PAIRS OF QUADRATIC FORMS MODULO ONE

by R. C. BAKER and J. BRÜDERN

(Received 23 July, 1991)

1. Introduction. Let s be a natural number, $s \geq 2$. We seek a positive number $\lambda(s)$ with the following property:

Let $\varepsilon > 0$. Let $Q_1(x_1, \ldots, x_s)$, $Q_2(x_1, \ldots, x_s)$ be real quadratic forms, then for $N > C_1(s, \varepsilon)$ we have

$$\max(\|Q_1(\mathbf{n})\|, \|Q_2(\mathbf{n})\|) < N^{-\lambda(s)+\varepsilon} \tag{1.1}$$

for some integers $n_1, \ldots, n_s$,

$$0 < \max(|n_1|, \ldots, |n_s|) \leq N. \tag{1.2}$$

Here $\|\theta\|$ denotes the distance from θ to the nearest integer.

The first result of this kind was obtained by Danicic **[6]**, who showed that one may take

$$\lambda(s) = \left(3 + \frac{4}{s} + \frac{2}{s} \sum_{r=1}^{s} \frac{1}{r}\right)^{-1}. \tag{1.3}$$

Thus $\lambda(2) = 2/13$ and $\lambda(3) = 9/50$ are admissible. In 1976, however, Schmidt **[11]** showed that, given real α, β,

$$\min_{1 \leq n \leq N} \max(\|\alpha n^2\|, \|\beta n^2\|) < C_2(\varepsilon) N^{-1/6+\varepsilon}.$$

This trivially permits one to take $\lambda(2) = 1/6$.

Baker and Harman **[2]** showed that one may take

$$\lambda(s) = 1 - \delta(s)$$

where $\delta(s) \to 0$ as $s \to \infty$, although $\delta(s)$ was not calculated explicitly. The method of **[2]** is weaker than Danicic's for small s, but obviously stronger for large s.

In the present paper we improve (1.3) for all $s \geq 2$. It is convenient to state our result in terms of the corresponding exponent for a single quadratic form. We write $\alpha(s)$ for a number with the following property: *given a real quadratic form $Q(x_1, \ldots, x_s)$, then for $\varepsilon > 0$ and $N > C_3(s, \varepsilon)$ we have*

$$\|Q(\mathbf{n})\| < N^{-\alpha(s)+\varepsilon}$$

for some integers $n_1, \ldots, n_s$ satisfying (1.2).

For $s \geq 1$, we may take

$$\alpha(s) = s/(s+1) \tag{1.4}$$

(Danicic **[5]**). We shall need a generalization of (1.4), which we establish in Section 2. For $s \geq 4$, results stronger than (1.4) have been obtained **[10, 3, 9]**. In particular, we may take

$$\alpha(4) = 8/9[\mathbf{9}],\ \alpha(5) = 1[\mathbf{3}],\ \alpha(6) = 78/71[\mathbf{9}], \tag{1.5}$$

$$\alpha(s) = 2 - 8/s[\mathbf{9}]. \tag{1.6}$$

Glasgow Math. J. **35** (1993) 51–61.

https://doi.org/10.1017/S0017089500009551 Published online by Cambridge University Press

THEOREM 1. *We may take*

$$\lambda(s)=\left(2+\frac{6}{s}\right)^{-1} \quad \text{for} \quad s\le 5, \tag{1.7}$$

$$\lambda(s)=\frac{s}{s+11+\alpha(6)^{-1}+\ldots+\alpha(s)^{-1}} \quad \text{for} \quad s\ge 6. \tag{1.8}$$

In particular, we may take $\lambda(2)=\frac{1}{5}$, $\lambda(3)=\frac{1}{4}$.

Clearly the limiting value of $\lambda(s)$ in (1.8) is 2/3; thus **[2]** is stronger for large s. We also observe that stronger results hold for additive quadratic forms **[4]**.

