z}_{m-1}, \tilde{z}_{m+2}])_p;
\]

where all norms are taken over \([\tilde{z}_{m-1}, \tilde{z}_{m+2}]\).

Having constructed the overlapping local polynomials which are "intertwining" with \( f \) and have the right approximation order, we now blend them for smooth spline approximants \( \tilde{S} \) and \( S \) on the original knot sequence \( T_a \) with the same properties. If both \( \tilde{I}_{m-1} \) and \( I_i \) are non-contaminated and \( i < m \), then \( p_{i-1} \) and \( p_i \) overlap on \( I_i \), which contains \( d - 1 \) interior knots from \( T_a \). By Beatson's Lemma (see Lemma 3.2 of [2]), there exists a spline \( \tilde{S}_i \) of order \( r \) on \( I_i \) on these knots that connects with \( p_{i-1} \) and \( p_i \) in a \( C^{r-1} \) manner at \( \tilde{z}_i = z_{d_i} \) and \( \tilde{z}_{i+1} = z_{d_{i+1}} \), respectively. Moreover, the graph of \( \tilde{S}_i \) lies between those of \( p_{i-1} \) and \( p_i \), and, hence, \( \text{sgn}(p_{i-1}(x) - f(x)) = \text{sgn}(p_i(x) - f(x)) = \text{sgn}(\tilde{S}_i(x) - f(x)) \), \( x \in I_i \).

Similarly, considering the overlapping polynomials \( q_{i-1} \) and \( q_i \), we construct a spline \( S_i \) satisfying

\[
\int_{I_i} |\tilde{S}_i - S_i|_p \leq 2^{p} (\int_{I_i} |p_{i-1} - q_{i-1}|^p + \int_{I_i} |p_i - q_i|^p).
\]

By (2.3) and a property of the \( \tau \)-modulus, this gives

\[
|\tilde{S}_i - S_i|_{\mathbb{L}^p(I_i)} \leq C r^{-1}(f', |\tilde{I}_{i}|, [\tilde{z}_{i-1}, \tilde{z}_{i+2}])_p \leq C |\tilde{I}_i| r^{-1}(f', |\tilde{I}_i|, [\tilde{z}_{i-1}, \tilde{z}_{i+2}])_p.
\]
The blending of overlapping polynomial pieces involving contaminated intervals can be done in the same way. The spline pieces ̃S_i and ̃S_j thus produced also satisfy the estimate above with a slightly larger interval in place of [ ̃I_i, ̃I_j] on the right-hand side, ([ ̃z_{i-2}, ̃z_{i+2}] at worst), which will make no difference in the rest of the proof.

We define the final spline ̃S on each ¯I_i as follows: if there is only one local polynomial p_i over ¯I_i, set ̃S to this polynomial; if there are two polynomials overlapping on ¯I_i, then there must be a blending local spline ̃S_i set ̃S to ̃S_i. It is clear from its construction that ̃S − f ∈ Δ̂0(Y_s) on the whole interval [−1, 1], and ̃S ∈ C^{r−2}. Similarly, we construct S ∈ C^{r−2} such that f − S ∈ Δ̂0(Y_s). Now, recall that all neighboring intervals ¯I_i := [z_i, z_{i+1}] in the original partition T_n are comparable in size and each interval ¯I_i = [z_d i, z_{d(i+1)}] contains no more than d such intervals. Therefore, (1.14) follows directly from (2.3) and (2.6). Now (1.15) is a direct consequence of (1.14) and (1.2).

The above proof also yields the following result on onesided spline approximation.

**Lemma 14.** Let f ∈ L_p[−1, 1], 1 ≤ p ≤ ∞, and let r ≥ 2 be an integer. Then there exist splines ̃S_n and S_n of order r on the knot sequence T_n such that ̃S_n(x) ≥ f(x) ≥ S_n(x), x ∈ [−1, 1], and for i = 1, . . . , n − 1

\[ \| ̃S_n − S_n \|_{L_p(J_i)} ≤ C r(f, [J_i], J_i)_p, \]

where C is a constant depending on r and the maximum ratio \( \rho := \max_{i=0}^{n-1} |J_i| / |J_i| \), and J_i is an interval such that J_i ⊆ [z_i−(r−1)², z_i+(r−1)²].

(Note that this lemma is probably known. For example, a similar result follows from Andreev, Popov and Sendov [1] and Popov [27].)

### 2.2. Copositive spline approximation

The next theorem is an improvement to Theorem 4 in Hu [12], where it was proved for C^1 quadratic splines on equidistant knots. The improvement is needed in the proof of Theorem 10.

**Theorem 15.** Let f ∈ L_p[−1, 1] ∩ Δ̂0(Y_s), 1 ≤ p < ∞, s ≥ 0, and let r ≥ 3 be an integer. Let T_n be a given knot sequence such that there are at least 4(r−1)² knots in each open interval (y_j, y_{j+1}), j = 0, . . . , s. Then there exists a spline S_n ∈ C^{r−2}[−1, 1] ∩ Δ̂0(Y_s) of order r on the knot sequence T_n such that for i = 1, . . . , n − 1

\[ \| f − S_n \|_{L_p(J_i)} ≤ C r^3(f, [J_i], J_i)_p, \]

where C and J_i are as in Theorem 3.

**Proof.** The theorem is proved in Hu [12] for the case of quadratic splines with equidistant knots. We now generalize it to splines of any order r ≥ 3 on unequally spaced knots. The proof is similar to that of Theorem 3. We shall use most of the notation and only indicate the differences of the two proofs.

The first two paragraphs in the proof of Theorem 3 also work here, except that we need to change (2.3) to:

\[ \| f − p_i \|_{L_p[ ̃I_i, ̃I_j]} ≤ C r^3(f, [ ̃I_i], [ ̃z_{i+2}, ̃z_{i+2}])_p ≤ C r^3(f, [ ̃I_i], [ ̃z_{i+2}, ̃z_{i+2}])_p, \]
The main change of the proof is near each point \( y_j \) of sign change of \( f, j = 1, \ldots, s \), where we now construct a local quadratic polynomial \( p_m \) on \([\bar{z}_{m-1}, \bar{z}_{m+2}]\) interpolating \( f \) at \( \bar{z}_{m-1}, y_j \) and \( \bar{z}_{m+2} \). Since (2.1) is now true for \( j = 0, \ldots, s \), these points are separated by at least \( d = 2(r - 1)^2 \) knots in \( T_n \). It is proved in [12] that \( p_m \) is copositive with \( f \) and satisfies
\[
||f - p_m||_{L_p([\bar{z}_{m-1}, \bar{z}_{m+2}])} \leq C r^2 \left(|I_m|, [\bar{z}_{m-1}, \bar{z}_{m+2}]\right). 
\]

The rest of the proof is analogous to that of Theorem 3.

