TWISTED GROUP RINGS WHICH ARE SEMI-PRIME
GOLDIE RINGS

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In this paper we examine when a twisted group ring, \( R^r(G) \), has a semi-simple, artinian quotient ring. In §1 we assemble results and definitions concerning quotient rings, Ore sets and Goldie rings, and then, in §2, we define \( R^r(G) \). We prove a useful theorem for constructing a twisted group ring of a factor group and establish an analogue of a theorem of Passman. Twisted polynomial rings are discussed in §3 and I am indebted to the referee for informing me of the existence of [4]. These are used as a tool in proving results in §4.

A group \( G \) is a poly- (torsion-free abelian or finite) group if \( G \) has a series of subgroups \( \{e\} = H_0 \triangleleft H_1 \triangleleft H_2 \triangleleft \ldots \triangleleft H_n = G \) such that \( H_i/H_{i-1} \) is either torsion-free abelian or finite \( (i = 1, 2, \ldots, n) \). These groups are considered here and we prove (Theorem 4.5) that if such a group \( G \) has only a finite set \( S \) of periodic elements with \( |S| \) regular in \( R \) and \( R \) is semi-prime, left Goldie, then \( R^r(G) \) is semi-prime, left Goldie.

In §5 we define a class of groups \( \mathcal{G} \) such that if \( G \) is a torsion-free element of \( \mathcal{G} \) and \( D \) is a division ring then \( D^r(G) \) is an Ore domain. We call these groups Ore groups and prove a theorem similar to Theorem 4.5 for this class of groups.

Throughout, \( R \) will denote a ring with identity element 1 and \( G \) a multiplicative group with identity \( e \). By artinian and noetherian we mean left artinian and left noetherian.

1. Goldie rings.

We restate the following definitions which appear in [2, pp. 228, 229].

An element of a ring \( R \) is \textit{regular} if it is neither a left nor a right zero divisor. A set \( T \) of regular elements of \( R \) which is multiplicatively closed is a \textit{left Ore set} if, whenever \( a \in R \), \( c \in T \), there exist \( a' \in R \), \( c' \in T \) such that \( c'a = a'c \).

A ring \( Q \) is a \textit{left quotient ring} of \( R \) with respect to a set \( T \) of regular elements of \( R \) if

(i) \( Q \supseteq R \),

(ii) the elements of \( T \) are units in \( Q \),

(iii) the elements of \( Q \) have the form \( c^{-1}a \) where \( c \in T \), \( a \in R \).

If such a ring \( Q \) exists, it will be denoted by \( R_T \). When \( T \) is the set of all regular elements of \( R \) we say that \( Q \) is the \textit{left quotient ring} of \( R \).

\textbf{Theorem 1.1.} \textit{Let \( T \) be a set of regular elements of \( R \). Then \( R_T \) exists if and only if \( T \) is a left Ore set in \( R \).}

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A ring $R$ has finite left Goldie rank if it contains no infinite direct sum of non-zero left ideals. Let $S$ be a non-empty subset of $R$; then $\ell(S)$, the left annihilator of $S$, is the left ideal \{a \in R : as = 0 \text{ for all } s \in S\}. A ring $R$ is a left Goldie ring if (i) $R$ has finite left Goldie rank and (ii) $R$ has ascending chain condition on left annihilators.

**Goldie's Theorem** [2, Theorem 1.37]. A ring $R$ has a semi-simple artinian left quotient ring if and only if $R$ is a semi-prime left Goldie ring.

**Lemma 1.2** [11, Corollary 2.5]. Let $Q$ be an artinian ring with subring $R$ such that every element of $Q$ has the form $c^{-1}a$, where $c, a \in R$. Then $Q$ is the left quotient ring of $R$.

For convenience, we formulate the following straightforward lemmas.

**Lemma 1.3.** Let $R$ be a ring and let $T \subseteq R$ be a left Ore set.

(i) Let $L$ be a left ideal and let $L_T = R_T L$, the left ideal in $R_T$ generated by $L$. Then $L_T = \{c^{-1}r : c \in T, r \in L\}$.

(ii) Let $L$ and $J$ be left ideals in $R$. Then $L_J \cap J_T = (L \cap J)_T$.

(iii) If $L$ is a left annihilator in $R$, then $L_T$ is a left annihilator in $R_T$ and $L_T \cap R = L$.

(iv) If $R_T$ is a left Goldie ring, then $R$ is a left Goldie ring.

**Lemma 1.4.** Let $R_1, R_2, \ldots, R_n$ be a finite number of left Goldie rings. Then $R = R_1 \oplus R_2 \oplus \ldots \oplus R_n$ is also a left Goldie ring.