In our proof we use ideas from the lattice method of Schmidt **[11]**, **[1]**. A key role is also played by estimates for

$$\sum_{m=1}^{M}|S(mQ)|^2$$

where Q is a real quadratic form and

$$S(mQ)=\sum_{x_1=1}^{N_1}\ldots\sum_{x_s=1}^{N_s}e(mQ(x_1,\ldots,x_s)). \tag{1.9}$$

Here $e(\theta)$ denotes $e^{2\pi i\theta}$. Davenport **[7, 8]** studied the case $M=1$, $N_1=\ldots=N_s$ and Danicic **[5]** treated the case $M>1$, $N_1=\ldots=N_s$. We discuss the general case in Section 2.

Constants implied by $\ll$ and $\gg$ depend at most on ε, s. We suppose, as we may, that ε is sufficiently small and write $\delta=\varepsilon^2$. We write $|\mathscr{A}|$ for the cardinality of a finite set $\mathscr{A}$. The fractional part of θ is written $\{\theta\}$.

2. Successive minima. Let

$$Q(x_1,\ldots,x_s)=\sum_{i=1}^{s}\ldots\sum_{j=1}^{s}\lambda_{ij}x_ix_j$$

with $\lambda_{ij}=\lambda_{ji}$, and write

$$L_i(x_1,\ldots,x_s)=\sum_{j=1}^{s}\lambda_{ij}x_j.$$

Given positive integers $M, N_1,\ldots,N_s$, we define $S(mQ)$ by (1.9) $(m=1,\ldots,M)$.

Just as on p. 107 of **[1]**, we have

$$\begin{aligned}|S(mQ)|^2&=\sum_{x_1=1}^{N_1}\ldots\sum_{x_s=1}^{N_s}\sum_{1\le x_j+z_j\le N_j}e(2mx_1L_1(\mathbf{z})+\ldots+2mx_sL_s(\mathbf{z})+mQ(\mathbf{z}))\\&\le\sum_{z_1=-(N_1-1)}^{N_1-1}\ldots\sum_{z_s=-(N_s-1)}^{N_s-1}\prod_{j=1}^{s}\min((N_j,\|2mL_j(\mathbf{z})\|^{-1}).\end{aligned} \tag{2.1}$$

In order to estimate the right hand side of (2.1) we define $2s$ linear forms as follows:

$$\left.\begin{aligned}\zeta_j(x_1,\ldots,x_{2s})&=2M^{1/2}N_j(L_j(x_1,\ldots,x_s)-x_{s+j}\\ \zeta_{s+j}(x_1,\ldots,x_{2s})&=(2M^{1/2}N_j)^{-1}x_j\end{aligned}\right\}\quad (j=1,\ldots,s).$$

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Throughout this section, let $\pi_1, \ldots, \pi_{2s}$ denote the successive minima of the lattice

$$\Gamma = \{(\zeta_1(\mathbf{x}), \ldots, \zeta_{2s}(\mathbf{x})) : \mathbf{x} \in \mathbb{Z}^{2s}\}$$

with respect to the unit ball.

LEMMA 1. *We have*

$$1 \ll \pi_j \pi_{2s+1-j} \ll 1 \qquad (j = 1, \ldots, 2s). \tag{2.2}$$

Proof. See [8], formula (20).

LEMMA 2. *Let $B > 0$. The number of integer solutions of*

$$\left.\begin{array}{c} \|L_j(x_1, \ldots, x_s)\| < B(2M^{1/2}N_j)^{-1} \\ |x_j| < B(2M^{1/2}N_j) \end{array}\right\} \qquad (j = 1, \ldots, s). \tag{2.3}$$

is

$$\ll 1 + (\pi_1 \ldots \pi_l)^{-1} B^l \tag{2.4}$$

for some l, $1 \le l \le 2s$.

Proof. The number of solutions of the inequalities (2.3) is at most the number of lattice points $\mathbf{p}$ in Γ with $|\mathbf{p}| \le \sqrt{2s}B$. As in the proof of Lemma 7.1 of [1], the number of such points $\mathbf{p}$ is 1 if $\sqrt{2s}B < \pi_1$ and is

$$\ll (\pi_1 \ldots \pi_l)^{-1} B^l$$

otherwise, where l is maximal with $\pi_l \le \sqrt{2s}B$.