3. Constrained polynomial approximation.

3.1. Intertwining approximation of truncated power functions. In this section, we consider intertwining polynomial approximation of the truncated power functions \((x - \Lambda)^2\), i.e., we construct polynomials \( P_1 \) and \( P_2 \) which sufficiently approximate \((x - \Lambda)^2\) and such that \( P_1(x) - (x - \Lambda)^2 \in \Delta^0(Y_s) \) and \((x - \Lambda)^2 - P_2(x) \in \Delta^0(Y_s)\). After that we use the well known procedure involving analytic representations of splines to construct an intertwining pair of polynomials for an arbitrary function \( f \).

Let
\[
x_j := \cos \frac{j\pi}{n}, \quad 0 \leq j \leq n; \quad \bar{x}_j := \cos \left(\frac{j\pi}{n} - \frac{\pi}{2n}\right), \quad 1 \leq j \leq n; \\
x_j^0 := \cos \left(\frac{j\pi}{n} - \frac{\pi}{4n}\right) \quad \text{if } j < n/2, \quad x_j^0 := \cos \left(\frac{j\pi}{n} - \frac{3\pi}{4n}\right) \quad \text{if } j \geq n/2; \\
L_j := [x_j, x_{j-1}], \quad h_j := x_{j-1} - x_j, \quad 1 \leq j \leq n
\]
(note that \( h_{j+1} < 3h_j \) and \( \Delta_n < h_j < 5\Delta_n \) for \( x \in L_j \)).

Also,
\[
t_j(x) := (x - x_j^0)^{-2} \cos^2 2n \arccos x + (x - \bar{x}_j)^{-2} \sin^2 2n \arccos x
\]
is the algebraic polynomial of degree \( 4n - 2 \) (see [30], for example).

We also denote
\[
\psi_j(x) := \frac{h_j}{|x - x_j| + h_j}, \quad \chi[a, b](x) := \begin{cases} 1 & \text{if } x \in [a, b], \\ 0 & \text{otherwise}, \end{cases} \\
\chi_j(x) := \chi[x_j, 1](x), \quad \text{sgn}(f(x)) := \begin{cases} -1 & \text{if } f(x) < 0, \\ 0 & \text{if } f(x) = 0, \\ 1 & \text{if } f(x) > 0. \end{cases}
\]

Now, let
\[
T_{\sigma, \beta}(x) := T_{\sigma, \beta}(j, n, \mu, \xi, \zeta)(x) := \frac{\int_{\alpha - \zeta}^{\beta - \zeta}(y - \alpha)(\beta - y)(y - x_j^0)(x_{j-1} - y)^\mu dy}{\int_{\alpha - \zeta}^{\beta - \zeta}(y - \alpha)(\beta - y)(y - x_j^0)(x_{j-1} - y)^\mu dy}
\]
where \( \alpha \in [-1, x_j] \) and \( \beta \in [x_{j-1}, 1] \). If \( \mu \) is sufficiently large in comparison with \( \xi \) and \( \zeta \) (for example, \( \mu \geq 5\xi + 5\zeta + 15 \) will do), then \( T_{\sigma, \beta} \) is a polynomial of degree \( \leq c(\mu)n \), and the denominator \( \int_{\alpha - \zeta}^{\beta - \zeta}(y - \alpha)(\beta - y)(y - x_j^0)(x_{j-1} - y)^\mu dy \) is a positive number (see [18], for example).
For the sake of brevity, we introduce the following convention:

\[ \tau(y) := (y - x_j)^\ell (x_{j-1} - y)\psi(y)^\mu. \]

First, we consider intertwining approximation of the step function \( \chi_j(x) \). The following lemma and its corollary contain the basis for all our further constructions.

**Lemma 16.** For any \( A \in [-1, x_j] \) and \( B \in [x_{j-1}, 1] \) there exist \( \alpha \in [-1, A] \) and \( \beta \in [B, 1] \) such that

(3.1) \[ T_{\alpha, \beta}(A) = 0 \]

and

(3.2) \[ T_{\alpha, \beta}(B) = 1. \]

**Proof.** If \( A = -1 \) or \( B = 1 \), then the choice of \( \alpha \) and \( \beta \) is obvious. Now, let \( A \in (-1, x_j) \) and \( B \in [x_{j-1}, 1) \) be fixed. Then for any \( \beta \in [B, 1] \) there exists a unique \( \alpha \in [-1, A] \) such that (3.1) is satisfied. Indeed,

\[ T_{\alpha, \beta}(A) = 0 \iff \alpha = \gamma(\beta) := \frac{\int_{-1}^{\beta} y(\beta - y)\tau(y) \, dy}{\int_{-1}^{\beta} (\beta - y)\tau(y) \, dy}. \]

Since \( \int_{-1}^{\beta} (\beta - y)\tau(y) \, dy \neq 0 \) for \( \beta \in [B, 1] \), then the function \( \gamma \in C[B, 1] \). Also, for any \( \beta \in [B, 1] \) we have \(-1 \leq \gamma(\beta) \leq A\). The first inequality is obvious. The second one holds since

\[ \gamma(\beta) \leq A \iff \frac{\int_{-1}^{\beta} (A - y)(\beta - y)\tau(y) \, dy}{\int_{-1}^{\beta} (\beta - y)\tau(y) \, dy} \geq 0, \]

which is true since

\[ \text{sgn}\left(\int_{-1}^{\beta} (A - y)(\beta - y)\tau(y) \, dy\right) = \text{sgn}\left(\int_{-1}^{\beta} (\beta - y)\tau(y) \, dy\right) = (-1)^\ell. \]

Similarly, it can be shown that for any \( \alpha \in [-1, A] \) there exists a unique \( \beta \in [B, 1] \) such that (3.2) is satisfied, i.e., there exists a function \( \delta \in C[-1, A] \) with the range \([B, 1] \) such that (3.2) is satisfied for \( \beta = \delta(\alpha) \).

