1 Numbers in a Nutshell

This is a book about what numbers are and where they come from, as understood through their *materiality*, the material devices used to represent and manipulate them: things like fingers, tallies, tokens, and symbolic notations. This book is concerned with the *natural* or *counting numbers* – the sequence *one*, *two*, *three*, *four*, and so on, and maybe as high as *ten* or *twenty* or *hundred* – that are the basis of arithmetic and mathematics. While the book focuses on how concepts of number emerge and ultimately become elaborated as arithmetic and mathematics through the use of material devices, it will also examine related phenomena, like the way numbers vary cross-culturally.

This book examines numbers through the lens of archaeology. Why *archaeology*, of all things, is a reasonable question, since numbers are not the sort of thing that can be dug up from the ground or analyzed in the lab, the activities typically performed by most archaeologists. However, archaeology is also the science of material objects, and here we are looking at numbers through their material component, the counting devices used to represent and manipulate them. These devices include *distributed exemplars* (these are objects like the arms or the hand, whose dependable quantity is used to express quantities like *two* and *five*); the fingers used in counting; tallies and other devices that accumulate quantity; tokens and forms like the abacus that accumulate, group, and permit the manipulation of quantity; and numerical notations. As noted in the preface, some of these forms are unconventional as material devices, but will be treated as such for the purposes of this analysis.

We are also taking a *cognitive* approach to material objects. Accordingly, we will consider how and why material objects

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contribute to numerical concepts and numerical thinking, past and present. This will require us to consider the material devices used in numbers as having a role in conceptualizing and thinking about numbers. We will consider material devices to be an implicit part of the cognitive system for numbers, and this approach and the theoretical framework used are explained in later chapters.

To understand what material devices do in numerical conceptualization and thinking, we will also need to look beyond the archaeological data and consider data from other disciplines, particularly psychology and neuroscience, paleoneurology, biological anthropology and zoology, linguistics, and ethnography. The interdisciplinary data provide information that is useful for attesting or explaining how material forms function in numbers. For example, contemporary languages often attest to ancient finger-counting in forms like *six* that mean *five and one* and in productive terms that show counting structured by the number of fingers, like *ten* (the number of the fingers) and *hundred* (the number of the fingers counted by the same amount). Similarly, neuropsychology provides insight into neural interconnections within the brain that explain why finger-counting is ubiquitous and cross-culturally prevalent. Such data are also useful for understanding what numbers are as concepts. This understanding is vital when investigating the questions of how, when, and why numbers began, as it necessarily informs what we look for in the archaeological record and how we interpret what we find there. Thus, we will begin by looking at what numbers are as concepts.

WHAT NUMBERS ARE AS CONCEPTS

> *Number* is formally defined as "a unit belonging to an abstract mathematical system and subject to specified laws of succession, addition, and multiplication; an element (such as π) of any of many mathematical systems obtained by extension of or analogy with the natural number system."
>
> (Merriam-Webster, *2014, def. 1c2 and 1c3*)

As formally if somewhat circularly defined, a *number* is an element of a mathematical system obtained by extending or analogizing the *natural numbers*,[1] which are also known as the *counting numbers*, the *whole numbers*, or the *integers* – *one*, *two*, *three*, and so on. Numbers are the basic elements of a mathematical system, so all of the things that we think of as arithmetic and mathematics develop – or have the potential to develop – once a basic counting sequence is available.

As stated, the formal definition is arguably an unsatisfactory basis for our stated goal, which is understanding numerical emergence and elaboration through the material devices used for representing and manipulating numbers. We need a definition that specifies numbers in terms of their properties – particularly those properties that can be associated with and explained by the material devices used, and which can be empirically established through the devices and properties of different cultural number systems.

We will start by considering the old and deeply philosophical questions of what numbers *are* as concepts – what the Greek philosopher Aristotle might have called their *essence*, the properties that give an entity or a substance its identity and nature. Here we will examine what numbers are as concepts by specifying their properties.

A number, first and foremost, is the idea of how many of something there are, a distinct or *discrete* amount. This is *cardinality*, or how many of something there are in a group of objects. For example, a trio has three members, a property of *threeness*, and the number *three* is how many members all trios have in common.[2] In offering this definition, the philosopher and mathematician Bertrand Russell distinguished a property of a particular trio (*threeness*) from a property shared and instantiated by all trios (the number *three*). The former is the property of having three members and is applicable to a particular trio. The latter is a number, a property of all the sets with that many members. The distinction between the quantity of a particular set and

[1] *Merriam-Webster*, 2014. [2] Russell, 1920.

the idea that a number is a quantity shared by two or more sets is consistent with the idea that number begins as the perception of quantity: The first is something we can appreciate through the perceptual system for quantity when there are no more than three or four members, while the second is the conceptualization of that quantity as a number. We will look at how material forms are used as a second (or "reference") set to express perceptible quantity, which helps us visualize, understand, and express quantities that lie beyond the perceptible range of about three or four.