LEMMA 3. *We have*

$$\sum_{m=1}^{M} |S(mQ)|^2 \ll M^{\delta}(N_1 \ldots N_s)^{1+2\delta}(1 + M^{l/2}(\pi_1 \ldots \pi_l)^{-1})$$

for some l, $1 \le l \le 2s$.

Proof. By (2.1), and a standard divisor argument,

$$\sum_{m=1}^{M} |S(mQ)|^2 \ll M^{\delta}(N_1 \ldots N_s)^{\delta} \sum_{x_1=1}^{2MN_1} \ldots \sum_{x_s=1}^{2MN_s} \prod_{j=1}^{s} \min(N_j, \|L_j(\mathbf{x})\|^{-1}). \tag{2.5}$$

Let $k_1, \ldots, k_s$ be integers satisfying $0 \le k_i < N_i$ and let $\mathscr{E}(k_1, \ldots, k_s)$ be the set of $\mathbf{x}$ in $\mathbb{Z}^s$ with

$$\left.\begin{array}{c} \dfrac{k_j}{N_j} \le \{L_j(\mathbf{x})\} < \dfrac{k_j+1}{N_j} \\ |x_j| < 2MN_j \end{array}\right\} \qquad (j = 1, \ldots, s).$$

Fix $\mathbf{x}'$ in $\mathscr{E}(k_1, \ldots, k_s)$. Then for $\mathbf{x}'' = \mathbf{x}' + \mathbf{x}$ in $\mathscr{E}(k_1, \ldots, k_s)$,

$$\left.\begin{array}{c} \|L_j(\mathbf{x})\| < N_j^{-1} \\ |x_j| < 4MN_j \end{array}\right\} \qquad (j = 1, \ldots, s).$$

Applying Lemma 2 with $B = 2M^{1/2}$, we obtain

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$$\max_{k_1,\dots,k_s} |\mathscr{E}(k_1,\dots,k_s)| \ll 1 + M^{l/2}(\pi_1 \dots \pi_l)^{-1} \tag{2.6}$$

for some l, $1 \le l \le 2s$.

Combining (2.5) and (2.6), and writing $k_j' = \min(k_j, N-1-k_j)$,

$$\sum_{m=1}^{M} |S(mQ)|^2 \ll M^{\delta}(N_1 \dots N_s)^{\delta} \sum_{k_1=0}^{N_1-1} \dots \sum_{k_s=0}^{N_s-1} |\mathscr{E}(k_1,\dots,k_s)| \prod_{j=1}^{s} \min\left(N_j, \frac{N_j}{k_j'}\right)$$

$$\ll M^{\delta}(N_1 \dots N_s)^{\delta}(1 + M^{l}(\pi_1 \dots \pi_l)^{-1}) \sum_{k_1=0}^{N_1-1} \dots \sum_{k_s=0}^{N_s-1} \prod_{j=1}^{s} \min\left(N_j, \frac{N_j}{k_j'}\right).$$

The lemma follows at once.

We can now prove the generalization of (1.4) mentioned in Section 1.

THEOREM 2. *Let $Q(x_1, \dots, x_s)$ be a real quadratic form. Let $N > C_4(s, \varepsilon)$ and let $N_1, \dots, N_s$ be positive real numbers satisfying*

$$N_1 \dots N_s \geqslant N^s. \tag{2.7}$$

Then there are integers $n_1, \dots, n_s$, not all zero, satisfying

$$\|Q(n_1, \dots, n_s\| < N^{-s/(s+1)+\varepsilon}, \tag{2.8}$$

$$|n_j| \le N_j \qquad (j = 1, \dots, s). \tag{2.9}$$

Proof. We consider first the case of positive integral $N_1, \dots, N_s$. Suppose if possible that (2.8) has no nonzero solution satisfying (2.9). Let $M = [N^{s/(s+1)-\varepsilon}] + 1$. Applying Theorem 2.2 of **[1]**, together with Cauchy's inequality, we obtain

$$\sum_{m=1}^{M} |S(mQ)|^2 \ge M^{-1}\Big(\sum_{m=1}^{M} |S(mQ)|\Big)^2 \gg (N_1 \dots N_s)^2 M^{-1}.$$