Thus, there exists \((\alpha, \beta) \in [-1, A] \times [B, 1] \) such that \( \alpha = \gamma(\beta) \) and \( \beta = \delta(\alpha) \), i.e., (3.1) and (3.2) are satisfied simultaneously. The proof of the lemma is complete.

**Corollary 17.** Let an index \( 1 \leq j \leq n - 1 \) be fixed. For any \( A \in [-1, x_{j+1}] \) and \( B \in [x_{j-1}, 1] \) there exist polynomials \( T_i(A, B)(x) \), \( i = 1, 2, 3, 4 \) of degree \( \leq C(\mu)n \) such that \( T_i(A, B)(x) = \chi_j(x) \) for \( x = -1, A, B, 1 \), and also satisfying

(3.3) \[ |\chi_j(x) - T_i(A, B)(x)| \leq 1/3, \quad x \in [-1, 1] \setminus [x_{j+1}, x_{j-1}], \quad i = 1, 2, 3, 4, \]

(3.4) \[ |\chi_j(x) - T_i(A, B)(x)| \leq C\psi(x)^\mu, \quad x \in [-1, 1], \quad i = 1, 2, 3, 4, \]

(3.5) \[ \text{sgn}\{T_i(A, B)(x) - \chi_j(x)\} = \text{sgn}\{-x(x - A)(x - x_j)(B - x)\}, \quad x \in (-1, 1) \setminus \{x_j\}, \]

(3.6)
of the above statement for completeness, since the authors believe that Corollary 17 is an interesting and important result by itself.

**Proof.** It is sufficient to choose

\[
T_1(A, B)(x) := T_{\sigma, \delta}(M_j, M_n, \mu, 0, 0)(x),
\]

\[
T_2(A, B)(x) := T_{\sigma, \delta}(M_j + 1, M_n, \mu, 0, 1)(x),
\]

\[
T_3(A, B)(x) := T_{\sigma, \delta}(M_j, M_n, \mu, 1, 0)(x)
\]

and

\[
T_4(A, B)(x) := T_{\sigma, \delta}(M_j, M_n, \mu, 1, 1)(x),
\]

with sufficiently large constant \(M = M(\mu) \in \mathcal{N}\).

The equalities (3.5)–(3.9) are obvious, and (3.4) and (3.3) follow, respectively, from [18, Lemma 5] and the proof of [18, Lemma 6].

**Lemma 18.** Let an index \(1 \leq j \leq n - 1\) be fixed, and let the numbers \(\{a_i\}_{i=1}^k\) and \(b\) be such that \(-1 \leq a_1 < a_2 \leq \cdots < a_k \leq x_{j+1}\) and \(x_{j-1} \leq b \leq 1\). Then there exist polynomials \(Q_i(x) := Q_i(x, \mu; a_1, \ldots, a_k; b)(x), i = 1, 2, 3,\) of degree \(\leq C(\mu)n\) such that \(Q_i(x) = \chi_j(x),\) \(i = 1, 2, 3,\) for \(x = -1, a_1, \ldots, a_k, b, 1,\) \(Q_i(x) \geq 0, x \geq b, Q_i(x) \leq 1, x \leq a_k,\) and also satisfying

\[
|\chi_j(x) - Q_i(x)| \leq C(\mu, k)\psi_j(x)^p, \quad i = 1, 2, 3,
\]

\[
\text{sgn}\{Q_i(x, \mu; a_1, \ldots, a_k; b)(x) - \chi_j(x)\}
\]

\[
= \text{sgn}\left\{-\prod_{j=1}^k (x - a_j)(x - x_j)(b - x)\right\}, \quad x \in [-1, 1] \setminus \{x_j\},
\]

\[
\text{sgn}\{Q_i(x, \mu; a_1, \ldots, a_k; b)(x) - \chi_j(x)\}
\]

\[
= \text{sgn}\left\{-\prod_{j=1}^k (x - a_j)(b - x)\right\}, \quad x \in [-1, 1],
\]

and

\[
\text{sgn}\{Q_3(x, \mu; a_1, \ldots, a_k; b)(x) - \chi_j(x)\}
\]

\[
= \text{sgn}\left\{\prod_{j=1}^k (x - a_j)(b - x)\right\}, \quad x \in [-1, 1].
\]
PROOF. Let

\[ Q_1(x_j, \mu; a_1, \ldots, a_k; b)(x) := \prod_{p=1}^{k} T_1(a_p, b)(x), \]

\[ Q_2(x_j, \mu; a_1, \ldots, a_k; b)(x) := \prod_{p=1}^{k-1} T_1(a_p, b)(x)T_2(a_k, b)(x) \]

and

\[ Q_3(x_j, \mu; a_1, \ldots, a_k; b)(x) := \prod_{p=1}^{k} T_2(a_p, b)(x). \]

It is straightforward to check that the inequalities (3.11)–(3.13) are satisfied. Inequality (3.10) follows from the observation that if \( \tilde{q}(x) = \prod_{p=1}^{k} q_p(x) \) and \( |\chi_j(x) - q_p(x)| \leq C(\mu)\psi_j(x)^{\mu}, \ x \in [-1, 1], \) then \( |\chi_j(x) - \tilde{q}(x)| \leq C(\mu, k)\psi_j(x)^{\mu} \) for all \( x \in [-1, 1]. \) Indeed, for \( x \leq x_j \) we have

\[ |\tilde{q}(x) - \chi_j(x)| = \prod_{p=1}^{k} |q_p(x)| \leq C(\mu, k)\psi_j(x)^{\mu}. \]

If \( x \geq x_j, \)

\[ |\tilde{q}(x) - \chi_j(x)| = \left| \prod_{p=1}^{k} q_p(x) - 1 \right| = \left| \sum_{p=1}^{k-1} \left( \prod_{p=1}^{k} q_p(x) \right) (q_{p+1}(x) - 1) + q_1(x) - 1 \right| \leq C(\mu, k)\psi_j(x)^{\mu}. \]

Now, consider the “flipped” functions \( \tilde{Q}_i(x_j, \mu; a; b_1, \ldots, b_k)(x), i = 1, 2, 3, \) defined as follows:

\[ \tilde{Q}_i(x_j, \mu; a; b_1, \ldots, b_k)(x) := 1 - Q_i(x_{m-j}, \mu; -b_1, \ldots, -b_k; -a)(-x) \]

for \(-1 \leq a \leq x_{j+1} \) and \( x_{j-1} \leq b_1 < b_2 < \cdots < b_k \leq 1. \)

