NORMAL LIGHT INTERIOR FUNCTIONS
DEFINED IN THE UNIT DISK

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1. Preliminaries

Let $D$ be the unit disk, $C$ the unit circle, and $f$ a continuous function from $D$ into the Riemann sphere $W$. We say that $f$ is normal if $f$ is uniformly continuous with respect to the non-Euclidean hyperbolic metric in $D$ and the chordal metric in $W$. Let $\chi(w_1, w_2)$ denote the chordal distance between the points $w_1, w_2 \in W$; and let $\rho(z_1, z_2)$ denote the non-Euclidean hyperbolic distance between the points $z_1, z_2 \in D$ [6]. If $\{z_n\}$ and $\{z'_n\}$ are two sequences of points in $D$ with $\rho(z_n, z'_n) \to 0$, we say that $\{z_n\}$ and $\{z'_n\}$ are close sequences.

Let $A$ be an open subarc of $C$, possibly $C$ itself. A Koebe sequence of arcs relative to $A$ is a sequence $\{J_n\}$ of Jordan arcs such that: (a) for every $\varepsilon > 0,$

$$J_n \subset \{z \in D : |z - a| < \varepsilon \text{ for some } a \in A\}$$

for all but finitely many $n$, and (b) every open sector $\Delta$ of $D$ subtending an arc of $C$ that lies strictly interior to $A$ has the property that, for all but finitely many $n$, the arc $J_n$ contains a subarc $L_n$ lying wholly in $\Delta$ except for its two end points which lie on distinct sides of $\Delta$.

We say that the function $f$ has the limit $c$ along the sequence of arcs $\{J_n\}$ (denoted by $f(J_n) \to c$) provided that, for every $\varepsilon > 0$, $\chi(c, f(J_n)) < \varepsilon$ for all but finitely many $n$.

2. Factorization of light interior functions

Let $f$ be a light interior function from $D$ into $W$, i.e. $f$ is an open map which does not take any continuum into a single point. Church [4, p. 86] has pointed out that $f$ has the representation $f = g \circ h$ where $h$ is a
homeomorphism of $D$ onto a Riemann surface $R$ and $g$ is a non-constant meromorphic function defined on $R$. In view of the uniformization theorem [1, p. 181], there exists a conformal mapping $\varphi$ of $R$ onto either the unit disk or the finite complex plane. We will be concerned with the case when the range of $\varphi$ is the unit disk, but remark that similar results hold when the range is the complex plane. Therefore, if $f$ is a light interior function from $D$ into $W$ then $f$ has a factorization $f = g \circ h$ where $h$ is a homeomorphism of $D$ onto $D$ and $g$ is a non-constant meromorphic function in $D$. Conversely, if $h$ is a homeomorphism of $D$ onto $D$ and $g$ is a non-constant meromorphic function in $D$ then the function $f = g \circ h$ is light interior.

**Definition 1.** Let $h$ be a homeomorphism of $D$ onto $D$. If $h$ is uniformly continuous with respect to the non-Euclidean hyperbolic metric in both its domain and range then we say that $h$ is HUC.

**Definition 2.** Let $f$ be a light interior function in $D$ with factorization $f = g \circ h$. If $h$ is HUC then $f$ has a type I factorization; otherwise $f$ has a type II factorization.

**Theorem 1.** If $f$ is a light interior function in $D$ then $f$ has a unique factorization type.

**Proof.** Let $f$ have the factorization $f = g \circ h$. Suppose $f$ also has the factorization $f = G \circ H$. Then as pointed out by Church [4, p. 86] $h \circ H^{-1}$ is a conformal homeomorphism. In view of Pick's theorem [6, Theorem 15, 1.3, p. 239] both $h \circ H^{-1}$ and $h^{-1} \circ H$ are HUC. Since the composition of two uniformly continuous functions is uniformly continuous, it follows that $h$ is HUC if and only if $H$ is HUC; and the proof of the theorem is complete.

**3. Necessary conditions for both $f$ and $g$ normal**

Noshiro [10, p. 154] has divided the class of normal meromorphic functions in $D$ into two categories which are defined as follows: A normal meromorphic function $g$ in $D$ is of the first category if the normal family $\left\{ g\left( \frac{a - z}{1 - \bar{a}z} \right) : a \in D \right\}$ admits no constant limit; otherwise $g$ is of the second category.