A number also has a specific place in a counting sequence. This is *ordinality*, numbers in order. For example, *six* is the number between *five* and *seven*. In any counting sequence, numerical order is based on increasing size: It is *five*, *six*, *seven*, *eight*, and never *six*, *five*, *eight*, *seven* or any of the other permutations possible – though granted, the sequence *eight*, *seven*, *six*, *five* might preface an annual cheering of *Happy New Year!* in Times Square or follow the phrase "ten seconds to liftoff" at NASA. When whole numbers or integers are counted in sequence, each number is *one* more than the one it follows. In the sequence *one*, *two*, *three*, *three* follows *two* and is *one* more than *two*, and *two* follows *one* and is *one* more than *one*. While the relation of *one-more* is implicit to an ordinal sequence of counting numbers, it is not necessarily explicit. After all, ordinality is no more than ordering, and as such, is as equally applicable to sequences like the letters of the alphabet or the days of the week as it is to a sequence of counting numbers. Ordinality does not fix the interval between any of the members of any sequence. Discovering that the interval between counting numbers is *one* is a matter of using material devices, where each new notch on a tally, for example, can be visually discerned as *one-more* than the previous notch in the process of making them.

Numbers have the potential for many more relations between them than just *one-more*. For example, *six* is the result of adding *four* and *two*, one of the many additive combinations that produce this number; others are *three* plus *three*, *five* plus *one*, *eight* minus *two*, and *thirteen* minus *seven*. Even *one*, *two*, and *three* are potentially related to each other in more ways than just the *one-more* of an ordinal counting sequence, since for example, *three* is *two* more than *one*. Just like the explicit *one-more* relation between sequential numbers was a matter of elaboration, so too are any other explicit relations between numbers. What is required for such elaboration is a manipulable technology like pebbles or tokens, objects that can be rearranged into different subgroups.

Numbers – or rather, the relations between them – have the potential to be manipulated by means of *operations* like addition and subtraction. Operations can involve explicit relations between numbers. For example, knowing the relations between *two*, *four*, and *six* permits the addition of *two* and *four* to obtain *six*, the subtraction of *two* from *six* to obtain *four*, and the subtraction of *four* from *six* to obtain *two*. It is also possible to add and subtract without explicit relations. For example, two groups of like objects can be commingled, and the whole counted to obtain the total without knowing any relations between numbers. This is true of numerical counters as well, since the beads on an abacus can be moved without the numerical relations being explicit. In any case, when relations are explicit, they facilitate the ability to compute mentally, rather than mechanically. Such relations are essential to mental – or, more accurately, knowledge-based – calculation. The corollary to that thought is this: When such relations do not yet exist, knowledge-based calculation is not yet possible. We will look at how material forms support the emergence of mechanical and knowledge-based calculation.

Not all numbers have attributes like the meshwork of potential relations – for example, *two* being the square root of *four* and the difference between 1,245,762 and both 1,245,760 and 1,245,764 – that characterize Western numbers. These are numbers in a decimal or base 10 system typically written in the familiar Hindu–Arabic notations (0 through 9). We are particularly interested in the differences between cultural number systems, not only because they are

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fascinating, but also because they are potential clues to where numbers come from and how they become elaborated over time.

THE WORKING DEFINITION OF NUMBER

The working definition of number used here is this: *Numbers* are concepts of discrete quantity, arranged in magnitude order, with relations between them, and operations that manipulate the relations (Fig. 1.1). As a system of numbers elaborates, it will also acquire a *productive base*, a number upon which other numbers are built. For example, in Western numbers, the number *ten* serves as the