Then \( \tilde{Q}_i(x) = \chi_j(x), i = 1, 2, 3, \) for \( x = -1, a, b_1, \ldots, b_k, 1, \) \( \tilde{Q}_i(x) \geq 0, x \geq b_1, \)

\( \tilde{Q}_i(x) \leq 1, x \leq a, \) and also

\[ |\chi_j(x) - \tilde{Q}_i(x)| \leq C(\mu, k)\psi_j(x)^{\mu}, \ i = 1, 2, 3, \]

\[ \text{sgn}\left\{ \tilde{Q}_1(x_j, \mu; a; b_1, \ldots, b_k)(x) - \chi_j(x) \right\} \]

\[ = \text{sgn}\left\{ -(x-a)(x-x_j) \prod_{p=1}^{k} (b_p - x) \right\}, \ x \in [-1, 1] \setminus \{x_j\}, \]

\[ \text{sgn}\left\{ \tilde{Q}_2(x_j, \mu; a; b_1, \ldots, b_k)(x) - \chi_j(x) \right\} \]

\[ = \text{sgn}\left\{ (x-a) \prod_{p=1}^{k} (b_p - x) \right\}, \ x \in [-1, 1]. \]
and
\[\text{sgn}\{\bar{Q}_i(x_j, \mu; a; b_1, \ldots, b_k)(x) - \chi_j(x)\}\]
(3.17)
\[= \text{sgn}\left\{-(x-a) \prod_{\nu=1}^{k} (b_{\nu} - x)\right\}, \quad x \in [-1, 1].\]

Let
\[R_i(x) := R_i(x_j, \mu; a_1, \ldots, a_k; b_1, \ldots, b_l)(x)
:= \prod_{\nu=1}^{k-1} \bar{Q}_i(x_j, \mu; a_{\nu}; b_1, \ldots, b_l)(x) \bar{Q}_3(x_j, \mu; a_k; b_1, \ldots, b_l)(x)\]
and
\[R_2(x) := R_2(x_j, \mu; a_1, \ldots, a_k; b_1, \ldots, b_l)(x) := \prod_{\nu=1}^{k} \bar{Q}_3(x_j, \mu; a_{\nu}; b_1, \ldots, b_l)(x).\]

Then
\[|\chi_j(x) - R_i(x)| \leq C(\mu, k, l)\psi_j(x)^{\mu}, \quad i = 1, 2,
\[= \text{sgn}\left\{-(x-a) \prod_{\nu=1}^{l} (b_{\nu} - x)\right\}, \quad x \in [-1, 1]\]
(3.19)
and
\[\text{sgn}\{R_2(x_j, \mu; a_1, \ldots, a_k; b_1, \ldots, b_l)(x) - \chi_j(x)\}
= \text{sgn}\left\{\prod_{\nu=1}^{k} (x-a_{\nu}) \prod_{\nu=1}^{l} (b_{\nu} - x)\right\}, \quad x \in [-1, 1].\]
(3.20)

Finally, multiplying \(R_i, i = 1, 2,\) by \((x - \Lambda_x^{2m})\) we obtain an intertwining pair of polynomials for \((x - \Lambda_x^{2m})\) with respect to \(\{a_{j=1}^{k}\} \cup \{b_{j=1}^{l}\}\) with good approximation properties.

**Lemma 19.** Let an index \(1 \leq j \leq n - 1\) be fixed, \(\{a_{j=1}^{k}\}\) and \(\{b_{j=1}^{l}\}\) be such that \(-1 \leq a_1 < a_2 < \cdots < a_k \leq x_{j+1} < x_{j-2} \leq b_1 < \cdots < b_l \leq 1, \Lambda \in [x_j, x_{j-1}]\) and \(m \in N \cup \{0\}.\) Then the polynomials
\[R_i(x) := R_i(x_j, \Lambda; \mu; a_1, \ldots, a_k; b_1, \ldots, b_l)(x)
:= (x - \Lambda)^{2m} R_i(x_{j+1-2}, \mu; a_1, \ldots, a_k; b_1, \ldots, b_l)(x), \quad i = 1, 2\]
of degree \(\leq C(\mu)n\) are such that
\[|\bar{R}_i(x) - (x - \Lambda_x^{2m})| \leq C(\mu, k, l)\psi_j(x)^{\mu-2m}b_1^{2m}, \quad x \in [-1, 1], \quad i = 1, 2,
\]
(3.21)
\[(-1)^{i+1} \bar{R}_i(x) - (x - \Lambda_x^{2m}) \in \Delta^i(\{a_{j=1}^{k}\} \cup \{b_{j=1}^{l}\})\]
(3.22)
and
\[(-1)^i \bar{R}_i(x) - (x - \Lambda_x^{2m}) \in \Delta^i(\{a_{j=1}^{k}\} \cup \{b_{j=1}^{l}\}).\]
(3.23)
Proof. The assertion of the lemma follows from (3.18)–(3.20), the observation that

\[
\tilde{R}_i(x) = (x - \Lambda_i)^{2m} \\
= (x - \Lambda)^{2m} \left( R_i(x_{i+2}, \mu, a_1, \ldots, a_k; b_1, \ldots, b_l)(x) - \chi[\Lambda, 1](x) \right) \\
= (x - \Lambda)^{2m} \left( R_i(x_{i+2}, \mu, a_1, \ldots, a_k; b_1, \ldots, b_l)(x) - \chi_{J}(x) \right), \quad i = 1, 2,
\]

for \( x \in [-1, 1] \setminus [x_j, x_{j-1}] \), the inequalities

\[
R_2(x_j, \mu, a_1, \ldots, a_k; b_1, \ldots, b_l)(x) \geq \chi(x) \geq \chi_{J}(x) \geq R_1(x_{j-1}, \mu, a_1, \ldots, a_k; b_1, \ldots, b_l)(x)
\]

for \( x \in [x_j, x_{j-1}] \), and the fact that \( \psi(x) \sim \psi_{j, 1}(x) \) for \( x \in [-1, 1] \).

3.2. Proof of Theorem 4. For the proof of Theorem 4 we need the following classical result on analytic representation of splines in terms of the truncated power functions. Its proof can be found in Kornejchuk [23], for example.