Combining this with Lemma 3, we have

$$(N_1 \dots N_s)^2 M^{-1} \ll (1 + M^{l/2}(\pi_1 \dots \pi_l)^{-1}) M^{\delta}(N_1 \dots N_s)^{1+2\delta}. \tag{2.10}$$

From (2.2),

$$\pi_1 \dots \pi_{2s} \ll 1,$$

$$(\pi_1 \dots \pi_l)^{-1} \ll \pi_{l+1} \dots \pi_{2s} \ll \pi_1^{-(2s-l)}. \tag{2.11}$$

Combining (2.10), (2.11), we have

$$(N_1 \dots N_s)^{1-2\delta} M^{-1-\delta} \ll 1 + M^{l/2}\pi_1^{-(2s-l)}. \tag{2.12}$$

From the hypothesis that (2.7) has no solution satisfying (2.8), it follows that

$$\pi_1 \ge (4sM^{1/2})^{-1}. \tag{2.13}$$

For suppose the contrary; then there is a nonzero integer point $(x_1, \dots, x_{2s})$ with

$$|L_j(x_1, \dots, x_s) - x_{s+j}| < (2M^{1/2}N_j)^{-1}(4sM^{1/2})^{-1}$$
$$< s^{-1}M^{-1}N_j^{-1}, \tag{2.14}$$

$$|x_j| < 2M^{1/2}N_j(4sM^{1/2})^{-1} < N_j \tag{2.15}$$

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for $j=1,\ldots,s$. Now $(x_1,\ldots,x_s)\neq \mathbf{0}$, since from $x_1=\ldots=x_s=0$ one obtains $x_{s+1}=\ldots=x_{2s}=0$ via (2.14). Thus there is a nonzero integer point $(x_1,\ldots,x_s)$ satisfying (2.15) and

$$\|Q(x_1,\ldots,x_s)\| = \left\|\sum_{j=1}^{s} x_j L_j(\mathbf{x})\right\|$$

$$\leq \sum_{j=1}^{s} |x_j|\, \|L_j(\mathbf{x})\| < \sum_{j=1}^{s} N_j s^{-1} M^{-1} N_j^{-1} = M^{-1},$$

which is a contradiction. This establishes (2.13).

Combining (2.12) and (2.13) yields

$$(N_1\ldots N_s)^{1-2\delta} M^{-1-\delta} \ll M^s,$$

$$M^{s+1+\delta} \gg (N_1\ldots N_s)^{1-2\delta} \gg N^{s-2\delta s}$$

from (2.7). This contradicts the definition of M, and Theorem 2 is proved in the case of positive integral N_j.

The case $N_1\geq 1,\ldots,N_s\geq 1$ follows at once. For (2.7) implies

$$[N_1]\ldots[N_s]\geq (N/2)^s.$$

Since

$$(N/2)^{-s/(s+1)+\varepsilon/2} < N^{-s/(s+1)+\varepsilon}$$

for large N, this permits us to solve (2.8) subject to (2.9) on enlarging C_4.

We may now prove the general case by induction on s. The case $s=1$ has been done. Now let $s\geqslant 2$ and suppose the theorem known for forms in $s-1$ variables. Let $N_1>0,\ldots,N_s>0$ satisfy (2.7) with $N\geq C_4(s,\varepsilon)$. We may suppose some N_j is less than 1; say

$$N_s<1.$$

But then

$$N_1\ldots N_{s-1}\geq H^{s-1}$$

where $H=N^{s/(s-1)}$. Since H is large, there is a nonzero solution of

$$\|Q(n_1,\ldots,n_{s-1},0)\| < H^{-(s-1)/s+\varepsilon}$$

in integers $n_1,\ldots,n_{s-1}$ with $|n_j|\leq N_j$. Since

$$H^{-(s-1)/s+\varepsilon} = N^{-1}H^{\varepsilon} \leqslant N^{-1+2\varepsilon},$$

this completes the induction step, and Theorem 2 is proved.