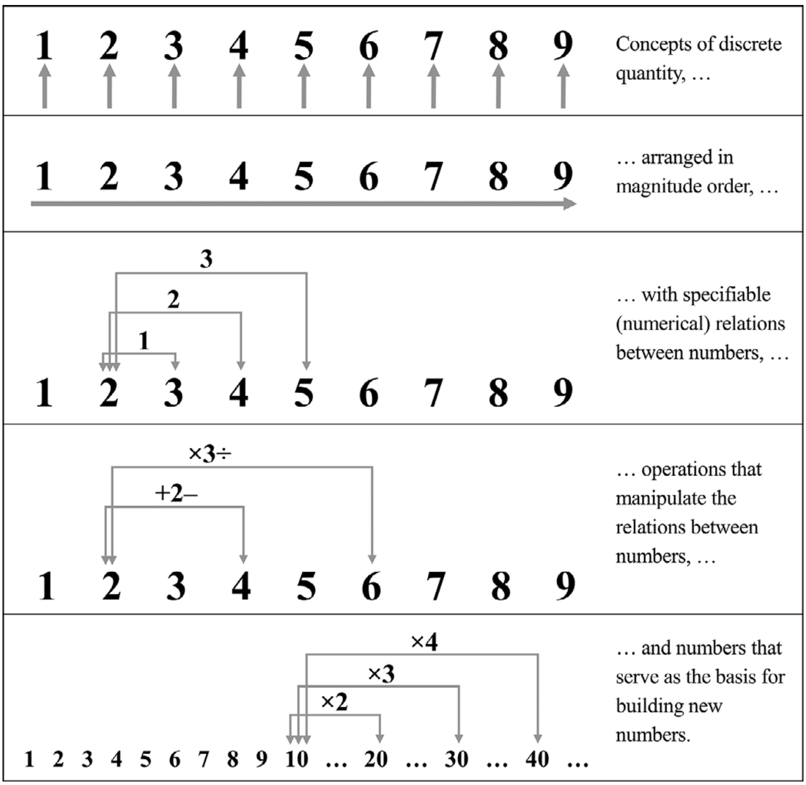

FIG. 1.1 The working definition of number. The definition focuses on five key properties: discreteness, magnitude ordering, relatedness, operational manipulability, and productive grouping. Image by the author.

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productive base, as it is repeatedly added or multiplied to produce values like *twenty* (either 10 + 10 or 10 × 2), *thirty* (10 + 10 + 10 or 10 × 3), *hundred* (ten tens or 10 × 10), and *thousand* (ten hundreds or 10 × 10 × 10).

These qualities are simply, no more and no less, what Western readers will already know about numbers from what they have been exposed to through culture and language and have learned through formal education. Granted, many readers may not have thought explicitly about numbers in terms of such properties before. Readers have also learned algorithms, or sequences of operations, that enable them to do things like add columns of numbers, divide one number into another regardless of which one is larger, and convert fractions from ratios to decimal format. While algorithmic insight will not be much called upon here – since our interest lies more in how such computations are performed, rather than performing such computations – readers can nonetheless use their existing knowledge of numbers and computations as a basis for gaining new insights into how such things become elaborated from a sequence of counting numbers, say, the numbers *one* through *ten*.

ANALYZING NUMBERS THROUGH AN EXISTING FAMILIARITY

People are enculturated into the numbers of their society from day one. For example, people in the Western tradition are exposed to objects that have quantity and can be counted; social behaviors like counting and finger-counting; social purposes like inventorying that involve numbers; material representations of numbers like written symbols and tally marks; and different forms of numbers in language. This means that most readers will have a considerable knowledge of numbers, whether or not that knowledge is explicit in the particular ways used here.

Something to keep in mind about our familiar Western numbers is that the Western numerical tradition is quite old. Its roots lie deep in the world's ancient mathematical traditions, those of Rome,

FIG. 1.2 Possible prehistoric counting devices. (Top) Notched bone from Border Cave, South Africa, dated to approximately 42,000 years ago. (Bottom) Shells punched to be strung from Blombos Cave, South Africa, dated to approximately 77,000 years ago.
Top image adapted from d'Errico et al. (2012, Supporting Information, Fig. 9, top image). Image © *PNAS* and used with permission. Bottom image adapted from one by Christopher Henshilwood and Francesco d'Errico, distributed under a Creative Commons license.

Greece, India, Egypt, and Babylon, traditions with even deeper temporal roots in counting sequences and practices that would have developed during the Neolithic and Upper Palaeolithic. The world's earliest known unambiguous numbers are numerical impressions in clay found in Mesopotamia in the mid-fourth millennium BCE.[3] Since Mesopotamian numbers are one of their roots, this makes Western numbers at least 5000 or 6000 years old. Undoubtedly, Western numbers are considerably older – perhaps 20,000 or 30,000 years old – given that the Mesopotamian numbers were already significantly elaborated by the time they first appear in the archaeological record. As if this timespan were not already impressive enough, Western numbers are likely to be older still, if archaeologists are correct in interpreting 42,000-year-old notched bones as tallies[4] (Fig. 1.2, top) and 77,000-year-old shell beads as rosaries[5] (Fig. 1.2, bottom). This impressive lifespan means that Western numbers have had a lot of time to change, and indeed, they have become highly elaborated, acquiring properties that are not necessarily shared by numbers in other cultural traditions.