Lemma 20. Let \( S(t) \) be a spline of order \( r \) on the knot sequence \(-1 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1\). Then, for \( t \in [-1, 1] \),

\[
S(t) = \sum_{\nu=0}^{r-1} \frac{S^{(\nu)}(-1)}{\nu!} (t + 1)^{\nu} + \sum_{j=1}^{n-1} \frac{S^{(\nu)}(t_j) - S^{(\nu)}(t_{j-1})}{(r-1)!} (t - t_j)^{r-1}.
\]

Let \( S \) satisfy the assertion of Theorem 4. Then

\[
S(x) = \sum_{\nu=0}^{2m} \frac{S^{(\nu)}(-1)}{\nu!} (x + 1)^{\nu} + \sum_{j \in J(Y)} \alpha_j (x - x_j)^{2m}
\]

\[
= \sum_{\nu=0}^{2m} \frac{S^{(\nu)}(-1)}{\nu!} (x + 1)^{\nu} + \sum_{j \in J^*} \alpha_j (x - x_j)^{2m} - \sum_{j \in J^-} \alpha_j (x - x_j)^{2m},
\]

where \( \alpha_j := \frac{S^{(\nu)}(x_j) - S^{(\nu)}(x_{j-1})}{(2m)!} \), \( J^* := \{ j \mid \alpha_j \geq 0 \} \cap J(Y) \), and \( J^- := \{ j \mid \alpha_j < 0 \} \cap J(Y) \). For each \( j \in J(Y) \) we define

\[
u(j) := \min \{ i \mid y_i \geq x_j \}
\]

(i.e., \( y_{\nu(j)} \) is the first point \( y_i \) on the right of \( x_j \)), and

\[
\nu(j) := \begin{cases} j, & \text{if } y_{\nu(j)} \geq x_{j-2}, \\ j + 1, & \text{if } x_{j-1} \leq y_{\nu(j)} < x_{j-2}. \end{cases}
\]

(In other words, we define \( \nu(j) \) to be an index satisfying \( |\nu(j) - j| \leq 1 \) and such that the interval \([x_{\nu(j)}, x_{\nu(j)-1}]\) is not “too close” to \( y_i \)’s (recall that \( j \in J(Y) \)). This is a technicality which is needed because \( a_i \)’s and \( b_i \)’s in the assertion of Lemma 19 “should be far” from \( x_i \). Also, we would like to emphasize that if \( \nu(j) = j + 1 \), then \( y_{\nu(j)-1} < x_{j+2} \), since there are at least 4 points \( x_j \) between \( y_{\nu(j)-1} \) and \( y_{\nu(j)} \).)
We show that the polynomials
\[
P_i(x) = \sum_{j=0}^{2m} \frac{S_j^{(p)}(-1)}{\nu!} (x + 1) \nu + \sum_{j \in J(Y_i)} \alpha_j \tilde{R}_j(Y_i)(x), \quad i = 1, 2,
\]
where \( \tilde{R}_j(Y_i)(x) = \tilde{R}_j(x_{(j)}, x_j, 2m + 30; Y_j)(x) \) if \( j \in J^+ \) and \((-1)^{s-j} > 0\), or \( j \in J^- \) and \((-1)^{s-j} < 0\), and \( \tilde{R}_j(Y_i)(x) \equiv \tilde{R}_j(x_{(j)}, x_j, 2m + 30; Y_j)(x) \) otherwise, \( i = 1, 2 \), form an intertwining pair for \( S \) satisfying the estimate (1.16).

It is relatively straightforward to verify (1.16). Using Markov’s inequality first and then Jensen’s inequality (note that \( \sum_{j=1}^{n-1} \psi_j(x)^{p-2m} < C \), as well as the fact that \( h_j \sim h_{j+1} \) and \( \psi_j(x) \sim \psi_{j+1}(x) \), we write for \( 0 < p < \infty \):
\[
\| P_1 - P_2 \|_p \leq C^p \int_{-1}^{1} \left( \sum_{j \in I(Y_i)} \| S_j(x) - S_{j+1}(x) \|_{C(I_j)} \right)^p dx \\
\leq C^p \int_{-1}^{1} \left( \sum_{j=1}^{n-1} \| S_j(x) - S_{j+1}(x) \|_{C(I_j)} \psi_j(x)^{p-2m} \right)^p dx \\
\leq C^p \sum_{j=1}^{n-1} h_j^{-1} \| S_j - S_{j+1} \|_{L_p(I_j)}^p \int_{-1}^{1} \psi_j(x)^{p-2m} dx \\
\leq C^p \sum_{j=1}^{n-1} \| S_j - S_{j+1} \|_{L_p(I_j)}^p,
\]
where \( S_j \) denotes a polynomial from \( P_{r-1} \) such that \( S_j(x) \equiv S_{[j]}(x), \) \( x \in I_j \) (i.e., \( S_j \) is a polynomial of degree \( \leq r-1 \) which coincides with \( S \) on \( I_j \)). A similar estimate is true in the case \( p = \infty \) as well (see [19], for example). It remains to show that
\[
\| S_j - S_{j+1} \|_{L_p(I_j)} \leq CE_{r-1}(S, I_j)_{p}, \quad 0 < p \leq \infty.
\]
Indeed, using the observation that \( S_{j+1} \) is the best approximant to \( S \) on \( I_{j+1} \) from \( P_{r-1} \) (this is, of course, true since \( S_{j+1} \equiv S_{[j+1]} \)) we write
\[
\| S_{j+1} - S_j \|_{L_p(I_j)} = \| S_{j+1} - S \|_{L_p(I_j)} \\
\leq \| S_{j+1} - S \|_{L_p(I_j \cup I_{j+1})} \\
\leq CE_{r-1}(S, I_j \cup I_{j+1})_p.
\]
In the last inequality, we used the fact that \( S_{j+1} \) is also a near-best \( L_p \) approximant to \( S \) on \( I_j \cup I_{j+1} \) from \( P_{r-1} \) (see DeVore and Popov [6, Lemma 3.3]).