We may describe the substitution of (2.9) for (1.2) as "replacing a cube by a box". It seems to be difficult to replace a cube by a box in the work of Baker and Harman **[3]**, and more difficult still in the work of Heath-Brown **[9]**. This is a pity, since such results would lead to an improvement of Theorem 1 for $s\geqslant 4$.

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3. Proof of Theorem 1. Define $\lambda(s)$ by (1.7), (1.8). Let $N > C_1(s, \varepsilon)$ and let

$$\Delta = N^{2(\lambda-\varepsilon)}, \qquad \Lambda = \Delta^{1/2}\mathbb{Z}^2, \qquad \Pi = \Delta^{-1/2}\mathbb{Z}^2.$$

Let Π^* denote the set of primitive points of Π. Let K_0 denote the unit ball in $\mathbb{R}^2$. Let

$$\mathbf{Q}(\mathbf{x}) = \Delta^{1/2}(Q_1(\mathbf{x}), Q_2(\mathbf{x})) = \sum_{i=1}^{s} \boldsymbol{\lambda}_{ij} x_i x_i,$$

where $\boldsymbol{\lambda}_{ij} = \boldsymbol{\lambda}_{ji}$. Let

$$\mathbf{L}_i(\mathbf{x}) = \sum_{j=1}^{s} \boldsymbol{\lambda}_{ij} x_j \qquad (i = 1, \ldots, s).$$

To prove the theorem, it suffices to find $\mathbf{n}$ satisfying (1.2) for which

$$\mathbf{Q}(\mathbf{n}) \in \Lambda + K_0. \tag{3.1}$$

Suppose that there is no such $\mathbf{n}$. Applying [**1**], Lemma 7.4, we find that

$$\sum_{\substack{\mathbf{p} \in \Pi \\ 0 < |\mathbf{p}| < N^{\delta}}} |S(\mathbf{p}\mathbf{Q})| \gg N^s. \tag{3.2}$$

Here

$$S(\mathbf{p}\mathbf{Q}) = \sum_{n_1=1}^{N} \ldots \sum_{n_s=1}^{N} e(\mathbf{p}\mathbf{Q}(n_1, \ldots, n_s)),$$

$\mathbf{ab}$ denotes dot product, and $|\mathbf{a}| = (\mathbf{aa})^{1/2}$.

We may rewrite (3.2) in the form

$$\sum_{\substack{\mathbf{p} \in \Pi^* \\ 0 < |\mathbf{p}| < N^{\delta}}} \sum_{m=1}^{[N^{\delta}/|\mathbf{p}|]} |S(m\mathbf{p}\mathbf{Q}| \gg N^s.$$

The outer summation is taken over $\ll \Delta N^{2\delta}$ points $\mathbf{p}$. We select $\mathbf{p}$ for which

$$\sum_{m=1}^{[N^{\delta}/|\mathbf{p}|]} |S(m\mathbf{p}\mathbf{Q})| \gg N^{s-2\delta}\Delta^{-1}. \tag{3.3}$$

It is helpful to note that

$$\Delta^{-1/2} \leq |\mathbf{p}| < N^{\delta}. \tag{3.4}$$

We now apply Lemma 3, taking $Q(\mathbf{x}) = \sum_{i=1}^{s}\sum_{j=1}^{s} \mathbf{p}\boldsymbol{\lambda}_{ij} x_i x_j$ and $L_i(\mathbf{x}) = \mathbf{p}\mathbf{L}_i(x)$; also $N_1 = \ldots = N_s = N$. With successive minima $\pi_1, \ldots, \pi_{2s}$ defined as in Section 2, and with

$$M = [N^{\delta}|\mathbf{p}|^{-1}], \tag{3.5}$$

we have

$$\sum_{m=1}^{M} |S(m\mathbf{p}\mathbf{Q})|^2 \ll N^{s+3\delta s}(1 + M^{l/2}(\pi_1 \ldots \pi_l)^{-1})$$

for some l, $1 \leq l \leq 2s$. In conjunction with (3.3) and Cauchy's inequality, this yields

$$N^{2s-4\delta}\Delta^{-2}M^{-1} \ll N^{s+3\delta s} + N^{s+3\delta s}M^{l/2}(\pi_1 \ldots \pi_l)^{-1}.$$