[3] Schmandt-Besserat, 1992a; Nissen et al., 1993; Overmann, 2016b, 2019b.
[4] Beaumont, 1973; d'Errico et al., 2012.
[5] Henshilwood et al., 2004; d'Errico et al., 2005.

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Readers' knowledge of the highly elaborated Western numerical concepts produced by this lengthy history and prehistory is a valuable resource for understanding the numbers of other cultural traditions. The key is thinking analytically about what is already known: This can help in understanding the ways in which other cultural number systems differ from the Western tradition, and in appreciating the principles of content, organization, and structure illuminated by the differences.

WHO HAS NUMBERS? AND DO WE ALL HAVE THE SAME NUMBERS?

Most, but not all, human societies have numbers. And while all societies that have numbers develop ones that are highly similar in their content, structure, and organization, no two societies develop identical number systems. We will look at differences and similarities between numerical traditions and the reasons for these differences and similarities. A major reason for similarity is that numbers emerge from the same perceptual experience of quantity and are represented with the same devices, things like the hands. Another reason, one that complicates the attempt to understand numerical emergence and elaboration, is that societies often borrow the numbers developed by another. Today, many societies have adopted Western numbers, just as the West once adopted the Hindu–Arabic notations and used them alongside the Roman numerals that have since become an archaic system retained for its prestige value.[6] The current prevalence of Western numbers reflects cultural contact, exposure, borrowing, and transfer through mechanisms like trade, conquest, and education. In many cases, the societies borrowing Western numbers had numbers that were similar to them; in other cases, the numbers differed, and this is one of the things that would have influenced the ease and speed with which the Western numbers were adopted. These matters would

[6] Chrisomalis, 2020.

have been true of number systems coming into contact in ancient times as well.

Western numbers differ from other cultural systems of number. As noted, they have become adopted by many contemporary societies on the planet, and they are quite old, so they have become highly elaborated, the basis for the complex mathematics that have developed in the West.[7] They also tend to be what we think of when we think about what a number is. Unfortunately, we also tend to superimpose this Western idea of what a number is onto all the other numbers we encounter, regardless of whether they are Western or not, contemporary or ancient, or elaborated or not.

One of the reasons for this "backward appropriation"[8] – our superimposing our idea of number onto all numbers, regardless of place or time – is that we have been taught to think of number as a thing that is well defined, fixed, and timeless. This idea goes back to another of the Greek philosophers, Plato. He thought numbers were real, by which he meant abstract, immaterial, invisible, intangible, nonmental, external, and eternal entities of the same kind as those designated by words like "beauty," "truth," and "justice." While no one, including Plato himself, has ever convincingly explained how we might come into contact with entities we can neither see nor touch, the idea that we somehow did has seemed to explain one of the most interesting qualities of numbers, their universality. That is, everyone has the same numbers that everyone else does, not personal or idiosyncratic systems of numbers. This is even true cross-culturally, despite the variability that is to be found there. While number is not a monolithic construct, a number is still recognizably a number, no matter how the details of its properties might differ.

Numbers also work the same for everyone. If we were to add several numbers together, we would get the same results that everyone else does: 2 + 2 equals 4, assuming that everyone performs the calculation correctly. If we were to prove that an equation or

[7] Gowers, 2008. [8] Rotman, 2000, p. 40.

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mathematical statement was correct, with a *mathematical proof* being an argument showing that the stated assumptions of the mathematical statement logically entail its conclusion, everyone would agree that the proof was indeed evidence that the statement was true, assuming they understood it. If we were to look for *prime numbers*, which are numbers that can be evenly divided only by themselves and *one*, everyone would find the identical prime numbers. At least part of the plausibility of Platonic realism flows from this universal quality: Not only do numbers work the same way for everyone, whatever we discover about numbers and regardless of whenever or wherever we discover it, we all discover the same things and we all agree that they are the same things. Since we all discover the same things and agree that they are the same things, there is a very real sense in which numbers are "out there" somewhere, waiting for us to discover them.

Another reason we superimpose our Western idea of number onto all numbers is that we can. We can because numbers are so highly similar between systems. They are so similar, we can even understand them in different notations (Roman numerals, for example) and with different bases (in the numbers used with computers, binary has a base of *two*, while octal and hexadecimal bases are *eight* and *sixteen*, respectively; Roman decimal numerals have a subbase of *five* that our Western decimal numbers lack). Numbers are so recognizable as numbers that we can even pick them out of unknown languages or scripts, like the still-untranslated writing known as Linear A used in Minos, modern Crete, about 4000 years ago,[9] or the still-mostly-untranslated Proto-Elamite script used in Elam, modern Iran, about 5000 years ago.[10] Numbers can also be recognizable when they are not written, which is why we understand the numerical component of the khipu, the Inka device of knotted strings. Nevertheless, identifiability as numbers depends on the degree of

[9] Packard, 1974; Corazza et al., 2020. [10] Englund, 1998a, 2004.