Finally, we verify that \( P_1(x) - S(x) \in \Delta^0(Y_i) \) (the proof of the inclusion \( S(x) - P_2(x) \in \Delta^0(Y_i) \) is similar). Let \( x \in [Y_i, Y_{i+1}) \) be fixed. Then denoting \( \tilde{R}_j(Y_i) := \tilde{R}_j(x_{(j)}, x_j, 2m + 30; Y_j)(x) \), \( i = 1, 2 \), and using Lemma 19 with \( l = s - u(j) + 1 \) for each \( j \in J(Y_i) \), we have
\[
P_1(x) - S(x) = \sum_{j \in J(Y_i)} \alpha_j \tilde{R}_j(Y_i)(x) - (x - x_j)^{2m}
\]
and, therefore, since if $f_j \in \Delta^0(Y_s)$ and $\beta_j \geq 0$ for all $j$, then $\sum_j \beta_j f_j \in \Delta^0(Y_s)$. The proof is now complete.

The proofs of Theorems 1, 5, 7, 8, and 10 will follow the same scheme. Namely, using Theorems 3, 15, and Lemma 14 we construct splines satisfying the appropriate constrains and having the right approximation order. Then, we use Theorem 4 to find polynomial(s) with similar characteristics.

**Proof of Theorem 1.** Let $f \in C[-1, 1]$ and $m \in \mathcal{N}$. It follows from Lemma 14 (with $\mathcal{T}_n = \{x_j\}$) that there exist splines $\tilde{S}$ and $S$ of an odd order $r, r \geq m$ (we choose $r = 2[\frac{m}{2}] + 1$), such that $\tilde{S}(x) \geq f(x) \geq S(x), x \in I$, and

$$\|\tilde{S} - S\|_{C(I)} \leq C \omega(f, [\mathcal{J}], \mathcal{J})_\infty.$$ 

Since $|\mathcal{J}| \sim |\mathcal{I}| = h_j \sim \Delta_n(x)$ for $x \in \mathcal{J}$, then

$$|\tilde{S}(x) - S(x)| \leq C \omega(f, h_j)_\infty \leq C \omega(f, \Delta_n(x))_\infty \leq C \omega^{m}(f, \Delta_n(x))_\infty, \quad x \in I,$$

and, therefore,

$$|\tilde{S}(x) - S(x)| \leq C \omega^{m}(f, \Delta_n(x))_\infty \quad \text{for all } x \in I.$$

Also,

$$E_{r-1}(\tilde{S}, I_j \cup I_{j+1})_\infty \leq E_{r-1}(\tilde{S} - f, I_j \cup I_{j+1})_\infty + E_{r-1}(f, I_j \cup I_{j+1})_\infty \leq \|\tilde{S} - S\|_{C(I_j, I_{j+1})} + C \omega^{m}(f, h_j)_\infty \leq C \omega^{m}(f, h_j)_\infty,$$

and, similarly,

$$E_{r-1}(S, I_j \cup I_{j+1})_\infty \leq C \omega^{m}(f, h_j)_\infty.$$

Theorem 4 (where $\mu = 31m$ is chosen) implies that there exist polynomials $\tilde{P}_1, \tilde{P}_2, P_1, P_2$ of degree $\leq C(m)n$ such that $\tilde{P}_1(x) \geq \tilde{S}(x) \geq \tilde{P}_2(x), P_1(x) \geq S(x) \geq P_2(x)$, and

$$|\tilde{P}_1(x) - \tilde{P}_2(x)| \leq C \sum_{j=1}^{n-1} \omega^{m}(f, h_j)_\infty \psi_j(x)^{r+2} \leq C \omega^{m}(f, \Delta_n(x))_\infty \sum_{j=1}^{n-1} \psi_j(x)^{2} \leq C \omega^{m}(f, \Delta_n(x))_\infty \leq C \omega^{m}(f, \Delta_n(x))_\infty.$$
and,
\[ |P_1(x) - P_2(x)| \leq C \omega^m \bigl( f, \Delta_n(x) \bigr)_{\infty}. \]
Here, we have used the inequality
\begin{equation}
(3.27) \quad h_j \leq C \Delta_n(x) \psi_j(x)^{-1}, \quad x \in I,
\end{equation}
which follows from the fact that \( \Delta_n(y)^2 \leq 4 \Delta_n(x) \bigl( |x - y| + \Delta_n(x) \bigr) \) and \( |x - y| + \Delta_n(x) \sim |x - y| + \Delta_n(y) \), \( x, y \in I \) (see [30] or [19], for example). Now, the polynomials \( \tilde{P}_1 \) and \( P_2 \) are what we are looking for since \( \tilde{P}_1 \geq \tilde{S} \geq f \geq S \geq P_2 \), and
\[ |\tilde{P}_1(x) - P_2(x)| \leq |\tilde{P}_1(x) - P_2(x)| + |\tilde{S}(x) - S(x)| + |P_1(x) - P_2(x)| \leq C \omega^m \bigl( f, \Delta_n(x) \bigr)_{\infty}. \]
The proof is complete for large \( n (n \geq C(m)) \). For \( m - 1 \leq n < C(m) \) the estimate (1.7) is a trivial corollary of (1.3).

**Proof of Theorem 5.** Theorem 3 implies the existence of the intertwining pair of splines \( \{ \tilde{S}, S \} \) of order \( r \) for \( f \) on the knot sequence \( \{ x_j \}_{j \in J(Y)} \) (recall that \( J(Y_i) = \{ 1, \ldots, n \} \setminus \{ \ell, j - 1 \mid x_{j} \leq y_{j+1} \text{ for some } 1 \leq i \leq s \} \) satisfying
\[ \| \tilde{S} - S \|_{C(J)} \leq Ch_{j} \omega^{-1}(f', h_j, J)_{\infty}, \]
where \( r \) is an odd integer such that \( m + 1 \leq r \leq m + 2 \).