**Theorem 2.** Let $f$ be a normal light interior function with factorization $f = g \circ h$. If $g$ is a normal meromorphic function then $h$ is normal. Furthermore, if $g$ is a normal meromorphic function of the first category then $h$ is HUC.
Proof. Let $f$ have the factorization $f = g \circ h$. If $h$ is not normal there exists close sequences $\{z_n\}$ and $\{z'_n\}$ such that $h(z_n) \to e^{i\alpha}$ and $h(z'_n) \to e^{i\beta}$ with $0 < \beta - \alpha < 2\pi$ [7]. For each integer $n$, let $J_n$ be the non-Euclidean geodesic joining $z_n$ to $z'_n$. Then $\{h(J_n)\}$ is a sequence of Jordan arcs such that for every $\varepsilon > 0$,

$$h(J_n) \subset \{ z \in D : 1 - \varepsilon < |z| < 1 \}$$

for all but finitely many $n$, and the end points of $h(J_n)$ tend to $e^{i\alpha}$ and $e^{i\beta}$. Choosing a subsequence of $\{h(J_n)\}$ if necessary, we may assume that there exists a Koebe sequence of arcs $\{L_n\}$ relative to either the open arc $(\alpha, \beta)$ or the open arc $(\beta, \alpha + 2\pi)$ with $L_n \subset h(J_n)$, and a constant $c$ such that $f(z_n) \to c$.

From the normality of $f$ we have $f(J_n) \to c$, and it follows that $g(L_n) \to c$. By a theorem of Bagemihl and Seidel [2, Theorem 1, p. 10], $g \equiv c$ in violation of our hypothesis. Therefore $h$ is normal and the proof of the first part is complete.

Now assume that $g$ is a normal meromorphic function of the first category. If $h$ is not HUC there exists close sequences $\{z_n\}$ and $\{z'_n\}$ and a $\delta > 0$ with $\rho(h(z_n), h(z'_n)) \geq \delta$, and a constant $c$ such that $f(z_n) \to c$.

Let $S_n(z) = (h(z_n) - z)/(1 - \overline{h(z_n)}z)$ and let $G_n(z) = g(S_n(z))$. Then the normal family $\{G_n\}$ has a subsequence which converges uniformly on each compact subset of $D$ to a meromorphic function $G$ [8, p. 53]. Let $J_n$ be the non-Euclidean geodesic joining $z_n$ to $z'_n$ and let $L_n = h(J_n)$. Then $d(L_n) = d(S_n^{-1}(L_n)) \geq \delta$, where $d(E)$ is the hyperbolic diameter of the set $E \subset D$. From the normality of $f$ we have $f(J_n) \to c$, so that $g(L_n) \to c$, and hence $G_n(S_n^{-1}(L_n)) \to c$. For $r$ ($0 \leq r \leq \delta$) fixed, there exists a point $Z_n \in S_n^{-1}(L_n)$ such that $\rho(0, Z_n) = r$. Let $Z_0$ be a cluster point of the sequence $\{Z_n\}$ on the circle $\{ z : \rho(0, z) = r \}$.

Choosing a subsequence of $\{G_n\}$ if necessary, we may assume that $Z_n \to Z_0$ and $G_n(Z_n) \to c$. A familiar argument (see e.g. [3, p. 179]) in the theory of continuous convergence shows that $G(Z_0) = c$. Since $r$ ($0 \leq r \leq \delta$) was arbitrary, $0$ is a limit point of values for which $G$ assumes $c$ and hence $G \equiv c$ in violation our hypothesis. Therefore $h$ is HUC and the proof of the theorem is complete.
4. Bounded non-normal light interior functions

Every bounded holomorphic function is normal, but the following result shows that boundedness is not sufficient for a light interior function to be normal.

**Theorem 3.** If a homeomorphism $h$ of $D$ onto $D$ is not HUC, then there exists a Blaschke product $B$ in $D$ such that the bounded light interior function $f = B \circ h$ is not normal.

**Proof.** If $h$ is not HUC there exists close sequences $\{z_n\}$ and $\{z'_n\}$ and a $\delta > 0$ such that $\rho(h(z_n), h(z'_n)) \geq \delta$. Let $h(z_n) = w_n$ and $h(z'_n) = w'_n$. Since $h$ is uniformly continuous on compact subsets we necessarily have that $|z_n| \to 1$, $|z'_n| \to 1$, $|w_n| \to 1$, and $|w'_n| \to 1$. Hence, choosing a subsequence of $\{w_n\}$ if necessary, we may assume that $\{w_n\}$ is a Blaschke sequence, i.e. $\sum_{n=1}^{\infty} (1 - |w_n|) < \infty$. There exists a Blaschke subsequence $\{w_{n_k}\}$ of $\{w_n\}$ and a corresponding subsequence $\{w'_{n_k}\}$ of $\{w'_n\}$ for which $\rho(R_{n_k}, r_k) \geq \tanh^{-1}(1-1/k^2)$ where $r_k = \min\{|w_{n_k}|, |w'_{n_k}|\}$ and $R_k = \max\{|w_{n_k}|, |w'_{n_k}|\}$.