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numerical elaboration. Ungrouped parallel linear marks, for example, are commonly found on prehistoric artifacts, but whether they meant numbers or something else is debatable.

Compare this accessibility with that of language, where a different language can be impenetrable, even when we know which one it is and even if it is written in the same alphabet we use. For example, that previous sentence, in Google Translate's best attempt at Greek, is "*Synkrínete aftín tin prosvasimótita me aftí tis glóssas, ópou mia diaforetikí glóssa boreí na eínai adiapérasti, akómi kai ótan xéroume poia eínai kai akóma ki an eínai gramméni sto ídio alfávito pou chrisimopoioúme*." While at least some combinations of letters are recognizable as meaningful and at least some words might be pronounceable because of an existing familiarity with the way the Latin alphabet works, the words themselves would not be intelligible without a knowledge of the Greek language.

By comparison, it is easy to understand that the Roman numeral XVII means 17, and this understanding occurs regardless of whether we also know that the word *seventeen* in Latin is *septemdecim*. This ability to understand different number systems is asymmetric. That is, when we look at other cultural systems of number, it is from the perspective of knowing and thinking in numbers that are highly elaborated, which means that they have acquired a lot of properties over their lifespan – discreteness, ordering, relatedness, manipulability, productive grouping, and conciseness. This elaboration enables us to recognize numbers in other cultural systems, regardless of whether they have the same properties or fewer. But the converse is unlikely to be true. For example, some indigenous number systems in South America are relatively unelaborated: The numbers might count no higher than *two* or *three*, and they might not be discrete, ordered, related, manipulable, productively grouped, or concise. This means that these numbers cannot provide a similar basis for recognizing the properties of other number systems. We will look at these matters in greater detail in later chapters.

VARIABILITY BETWEEN CULTURAL NUMBER SYSTEMS

Over thousands of years, Western numbers have become quite different from their counterparts in other cultural systems. For example,

- Numbers in the Western tradition are infinite, while numbers in many cultural systems are finite: They count to a certain point and then stop. Example: The Desana of the Amazonian Upper Rio Negro region count only as high as *twenty*.[11]
- Western numbers are entities defined by their relations, and they are not as meaningful in isolation as they are in relation to each other, just like the relations between notes are what make music *music* and the relations between sounds are what make speech *speech*.[12] In comparison, numbers in other cultural systems are equivalences or collections with fewer relations between them. Example: The Abipónes, a people who once inhabited the lowland Gran Chaco region in Argentina, expressed *four* as "the toes of an emu,"[13] an equivalence used to exemplify collections with the property of having four members. This number followed *three* in counting but would not necessarily have been understood as *one-more* than *three*, *two-more* than *two*, or *three-more* than *one*.
- Western numbers are discrete, while numbers in other systems can be approximate. Example: The Mundurukú of Amazonian Brazil count *one*, *two*, *about three*, and *about four*.[14]
- Western numbers can be used to count anything, while in other cultural systems, numbers might not be used to count animate beings like people, herd animals, and deities. Different types of objects might also be counted with different numbers. Example: The Nuer of Africa know their herd animals as individuals, and so they do not count them.[15] The Polynesian people of Tonga count 100 sugarcane as *au*, 100 coconuts as *fua*, 100 pieces of yam as *fuhi*, and 100 fish as *fulu*.[16]
- Western numbers count one thing each, while in other number systems, a single number might count a pair of objects together, so that counting to *ten* enumerates twenty objects. Example: Tongans count many objects one by one, but they count sugarcane, coconuts, pieces of yam, and fish in pairs.[17]

[11] Miller, 1999; Silva, 2012. [12] Plato, 1892. [13] Dobrizhoffer, 1822.
[14] Pica & Lecomte, 2008; Rooryck et al., 2017. [15] Evans-Pritchard, 1940.
[16] Bender & Beller, 2007. [17] Bender & Beller, 2007.