We need the following consequence of Lemma 5 of [22]:
Let \( [a, b] \subset [-1, 1] \) be such that \( |b - a| \sim \Delta_n(x) \) for \( x \in [a, b] \). Then for any \( r \in \mathcal{N} \) there exists a constant \( C(r) \) such that
\[ C(r)^{-1} \omega^r \bigl( f, \Delta_n(x), [a, b] \bigr)_{\infty} \leq \omega_{a}^r (f, n^{-1}, [a, b])_{\infty} \leq C(r) \omega^r \bigl( f, \Delta_n(x), [a, b] \bigr)_{\infty}, \quad x \in [a, b]. \]
In particular,
\begin{equation}
(3.28) \quad \omega^r (f', h_j, J)_{\infty} \leq C(r) \omega^{-1}(f, n^{-1})_{\infty}. \end{equation}
Therefore,
\[ |\tilde{S}(x) - S(x)| \leq C \Delta_n(x) \omega^{-1}(f', \Delta_n(x))_{\infty}, \quad x \in I_j \]
and
\[ \| \tilde{S} - S \|_{C(J)} \leq C n^{-1} \omega^{-1}(f', n^{-1})_{\infty}. \]
Theorem 4 (with sufficiently large \( \mu \), say, \( \mu = 31m \)) implies that there exist intertwining pairs of polynomials \( \{ \tilde{P}_1, \tilde{P}_2 \} \) and \( \{ P_1, P_2 \} \) for \( \tilde{S} \) and \( S \), respectively, satisfying the inequalities
\[ |\tilde{P}_1(x) - \tilde{P}_2(x)| \leq C \Delta_n(x) \omega^{-1}(f', \Delta_n(x))_{\infty}, \]
\[ |P_1(x) - P_2(x)| \leq C \Delta_n(x) \omega^{-1}(f', \Delta_n(x))_{\infty}, \]
\[ \| \tilde{P}_1 - \tilde{P}_2 \|_{C(J)} \leq C n^{-1} \omega^{-1}(f', n^{-1})_{\infty}, \]
and
\[ \|P_1 - P_2\|_{C(f)} \leq C n^{-1} \omega_{n^{-1}}(f', n^{-1})_{\infty}, \]
which can be verified similarly to (3.26) using (3.28), (3.27), and the fact that 
\[ \sum_{j=1}^{\infty} \psi_j(x) \leq C. \] Finally, \( \{P_1, P_2\} \) is an intertwining pair of polynomials for \( f \) satisfying (1.18) and (1.19).

**Proof of Theorems 7, 8 and 10.** The proofs of these theorems are similar to the above proofs. The needed modifications are obvious. We omit details and just emphasize that the inequalities (1.12) and (1.13) should be used.

Finally, we mention that the same proofs can be used to show the validity of the results of Stojanova [31] (ineq. (1.5)), Kopotun [20] (ineq. (1.9)) and the authors [13] (ineq. (1.11)).

4. **Counterexamples.**

**Proof of Theorem 11.** Let \( n \in \mathbb{N}, 0 < p < \infty, 0 < \varepsilon \leq 1 \) and \( A > 0 \) be fixed, and define
\[ g(x) := \ln(x^2 + e^{-b}), \quad f'(x) := bx^2 - \ln b - g(x) \]
and
\[ f(x) := \int_0^x f'(t) \, dt = \frac{b}{3} x^3 + (2 - \ln b)x - 2e^{-b/2} \arctan(xe^{b/2}) - x \ln(x^2 + e^{-b}), \]
where \( b \geq 1 \) is a parameter to be chosen later. Obviously, \( f \in C^\infty \) and \( f(0) = 0 \). Basic calculus shows that \( f' \) assumes its minimum \( 1 - be^{-b} > 0 \) at \( x = \pm \sqrt{-b}/be^{b} \in (-1, 1) \), thus \( f'(x) > 0 \) for all \( x \), and \( f \) is strictly increasing.

Since \( \ln 2 > g(x) > \ln x^2 > 2 \ln |x| \) on \([-1, 1]\), we have
\[
\begin{align*}
\|g\|_p^p &= \int_{-1}^{1} |g(x)|^p \, dx = 2 \int_0^1 |g(x)|^p \, dx \\
&\leq 2 \int_0^1 (2 \ln 2)^p \, dx + 2 \int_0^1 |2 \ln x|^p \, dx \\
&= 2(2 \ln 2)^p + 2^{p+1} \Gamma(p+1) =: M_1^p.
\end{align*}
\]
Hence
\[
\omega^3(f', 1)_p = \omega^3(g, 1)_p \leq 8^{\max(1,1/p)} \|g\|_p \leq 8^{\max(1,1/p)} M_1 =: M_2.
\]

We now prove there exists \( b \geq 1 \) for the given \( n, \varepsilon, p \) and \( A \) such that if any \( P_n(x) = a_0 + a_1 x + \cdots + a_n x^n \) satisfies
\[
\|f - P_n\|_{L_p[0,1]} \leq A \omega^3(f', 1)_p,
\]
then \( P_n'(0) < 0 \). The theorem follows immediately because if \( P_n \) were copositive with \( f \), \( P_n'(0) \) would be nonnegative. Let \( P_n(x) = (2 - \ln b)x - bx^3/3 \) be denoted by \( \bar{P}_n \). From (4.2)–(4.3) we have
\[
\|\bar{P}_n\|_{L_p[0,1]} \leq \|f - P_n\|_{L_p[0,1]} + \int_1^1 |2e^{-b/2} \arctan(xe^{b/2})|^p \, dx \\
+ \int_{-1}^{1} |x \ln(x^2 + e^{-b})|^p \, dx \leq A^p \omega^3(f', 1)_p + 2\pi^p + \|g\|_p^p \\
\leq A^p M_1^p + 2\pi^p + M_1^p =: M_3^p.
\]
It follows from Markov’s inequality and the equivalence of norms in \( P_n \) (see Theorem 2.2.7 of [5], for example) that
\[
a_1 - 2 + \ln b = \bar{P}_n(0) \leq n^2 \varepsilon^{-1} \| \bar{P}_n \|_{\mathcal{C}^0[0,1]} \leq C n^2 \varepsilon^{-1/p} M_3 =: M_4, \\
or
\[
a_1 \leq M_4 + 2 - \ln b.
\]
Since \( M_4 \) is independent of \( b \), (though it depends on \( n, \varepsilon, p \) and \( A \)), we can choose \( b > \exp(M_4 + 2) \), which gives \( P_n(0) = a_1 < 0 \), as desired.

**Proof of Theorem 12.** Let \( n \in \mathcal{N}, 0 < p < \infty, 0 < \varepsilon \leq 1 \) and \( A > 0 \) be fixed, and define
\[
g(x) := \ln((x - b^{-1})^2 + e^{-b}), \quad f(x) := bx + \ln(b^{-2} + e^{-b}) - g(x),
\]
where \( b \geq 3 \) is a parameter to be chosen later. Obviously, \( f \in C^\infty \) and \( f(0) = 0 \). Basic calculus shows that \( f \) increases on \([-1, x_1]\), where \( x_1 = \frac{2 \sqrt{1 - b^{-2}}}{b} > 0 \), and assumes a maximum at \( x_1 \); then it decreases and assumes a minimum
\[
2 + \sqrt{1 - b^2 e^{-b}} + \ln(1 + b^2 e^{-b}) - \ln(2 + 2 \sqrt{1 - b^2 e^{-b}}) > 2 - 2 \ln 2 > 0
\]
at \( x_2 = \frac{2 \sqrt{1 - b^{-2} e^{-2b}}}{b} \in (x_1, 1) \). On \([x_2, 1]\) it becomes monotone increasing again. Therefore, \( f(x) < 0 \) for \( x \in [-1, 0) \) and \( f(x) > 0 \) for \( x \in (0, 1] \).