It follows easily that

$$\rho(w_{n_k}, w'_{n_k}) \geq \begin{cases} \tanh^{-1}(1 - 1/(k + 1)^2) & (1 \leq k < j) \\ \tanh^{-1}(1 - 1/k^2) & (1 \leq j \leq k), \end{cases}$$

and hence

$$\left| \frac{w_{n_k} - w'_{n_j}}{1 - w_{n_k}w'_{n_j}} \right| \geq \begin{cases} 1 - 1/(k + 1)^2 & (1 \leq k < j) \\ 1 - 1/k^2 & (1 \leq j \leq k). \end{cases}$$

Recall that $\rho(w_{n_k}, w'_{n_k}) \geq \delta > 0$ ($k = 1, 2, \ldots$) so that

$$\left| \frac{w_{n_k} - w'_{n_k}}{1 - w_{n_k}w'_{n_k}} \right| \geq \tanh^{-1} \delta > 0 \ (k = 1, 2, \ldots).$$

Set $B(z) = \prod_{k=1}^{\infty} \frac{|w_{n_k}|(w_{n_k} - z)}{w_{n_k}(1 - w_{n_k}z)}$.

Consider $B(w'_{n_j})$ for $j \geq 1$,

$$|B(w'_{n_j})| = \prod_{k=1}^{j-1} \left| \frac{w_{n_k} - w'_{n_j}}{1 - w_{n_k}w'_{n_j}} \right| \cdot \left| \frac{w_{n_j} - w'_{n_j}}{1 - w_{n_j}w'_{n_j}} \right| \cdot \prod_{k=j+1}^{\infty} \left| \frac{w_{n_k} - w'_{n_j}}{1 - w_{n_k}w'_{n_j}} \right|$$
\[
\geq \left( \tanh^{-1} \theta \right) \prod_{k=1}^{j-1} \left( 1 - \frac{1}{(k+1)^2} \right) \prod_{k=j+1}^\infty \left( 1 - \frac{1}{k^2} \right)
\]

\[= \left( \tanh^{-1} \theta \right) \prod_{k=2}^\infty \left( 1 - \frac{1}{k^2} \right) = \frac{1}{2} \tanh^{-1}(\theta) > 0.\]

Let \( f = B \circ h \). By assumption \( \{z_n\} \) and \( \{z'_n\} \) are necessarily close sequences with

\[\lim f(z_n) = \lim B(h(z_n)) = \lim B(w_n) = 0\]

and \( |f(z'_n)| = |B(h(z'_n))| = |B(w'_n)| \geq 1/2 \tanh^{-1}(\theta) > 0.\) By a theorem of Lappan [7, Theorem 3, p. 156], \( f \) is not normal and the proof is complete.

The previous theorem suggests that the normality of \( g \) does not insure the normality of \( f \). An even stronger statement is the following result.

**Theorem 4.** There exists a homeomorphism \( h \) of \( D \) onto \( D \) with the property: If \( g \) is a normal meromorphic function in \( D \), which has two distinct asymptotic limits, then the light interior function \( f = g \circ h \) is not normal.

Since a bounded holomorphic function in \( D \) is normal and possesses uncountably many distinct radial limits we obtain the following corollary.

**Corollary.** There exists a homeomorphism \( h \) of \( D \) onto \( D \) with the property: If \( g \) is a non-constant bounded holomorphic function in \( D \), then the bounded light interior function \( f = g \circ h \) is not normal.

**Proof of Theorem 4.** Let \( \{R_n\} \) be a strictly increasing sequence of non-negative real numbers with \( R_1 = 0 \) for which \( \rho(R_n, R_{n+1}) = 1/n \). Define the mapping \( h \) in \( D \) by

\[h(z) = h(re^{i\theta}) = r \exp \left( i\theta + 2\pi i (r - R_n)/(R_{n+1} - R_n) \right)\]

for \( R_n \leq r < R_{n+1} \) \((n = 1, 2, \cdots)\). It is easy to verify that \( h \) is a homeomorphism of \( D \) onto \( D \).