In view of (3.4), (3.5), it is easily verified that

$$N^{s+3\delta s} \ll N^{2s-5\delta}\Delta^{-2}M^{-1},$$

so that

$$\pi_1 \ldots \pi_l \ll N^{-s+5\delta s}M^{1+l/2}\Delta^2. \tag{3.6}$$

Suppose for a moment that $l > s$. We apply (2.2). Cancelling $\pi_{2s+1-l}\pi_l, \ldots, \pi_s\pi_{s+1}$ from (3.6),

$$\pi_1 \ldots \pi_{2s-l} \ll N^{-s+5\delta s}M^{1+l/2}\Delta^2.$$

We deduce that in all cases there is a k, $1 \le k \le s$, such that

$$\pi_1 \ldots \pi_k \ll N^{-s+5\delta s}M^{1+s-k/2}\Delta^2. \tag{3.7}$$

We now find a lower bound for π_r $(1 \le r \le s)$. By definition of successive minima, there are points $\mathbf{z}_1, \mathbf{y}_1, \ldots, \mathbf{z}_s, \mathbf{y}_s$ in $\mathbb{Z}^s$ such that $(\mathbf{z}_1, \mathbf{y}_1), \ldots, (\mathbf{z}_s, \mathbf{y}_s)$ are linearly independent in $\mathbb{Z}^{2s}$ and, for $j = 1, \ldots, s$,

$$2M^{1/2}N|\mathbf{p}\mathbf{L}_j(\mathbf{z}_r) - y_{rj}| \le \pi_r, \tag{3.8}$$

$$(2M^{1/2}N)^{-1}|z_{rj}| \le \pi_r. \tag{3.9}$$

Here $\mathbf{z}_r = (z_{r1}, \ldots, z_{rs})$, $\mathbf{y}_r = (y_{r1}, \ldots, y_{rs})$.

In particular,

$$\|\mathbf{p}\mathbf{L}_j(\mathbf{z}_r)\| \ll M^{-1/2}N^{-1}\pi_r.$$

According to [**1**], Lemma 7.9, this implies

$$\mathbf{L}_j(\mathbf{z}_r) = \boldsymbol{l}_{jr} + \mathbf{s}_{jr} + \mathbf{b}_{jr}$$

with $\boldsymbol{l}_{jr} \in \Lambda$ and

$$|\mathbf{b}_{jr}| \ll |\mathbf{p}|^{-1}M^{-1/2}N^{-1}\pi_r, \tag{3.10}$$

where $\mathbf{s}_{jr}$ lies in the 1-dimensional space $\mathbf{p}^\perp$ orthogonal to $\mathbf{p}$.

Consider the points $\boldsymbol{\theta}_{uv}$ defined by

$$\mathbf{Q}(x_1\mathbf{z}_1 + \ldots + x_r\mathbf{z}_r) = \sum_{u=1}^{r}\sum_{v=1}^{r} \boldsymbol{\theta}_{uv}x_ux_v.$$

By an easy computation,

$$\begin{aligned}\boldsymbol{\theta}_{uv} &= \sum_{j=1}^{s} \mathbf{z}_{uj}\mathbf{L}_j(\mathbf{z}_v)\\ &= \sum_{j=1}^{s} \mathbf{z}_{uj}(\boldsymbol{l}_{jv} + \mathbf{s}_{jv} + \mathbf{b}_{jv}).\end{aligned}$$

Consequently, there is a form $\mathbf{Q}_0(x_1, \ldots, x_r)$ with coefficients in $\mathbf{p}^\perp$ such that

$$\mathbf{Q}(x_1\mathbf{z}_1 + \ldots + x_r\mathbf{z}_r) \equiv \mathbf{Q}_0(x_1, \ldots, x_r) + \sum_{u=1}^{r}\sum_{v=1}^{r} x_ux_v \sum_{j=1}^{s} z_{uj}\mathbf{b}_{jv} (\mathrm{mod}\ \Lambda) \tag{3.11}$$

whenever $\mathbf{x} \in \mathbb{Z}^r$.