(4.5) \[
\|g\|_p^p = \int_{-1}^1 \left| \ln((x - b^{-1})^2 + e^{-b}) \right|^p dx = \int_{-1 - b^{-1}}^{1 - b^{-1}} \left| \ln(x^2 + e^{-b}) \right|^p dx \\
\leq \int_{-1}^1 \left| \ln(x^2 + e^{-b}) \right|^p dx \leq \int_{-2}^1 (\ln 5)^p dx + \int_{-2}^1 2 \ln |x| \right|^p dx =: M_4^p.
\]

Hence
(4.6) \[
\omega^2(f, 1)_p = \omega^2(g, 1)_p \leq 4 \max(1, 1/p) \|g\|_p \leq 4 \max(1, 1/p) M_4 =: M_2.
\]

We now prove the theorem by showing that there exists \( b \geq 3 \) for the given \( n, \varepsilon, p \) and \( A \) such that if any \( P_n(x) = a_0 + a_1 x + \cdots + a_n x^n \) satisfies
(4.7) \[
\|f - P_n\|_{\mathcal{L}_p[0,1]} \leq A \omega^2(f, 1)_p,
\]
then \( P_n(0) < 0 \). Let \( P_n(x) = \ln(b^{-2} + e^{-b}) - bx \) be denoted by \( \bar{P}_n \). From (4.5)–(4.7) we have
(4.8) \[
\|\bar{P}_n\|_{\mathcal{L}_p[0,1]}^p \leq \|f - P_n\|_{\mathcal{L}_p[0,1]}^p + \|g\|_{\mathcal{L}_p[0,1]}^p \leq A^p M_2^p + M_4^p =: M_3^p.
\]
It follows from the equivalence of norms in \( P_n \) (see Theorem 2.2.7 of [5], for example) that
\[
a_0 - \ln(b^{-2} + e^{-b}) = \bar{P}_n(0) \leq \|\bar{P}_n\|_{\mathcal{C}[0,1]} \leq C \varepsilon^{-1/p} M_3 =: M_4,
\]
or
\[
a_0 \leq M_4 + \ln(b^{-2} + e^{-b}) \leq M_4 + \ln(2b^{-2}) = M_4 + \ln 2 - 2 \ln b.
\]
Since \( M_4 \) is independent of \( b \) (though it depends on \( n, \varepsilon, p \) and \( A \)), we can choose \( b > \exp((M_4 + \ln 2)/2) \), which gives \( P_n(0) = a_0 < 0 \), as desired.
PROOF OF THEOREM 13. Let \( n \in \mathcal{N} \), \( 0 < p \leq \infty \), \( 0 < \varepsilon \leq 1 \) and \( A > 0 \) be fixed, and define
\[
f(x) := \arctan(bx),
\]
where \( b \geq 1 \) is a parameter to be chosen later. Note that \( \|f\|_p \leq 2^{1/p-1} \pi \).

The inequality (1.31) can be proved in much the same way as the previous proofs in the section: suppose there exist \( P_n \in \mathbb{P}_n \) with \( P_n(0) = 0 \) and \( P_n(x) \geq f(x) \), \( x \in [0, \varepsilon] \), and \( A \in \mathcal{R} \) such that \( \|f - P_n\|_{L_p[0,\varepsilon]} \leq A\|f\|_{L_p[0,1]} \), then
\[
P_n'(0) \leq n^2 \varepsilon^{-1/2} \|P_n\|_{L_p[0,\varepsilon]} \leq Cn^2 \varepsilon^{-1/2} \|P_n\|_{L_p[0,1]}
\leq Cn^2 \varepsilon^{-1/2}(A + 1) \|f\|_{L_p[0,1]} \leq Cn^2 \varepsilon^{-1/2}(A + 1).
\]
Choosing \( b \) greater than the right hand side, which is independent of \( b \), gives the contradiction \( b = f'(0) \leq P_n'(0) < b \) (since \( P(0) = f(0) \) and \( P_n(x) \geq f(x) \), \( x \in [0, \varepsilon] \) imply \( P_n'(0) \geq f'(0) \)).

We now suppose \( 0 < p < \infty \) and prove (1.32). Since
\[
0 < f'(x) = \frac{b}{1 + b^2 x^2} \leq \left\{ \begin{array}{ll}
1, & \text{if } |x| \geq \sqrt{b - 1}/b,
\frac{1}{b}, & \text{otherwise}
\end{array} \right.
\]
we have
\[
\int_{-1}^{1} |f'(x)|^p \, dx = \left\{ \int_{|x| \leq \sqrt{b - 1}/b} + \int_{|x| > \sqrt{b - 1}/b} \right\} |f'(x)|^p \, dx \leq C(b^{p-1/2} + 1).
\]
If some polynomial \( P_n \in \mathbb{P}_n \) satisfies \( \|f - P_n\|_{L_p[0,\varepsilon]} \leq A\|f\|_{L_p[0,1]} \), then
\[
P_n'(0) \leq n^2 \varepsilon^{-1/2} \|P_n\|_{L_p[0,\varepsilon]} \leq Cn^2 \varepsilon^{-1/2} \|P_n\|_{L_p[0,1]}
\leq Cn^2 \varepsilon^{-1/2} (\|f - P_n\|_{L_p[0,1]} + \|f\|_{L_p[0,1]})
\leq Cn^2 \varepsilon^{-1/2} (A^{b^{-1/2p}} + 1) := M(b^{b^{-1/2p}} + 1),
\]
therefore choosing \( b > (2M)^{\max\{1,2p\}} \) gives \( P_n'(0) < b \), which contradicts the facts that \( P_n(0) = 0 \) and \( P_n(x) \geq f(x) \) for \( x \in [0, \varepsilon] \). \( \blacksquare \)

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