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ON SOLUBILITY OF GROUPS WITH FEW NORMALISERS

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Abstract

In this paper we prove that every group with at most 26 normalisers is soluble. This gives a positive answer to Conjecture 3.6 in the author's paper [On groups with a finite number of normalisers', *Bull. Aust. Math. Soc.* **86** (2012), 416–423].

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1. Introduction and results

The groups having exactly one normaliser are the Dedekind groups. All finite groups having exactly two normalisers were classified by Pérez-Ramos [5]; Camp-Mora [1] generalised that result to locally finite groups. In 2004, Tota [6] showed that every group with at most four normalisers of subgroups is soluble of derived length at most two.

The author [7] has shown that every finite group with at most 20 normalisers of subgroups is soluble. Here we show that every arbitrary group with at most 26 normalisers is soluble. This gives a positive answer to [7, Conjecture 3.6].

THEOREM 1.1. *Every arbitrary group with at most 26 normalisers of subgroups is soluble. This estimate is sharp.*

We say that a group G is an $\mathfrak{N}_n$-group if it has exactly n normalisers of subgroups.

Now a general question is posed: *let G be a soluble $\mathfrak{N}_n$-group of solubility length d. Is there a function $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that $d \leq f(n)$?* This is answered in the following theorem.

THEOREM 1.2. *Let G be a soluble $\mathfrak{N}_n$-group of derived length d. Then*

$$d \leq 5 \log_9(n-1) + 2.$$

2. Proofs

Proof of Theorem 1.1. Suppose that G is an $\mathfrak{N}_n$-group with $n \leq 26$. Then it is easy to see that the group G acts on the set $X = \{N_G(H) \mid H \text{ is a subgroup of } G\} \backslash \{G\}$ by conjugation. By assumption, $|X| = n - 1$ (note that $n \geq 2$). Put

$$Y = \bigcap_{T \in X} N_G(T).$$

Now the subgroup Y is the kernel of this action, so Y is normal in G and $G/Y \hookrightarrow S_{n-1}$. Therefore, G/Y is finite. Assume that L is a subgroup of Y. It follows, by definition of Y, that $N_Y(L) \trianglelefteq Y$ and so

$$L \trianglelefteq N_Y(L) \trianglelefteq Y.$$

Hence, every subgroup of Y is a 2-subnormal subgroup. But it is well known, see [2] and [3], that such a group is nilpotent of class at most three. On the other hand, it is easy to see that the group G/Y is an $\mathfrak{N}_m$-group with $m \leq 26$. Thus, as Y is a soluble normal subgroup of G, replacing G by the factor group G/Y, it can be assumed without loss of generality that G is a finite $\mathfrak{N}_n$-group with $n \leq 26$. Now suppose on the contrary that there exists a non-Abelian finite insoluble $\mathfrak{N}_k$-group of the least possible order, where $k \leq 26$. If there exists a nontrivial proper normal subgroup N of G, then the groups G/N and N are in an $\mathfrak{N}_r$-group with $r \leq 26$, and so they are soluble. It follows that G is soluble, which is a contradiction. Therefore, G is a minimal simple $\mathfrak{N}_n$-group with $n \leq 26$. Now [7, Theorem 4.6 and Corollary 4.7] completes the proof. Note that according to [7, Corollary 4.7], A_5 has exactly 27 normalisers of subgroups and so the estimate is sharp. □

Proof of Theorem 1.2. Assume that G is a soluble $\mathfrak{N}_n$-group. Now it is surely well known (see [4]) that the derived length of every soluble subgroup of the symmetric group of degree n ($n \neq 1$), S_n, is at most $5 \log_9 n$. Hence, G/Y, as mentioned in the proof of Theorem 1.1, has derived length at most $5 \log_9(n-1)$ and this completes the proof, as Y is nilpotent of class at most three and so it is soluble of length at most two. □

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