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The one-dimensional lattice $\Lambda' = 2\Lambda \cap \mathbf{p}^{\perp}$ has determinant $2|\mathbf{p}|\Delta$, as one easily verifies. By the definition of $\alpha(r)$ we may choose integers $x_1, \dots, x_r$ with

$$0 < \max_u |x_u| \ll d(\Lambda')^{\delta + 1/\alpha(r)}$$

$$\ll (|\mathbf{p}|\Delta)^{\delta + 1/\alpha(r)} \tag{3.12}$$

such that

$$2\mathbf{Q}_0(x_1, \dots, x_r) \in \Lambda' + K_0. \tag{3.13}$$

In particular, we have

$$\mathbf{Q}_0(x_1, \dots, x_r) \in \Lambda' + \tfrac{1}{2}K_0. \tag{3.14}$$

Suppose for a moment that

$$\sum_{u=1}^{r} x_u \mathbf{z}_u = \mathbf{0}. \tag{3.15}$$

Then, recalling (3.12), (3.8), we have

$$\left| \sum_{u=1}^{r} x_u y_{uj} \right| = \left| \sum_{u=1}^{r} x_u \{\mathbf{p}\mathbf{L}_j(\mathbf{z}_u) - y_{uj}\} \right|$$

$$\ll (|\mathbf{p}|\Delta)^{\delta + 1/\alpha(r)} M^{-1/2} N^{-1} \pi_r$$

for $j = 1, \dots, s$. Since $(\mathbf{z}_1, \mathbf{y}_1), \dots, (\mathbf{z}_s, \mathbf{y}_s)$ are linearly independent, not all the integers

$$\sum_{u=1}^{r} x_u y_{uj} \qquad (j = 1, \dots, s)$$

are zero. It follows that

$$\pi_r \gg M^{1/2} N (|\mathbf{p}|\Delta)^{-\delta - 1/\alpha(r)}. \tag{3.16}$$

Now suppose that

$$\sum_{u=1}^{r} x_u \mathbf{z}_u \neq \mathbf{0}. \tag{3.17}$$

Then either

$$\left| \sum_{u=1}^{r} x_u \mathbf{z}_u \right| > N \tag{3.18}$$

or

$$\left| \sum_{u=1}^{r} \sum_{v=1}^{r} x_u x_v \sum_{j=1}^{s} \mathbf{z}_{uj} \mathbf{b}_{jv} \right| > 1/2. \tag{3.19}$$

For, if both these inequalities fail, we can combine (3.11), (3.14) to obtain a non-zero integer vector

$$\mathbf{n} = \sum_{u=1}^{r} x_u \mathbf{z}_u$$

satisfying (1.2) and (3.1), which is a contradiction.

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If (3.18) holds then, recalling (3.9), (3.12), we have

$$(|\mathbf{p}|\Delta)^{\delta+1/\alpha(r)}\pi_r M^{1/2}N \gg N,$$

$$\pi_r \gg M^{-1/2}(|\mathbf{p}|\Delta)^{-\delta-1/\alpha(r)}.$$

If (3.19) holds, then recalling (3.9), (3.12), (3.10), we have

$$(|\mathbf{p}|\Delta)^{2\delta+2/\alpha(r)}|\mathbf{p}|^{-1}\pi_r^2 \gg 1,$$

that is

$$\pi_r \gg |\mathbf{p}|^{1/2}(|\mathbf{p}|\Delta)^{-\delta-1/\alpha(r)}.$$

Taking into account (3.5), we conclude that, in all cases

$$\pi_r \gg |\mathbf{p}|^{1/2}N^{-\delta}(|\mathbf{p}|\Delta)^{-\delta-1/\alpha(r)}. \tag{3.20}$$