Since \( g \) has two distinct asymptotic limits, a theorem of Lehto and Virtanen [8, Theorem 2, p. 53] implies that \( g \) has two distinct radial limits. Let \( \tau_a \) and \( \tau_b \) be the radii which terminate at the points \( e^{i\alpha} \) and \( e^{i\beta} \), respectively, for which \( g(re^{i\theta}) \to a \) and \( g(re^{i\theta}) \to b \) with \( b \neq a \).

Now the radii of \( D \) are mapped onto spirals by \( h^{-1} \). Let \( h^{-1}(\tau_a) \cap [R_n, R_{n+1}) = z_n \) and \( h^{-1}(\tau_b) \cap [R_n, R_{n+1}) = z'_n \). Then \( \rho(z_n, z'_n) < \rho(R_n, R_{n+1}) = 1/n \) with
5. Sufficient conditions for \( f \) normal

We now determine conditions on \( h \) and \( g \) which insure the normality of \( f \). Since the composition of two uniformly continuous functions is uniformly continuous the first result in this direction is obvious.

**Theorem 5.** Let \( h \) be a homeomorphism of \( D \) onto \( D \) which is HUC. If \( g \) is a non-constant normal meromorphic function, then the light interior function \( f = g \circ h \) is normal. Furthermore, if both \( h \) and \( h^{-1} \) are HUC, then \( g \) is normal if and only if \( f \) is normal.

Let \( f \) be a light interior function in \( D \) with factorization \( f = g \circ h \) with \( h \) a \( K \)-quasiconformal homeomorphism of \( D \) onto \( D \). We show that \( f \) is normal if and only if \( g \) is normal. This result was proved by Väisälä [11, Theorem 5, p. 20] whose proof is considerably different.

**Theorem 6.** If \( h \) is a \( K \)-quasiconformal homeomorphism of \( D \) onto \( D \), then both \( h \) and \( h^{-1} \) are HUC.

**Theorem 7.** Let \( f \) be a light interior function in \( D \) with factorization \( f = g \circ h \) with \( h \) a \( K \)-quasiconformal homeomorphism. Then \( f \) is normal if and only if \( g \) is normal.

**Proof of theorem 6.** Since \( h \) is \( K \)-quasiconformal, by a theorem of Mori [9] \( h^{-1} \) is also \( K \)-quasiconformal. Hersch and Pfluger [5] have shown that if \( h \) is \( K \)-quasiconformal then \( \rho(h(z), h(z')) \leq \mathcal{W}_K(\rho(z, z')) \) where \( \mathcal{W}_K \) is continuous and strictly increasing and defined for all \( z \geq 0 \) with \( \mathcal{W}_K(0) = 0 \). It follows easily that \( h \) is HUC. Similarly \( h^{-1} \) is HUC and the theorem is proved.

**Proof of theorem 7.** From Theorem 6 both \( h \) and \( h^{-1} \) are HUC. By Theorem 5, \( f \) is normal if and only if \( g \) is normal and the theorem is proved.

**Definition 3.** Let \( h \) be a homeomorphism of \( D \) onto \( D \). Define the set \( F(h) \) as follows: \( e^{i\theta} \in F(h) \) if there exist close sequences \( \{z_n\} \) and \( \{z'_n\} \) and a \( \delta > 0 \) for which \( \rho(h(z_n), h(z'_n)) \geq \delta \) and \( h(z_n) \to e^{i\theta} \).
THEOREM 8. Let $h$ be a normal homeomorphism of $D$ onto $D$. If $g$ is a non-constant normal meromorphic function which is continuous on $D \cup F(h)$, then the light interior function $f = g \circ h$ is normal.

Proof. If $f$ is not normal there exist close sequences $\{z_n\}$ and $\{z'_n\}$ such that $f(z_n) \to a$ and $f(z'_n) \to b$ with $b \neq a$ [7]. It follows from the normality of $g$ that $\{h(z_n)\}$ and $\{h(z'_n)\}$ are not close. Choosing a subsequence of $\{z_n\}$ and a corresponding subsequence of $\{z'_n\}$ if necessary, we may assume that $h(z_n) \to e^{i\theta}$ and $h(z'_n) \to e^{i\varphi}$ with $e^{i\varphi} \in F(h)$. But $g$ is continuous on $D \cup F(h)$ and hence $b = \lim f(z'_n) = \lim g(h(z'_n)) = \lim g(h(z_n)) = \lim f(z_n) = a$ which is a contradiction. Therefore $f$ is normal and the proof is complete.

REFERENCES