- Western numbers include *zero*, a concept that many cultural systems do not include. *Zero* emerged relatively late in the Western cultural tradition; about 4000 years ago, it began as a blank space used to align columns of numbers to maintain their place value.[18] Example: The Romans lacked a concept of *zero*, a characteristic for which their numerals are famous. One of the most interesting aspects of this "lack" is the idea that the number system was somehow impaired by it. On the contrary, Roman numerals were perfectly fitted to the abacus and counting boards in use at the time.[19]
- Western numbers are grouped by tens, and such productive grouping is known as a number system's base. Decimal organization is very common among the world's many number systems, though number systems can also be grouped by *twenty* (vigesimal), *five* (quinary), *twelve* (duodecimal), *four* (quaternary), *six* (senary), and *eight* (octal).[20] While these are not all the known bases, all of them appear to be based on the human hand. Example: The number system of the Yuki of California was organized by eights, which is understood as the effect of counting the spaces between the fingers, rather than the digits themselves, and using both hands.[21]
- Western numbers do not have a subbase, though many number systems do. Example: Roman numerals had a base of *ten* (X [10], C [100], M [1000]) and a subbase of *five* (V [5], L [50], D [500]), making them a quinary-decimal system. The numbers of Sumer, an ancient civilization in Mesopotamia, had a base of *ten* and a subbase of *six*, giving them productive cycles of *sixty*, a sexagesimal system.[22]
- Western numbers are added to produce the next higher number. Not all counting sequences add to produce the next number. Example: Some subtract: In Latin, the language of the Roman empire, *nineteen* (*undeviginti*) is *one from twenty*. Some overcount: In Ainu, the language of an East Asian group indigenous to Japan, *twenty-six* is *four from ten with twenty*.[23] And some anticipate: In Kakoli, a language of Papua New Guinea, *eighteen* is *two [in the next group of four above sixteen] toward twenty*.[24]
- Western numbers can be represented and/or manipulated with a variety of material forms, including the fingers, tallies, abacus beads, and written notations. While finger-counting appears to be a universal behavior – most

[18] Rotman, 1987; Kaplan, 2000. [19] Pullan, 1968; Schlimm & Neth, 2008.
[20] Comrie, 2011, 2013. [21] Dixon & Kroeber, 1907.
[22] Thureau-Dangin, 1939; Lewy, 1949; Powell, 1972. [23] Menninger, 1992.
[24] Bowers & Lepi, 1975.

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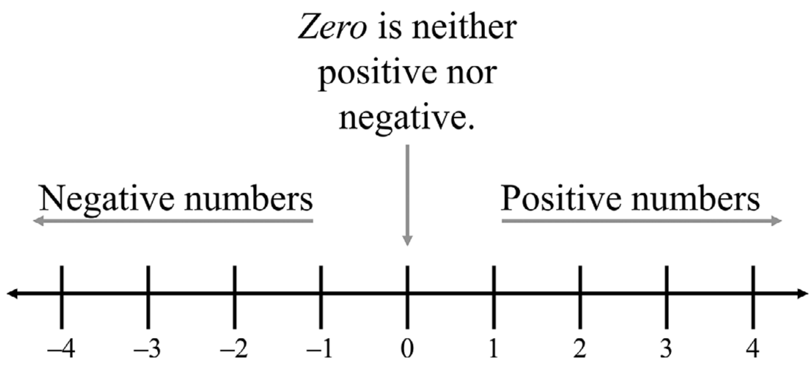

FIG. 1.3 The mental number line is a construct that envisions numbers as falling along a linear continuum. Opinions are divided on whether the mental number line represents a cultural invention or an innate biological disposition. Image by the author.

societies count on their fingers – not everyone uses the same devices. Today, Western numbers are commonly represented with Hindu–Arabic notations, but this was not always the case, as we will see, and we still use all the other forms just mentioned (like tally marks: 卌) and more. Example: The Oksapmin of Papua New Guinea count to 27 using their body as a tally; traditionally, the Oksapmin did not use an abacus or notations, though Western education and currency have introduced notations and decimal organization in the last decades.[25]

- Western numbers are often envisioned as being arranged on a linear continuum, something also known as the mental number line (MNL; see Fig. 1.3). It is an open question as to whether an MNL is innate or learned, and whether it is characteristic of all cultural systems of number. Example: Some investigations have found no evidence of the MNL in humans,[26] while other scholars have found evidence of the MNL in other species.[27]

It is worth noting that language does not readily distinguish between any of these numbers, regardless of the properties they have. So for example, a word is translated as *three* regardless of whether it concerns the fuzzy Mundurukú *about three*; the discrete and ordinally sequenced Oksapmin *three*; the Polynesian *three* that is related

[25] Saxe, 2012. [26] Núñez, 2011; Pitt et al., 2021. [27] Rugani et al., 2015.

to other numbers by twos, fours, and tens; or the infinitely related, notationally mediated Western *three*; and all the other variants described here and more. This lack of descriptiveness on the part of language has the effect of flattening the cross-cultural variability, reinforcing the impression that numbers are more similar than they actually are.