We can refine the above method to obtain a lower bound for $\pi_1 \dots \pi_r$ which is useful for small r. By Theorem 1, we may choose integers $x_1, \dots, x_r$, not all zero, such that

$$|x_u| < H\pi_u^{-1} \qquad (u = 1, \dots, r) \tag{3.21}$$

and (3.13) holds, provided that H satisfies

$$H\pi_1^{-1} \dots H\pi_r^{-1} \geqslant (|\mathbf{p}|\Delta)^{\delta+r+1}.$$

We take

$$H = (\pi_1 \dots \pi_r)^{1/r}(|\mathbf{p}|\Delta)^{\delta+(r+1)/r}. \tag{3.22}$$

Suppose now that (3.15) holds. Then from (3.8), (3.21),

$$\sum_{u=1}^{r} x_u y_{uj} = \sum_{u=1}^{r} x_u\{\mathbf{p}\mathbf{L}_j(\mathbf{z}_u) - y_{uj}\}$$

$$\ll \sum_{u=1}^{r} H\pi_u^{-1}M^{-1/2}N^{-1}\pi_u \ll HM^{-1/2}N^{-1}.$$

As in the proof of (3.16) we find that

$$H \gg M^{1/2}N. \tag{3.23}$$

Now suppose that (3.17) holds. As before, either (3.18) or (3.19) holds. If (3.18) holds, then

$$\sum_{u=1}^{r} H\pi_u^{-1}M^{1/2}N\pi_u \gg N$$

from (3.21), (3.9), that is,

$$H \gg M^{-1/2}.$$

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If (3.19) holds, then

$$\sum_{u=1}^{r}\sum_{v=1}^{r} H\pi_u^{-1}H\pi_v^{-1}M^{1/2}N\pi_u|\mathbf{p}|^{-1}M^{-1/2}N^{-1}\pi_v \gg 1$$

from (3.21), (3.9), (3.10). That is,

$$H \gg |\mathbf{p}|^{1/2}.$$

We deduce that

$$\pi_1 \ldots \pi_r \gg (|\mathbf{p}|\Delta)^{-r\delta-(r+1)}|\mathbf{p}|^{r/2}N^{-r\delta} \tag{3.24}$$

in all cases.

For $r \geqslant 6$, we can combine (3.24) (with r replaced by 5) and (3.20) (with $6, \ldots, r$ in place of r). The outcome can be written in a way that incorporates (3.24), namely

$$\pi_1 \ldots \pi_r \gg (|\mathbf{p}|\Delta)^{-r\delta-\mu(r)}|\mathbf{p}|^{r/2}N^{-r\delta} \tag{3.25}$$

for $r \geq 1$. Here $\mu(r) = r+1$ for $r \leq 5$, while

$$\mu(r) = 6 + \frac{1}{\alpha(6)} + \ldots + \frac{1}{\alpha(r)} < r + 1 (r \geq 6).$$

We now combine (3.7) with the case $r = k$ of (3.25), obtaining

$$(|\mathbf{p}|\Delta)^{-k\delta-\mu(k)}|\mathbf{p}|^{k/2}N^{-k\delta} \ll N^{-s+5\delta s}M^{1+s-k/2}\Delta^2.$$

Taking into account (3.4) and (3.5), we deduce that

$$\begin{aligned}\Delta^{(s+5+\mu(s))/2} &= \Delta^{2+\mu(s)}\Delta^{(s+1-\mu(s))/2}\\ &\gg \Delta^{2+\mu(s)}|\mathbf{p}|^{\mu(s)-1-s}\\ &\geq \Delta^{2+\mu(k)}|\mathbf{p}|^{\mu(k)-1-s} \gg N^{s-\varepsilon}.\end{aligned}$$

This contradicts the definition of Δ, and Theorem 1 is proved.

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Department of Mathematics
Royal Holloway and Bedford New College
Egham
Surrey TW20 0EX
UK

Mathematisches Institut
Georg-August-Universität
Bunsenstrasse 3–5
D-3400 Göttingen
Germany

Present address:
Department of Mathematics
Brigham Young University
Provo
Utah 84602
U.S.A.

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