EXPLAINING CROSS-CULTURAL VARIABILITY

Historically, the cross-cultural variability in number systems has been difficult to explain. This is because the brain has been considered to be the only place where conceptualization occurs, the brain-centered or *neurocentric* model of numerical cognition. When the brain is considered to perform all the work in conceiving numbers, cross-cultural variability of structure and organization are taken to indicate the range of things that the brain can potentially do. Nevertheless, why the brain does things differently in some cases but not in others has been difficult to explain in the neurocentric model, particularly when some societies have many highly elaborated numbers and others very few. For reasons that are similarly unclear in the neurocentric model, at some point, the brain decides to externalize its internal mental concepts onto external material forms like tallies and notations, with these devices acting as passive recipients of that mental content.

What is the alternative to the neurocentric view? The non-neurocentric model promoted in this book explains cross-cultural variation as simply the consequence of using different material forms to represent and manipulate numbers. The material forms used for these purposes are then considered to precede and inform the resultant numerical concepts and to act as an integral component of numerical thinking. In this model, the brain has less to do; rather than being responsible for all conceptualizing, its role becomes largely one of recognizing relations and patterns in the material forms used for representing and manipulating numbers. The brain remains a critical component of the cognitive system for numbers, and it is still very

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important in an evolutionary sense because leveraging material forms for cognitive purposes to the degree that humans do is unique among animal species.[28]

The non-neurocentric model starts with the visual experience of quantity and symbolic notations, recognizing both as involving material forms that are engaged manuovisually – that is, by means of the hands and eyes. The model then seeks to connect the dots between these two forms. Bridging the gulf between perceptual experience and symbolic notations are devices like fingers and tallies that are also engaged manuovisually to represent and manipulate number.[29] Rather than being the passive recipients of mental content, external representations have a constitutive role. Their material substance can be altered in ways that bring forth new meaning.[30] Cross-cultural variability in number systems – including the difference between highly elaborated and very few numbers – then becomes a relatively straightforward matter of whether material devices are used in counting, which ones are used, and how they are used.

CHANGE WITHIN ANY PARTICULAR CULTURAL TRADITION

Numbers also change over time within any particular cultural tradition. For now, we will stick with the one we know best, the Western tradition. We need not go back as far as their Mesopotamian roots to see that they have changed a lot over time. In fact, we will look at four changes that have occurred within just the last thousand years, selected from among many changes because they are relatively easy to understand and were likely to have made a difference to the average person using numbers:

- *Zero* became a number. About 4000 years ago, Babylonian mathematicians inserted blank spaces to align the values of columnar numbers. In India 2500 years later, these spaces became a metasign that meant the absence of any number ("no number goes here!"), and over the last 1000 years in the

[28] Overmann & Wynn, 2019a, 2019b; Overmann, 2021f; Wynn et al., 2021.
[29] Overmann, 2018a. [30] Malafouris, 2010a.

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West, this metasign acquired a new meaning as a sign for a number with a specified value, one less than *one*; a specific place in the ordinal sequence, exactly between the positive and negative integers; and unique characteristics, like its inability to divide any other number.[31]

- *One* became a number. The ancient Greeks certainly did not consider *one* to be a number. Instead, *one* represented the *unity*, which was not just the source of all numbers: It was the source of existence itself. As for numbers, they began with plurality, which effectively started with *two*. Be that as it may, the mathematician Nicomachus apparently doubted whether *two* was a number: Just like *one* was the *unity*, *two* represented the *dyad*, another metaphysical notion.[32] As recently as 1728 CE – only 300 years ago! – the encyclopedist Ephraim Chambers would observe that the status of *one* as a number was still a matter of debate. It did not help matters that any number multiplied by *one* yielded the very same number; this unique property likely reinforced the impression that *one* somehow differed from all the other numbers.[33]

> 'Tis difputed among Mathematicians, whether or no *Unity* be a *Number*. – The generality of Authors hold the Negative; and make *Unity* to be only inceptive of Number, or the Principle thereof; as a Point is of Magnitude, and *Unifon* of Concord.
>
> *Stevinus*[34] is very angry with the Maintainers of this Opinion: and yet, if Number be defin'd a Multitude of *Unites* join'd together, as many Authors define it,'tis evident *Unity* is not a Number.
>
> *(Chambers, 1728, p. 323)*

- Hindu–Arabic notations (0 through 9) replaced the Roman ones (I, II, III…). This transition was neither easy nor quick. Merchants and bankers were initially suspicious of the new notations, particularly of *zero*, since it seemed to make falsifying values far too easy: 10 could become 100 by simply adding another of the dodgy signs.[35] In comparison, Roman

[31] Rotman, 1987; Kaplan, 2000.
[32] Nicomachus, 1926; Evans, 1977.
[33] Nicomachus, 1926.
[34] *Stevinus* refers to the mathematician Simon Stevin, who helped influence the reconceptualization of *one* as a number in the sixteenth century CE.
[35] Ifrah, 1985; Rotman, 1987; Kaplan, 2000.

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numerals were harder to falsify, as it would be impossible to turn X [10] into C [100] by adding a *zero*.

- Calculating with algorithms involving learned relations, mental judgments, and handwritten notations supplanted the mechanical exchange of values on abaci and counting boards. This transition too was neither easy nor quick. The debate over whether it was better to calculate by means of knowledge-based algorithms, rather than by moving beads on an abacus or the counters known as *jettons* on a counting board, took centuries to resolve. The transition also involved contests like the one shown in Fig. 1.4, something that has persisted to recent decades as contests of speed and accuracy between the abacus and electronic calculators or computers.[36]

If we look further back in time, say, another thousand years or so, Greek philosophical ideas about numbers and other matters were even more influential. The idea of *zero* was inhibited by metaphysical ideas about *being* (existence, which was good) and *non-being* (non-existence, which was a horrifying possibility).[37] "Irrational" numbers like π and $\sqrt{2}$, which we know today as fractions that neither terminate nor repeat, so greatly challenged conceptions of what numbers were and how they were supposed to behave that the mathematician Hippasus, possibly their discoverer, is said to have drowned. Reports of the incident differ greatly regarding what really happened. Some accounts have Hippasus punished by the gods for impious behavior; others say his fellow mathematicians did him in. Some say he perished because he divulged the existence of irrational numbers, others because he told the secret of how they might be calculated to men who were not initiates of the philosophers' guild. One account has Hippasus throwing himself into the sea, driven incurably mad by the irrational nature of the numbers in question. Considering that numbers were foundational to Greek concepts of existence, the idea that numerical irrationality could be so thoroughly confounding at least has the virtue of consistency, if a trifle over-exacting of enthusiastic devotion.

[36] Pullan, 1968; Stone, 1972; Evans, 1977; Reynolds, 1993. [37] Rotman, 1987.

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FIG. 1.4 Woodcut from *Margarita Philosophica* [*Pearl of Wisdom*], originally published in 1503 as one of the first printed encyclopedias of general knowledge. To the left, the Roman mathematician Boethius calculates with algorithms and notations, while to the right, the Greek philosopher Pythagoras uses a counting board. Arithmetic personified as a lady looks on, turned to the left apparently to favor the algorithmic approach. Image in the public domain.

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ARE NUMBERS EVER THE SAME, AND IF SO, *HOW* ARE THEY THE SAME?

Despite all this variability between cultural traditions and within any particular tradition over time, numbers are astonishingly similar. For one thing, they all have the same cardinality. For another, they all have the same ordinal sequencing and ordering by increasing magnitude. Thirdly, each next number increases over the previous number by *one*, though the way in which each next number is derived can be quite variable, as shown earlier with examples from Latin, Ainu, and Kakoli. And all numbers demonstrate the same patterns: For example, the prime numbers are the prime numbers for everyone.

Another of those patterns is an anatomically derived base number. The *ten* in decimal comes from the fingers of both hands; the *twenty* in vigesimal comes from all the fingers and toes; and the *five* in quinary comes from the fingers of one hand.[38] A base of *four* (quaternary) might indicate counting the spaces between the fingers, rather than the digits themselves,[39] or considering the fingers separately (counted) from the thumb (not counted). A base of *eight* (octal) might simply double the method of counting by *fours*, much as decimal doubles quinary. A base of *six* (senary) might emerge from including the thumb joint along with the fingers, with *twelve* (duodecimal) emerging from its doubling or from using the three segments on each of the four fingers while omitting the thumb. A base of *fourteen* uses the three segments per finger and includes the thumb with its two segments. As for sexagesimal or the base *sixty* number system of Mesopotamia, no one really knows the reason for it. It had a base of *ten* and a subbase of *six*. While *ten* most likely was related to the fingers,[40] the reason for *six* as the next higher base has simply been lost to the passage of time.

Why consider all this variability between cultural traditions and within any particular cultural tradition over time? While we are

[38] Epps, 2006. [39] Dixon & Kroeber, 1907. [40] Overmann, 2019b.

interested in *what* changes, we are even more interested in *why* and *how* things change, as these can illuminate aspects of the processes whereby numbers emerge and become elaborated. And in the process of learning about these things, we will get an overview of how numbers work for human societies in all sorts of places and at a lot of different times: contemporary, historical, and prehistoric.