2. Twisted group rings.

**Definition.** Let $G$ be a group with identity element $e$, $R$ a ring with identity $1$, $R^*$ the group of central units of $R$ and $\gamma : G \times G \to R^*$ a 2-cocycle. [That is, $\gamma(g, h)\gamma(gh, k) = \gamma(g, hk)\gamma(h, k), g, h, k \in G$.] Let $R^G(G)$ be the free left $R$-module with basis $\{g : g \in G\}$. Define multiplication in $R^G(G)$ by

$$\tilde{g} \tilde{h} = \gamma(g, h)\tilde{g}\tilde{h} \quad (g, h \in G)$$

extending this, by linearity, to the whole of $R^G(G)$. Then $R^G(G)$ is an associative ring with identity element $\gamma(e, e)^{-1} \tilde{e}$. We call $R^G(G)$ the twisted group ring of $G$ over $R$ with twist $\gamma$.

We shall identify an element $r \in R$ with its image $r\gamma(e, e)^{-1} \tilde{e}$ in $R^G(G)$.

In this section we prove some results about $R^G(G)$ that we shall require later.

**Theorem 2.1.** Let $G$ be a group with a central normal subgroup $Z$ and $R^G(G)$ a twisted group ring such that $\gamma(g, z) = \gamma(z, g)$ for all $g \in G$ and $z \in Z$. Then there exists a twisted group ring of $G/Z$ over $R^G(G)$ with twist $\delta$ such that

$$R^G(G) \cong [R^G(Z)]^U(G/Z).$$
Proof. Let $T$ be a set of coset representatives for $Z$ in $G$. Then every element of $G$ is uniquely represented in the form $t \sigma$ for some $t \in T, \sigma \in Z$. Thus given $t_1, t_2 \in T$ there are a unique $\tau (t_1, t_2) \in T$ and $\sigma \in Z$ such that $t_1 t_2 = \tau (t_1, t_2) \sigma$. Then, in $R' (G)$,

$$l_1 l_2 = g(t_1, t_2) \overline{\tau (t_1, t_2) \sigma} = g(t_1, t_2) g(\sigma, \tau (t_1, t_2))^{-1} \overline{\sigma} \overline{\tau (t_1, t_2)}.$$

Thus

$$l_1 l_2 (\overline{\tau (t_1, t_2)})^{-1} = g(t_1, t_2) g(\sigma, \tau (t_1, t_2))^{-1} \overline{\sigma} \in \text{central units of } R' (Z).$$

Let $F = G/Z$. Then for each $f \in F$ there is a unique $t \in T$ such that $f = t \sigma$. Define $\delta: F \times F \to (R'(Z))^{*}$ by

$$\delta (f_1, f_2) = l_1 l_2 (\overline{\tau (t_1, t_2)})^{-1}, \text{ where } f_1 = t_1 \sigma, f_2 = t_2 \sigma, t_1, t_2 \in T.$$

Given $f_1, f_2$, then $t_1, t_2$ and $\tau (t_1, t_2)$ are uniquely determined. Thus $\delta$ is well-defined and it is readily verified that $\delta$ is a 2-cocycle.

Hence we have defined $[R'(Z)]^{\delta} (F)$. We shall denote by $f^*$ the image in $[R'(Z)]^{\delta} (F)$ of an element $f \in F$.

Now we construct an isomorphism between $R'(G)$ and $[R'(Z)]^{\delta} (F)$. As remarked earlier, given $g \in G$ there are a unique $t \in T$ and $\sigma \in Z$ with $g = t \sigma = \sigma t$. Then $g = g(t, \sigma)^{-1} \bar{\sigma} t$ in $R'(G)$. Define $\theta: R'(G) \to [R'(Z)]^{\delta} (F)$ to be the $R$-homomorphism defined by

$$\theta (g) = g(t, \sigma)^{-1} \bar{\sigma} t \mapsto g(t, \sigma)^{-1} \bar{\sigma} (t \sigma)^{*}.$$

We show that $\theta$ is also a ring homomorphism. To do this, it is sufficient to show that $\theta (g_1 g_2) = \theta (g_1) \theta (g_2) (g_1, g_2 \in G)$. Let $g_1 = z_1 t_1, g_2 = z_2 t_2$, where $z_1, z_2 \in Z, t_1, t_2 \in T$. Then

$$\bar{\sigma} (t \sigma)^{*} = g(z_1, t_1)^{-1} \bar{\sigma} (t \sigma)^{*} g(z_2, t_2)^{-1} \bar{\sigma} (t \sigma)^{*} = g(z_1, t_1)^{-1} \bar{\sigma} g(z_2, t_2)^{-1} \bar{\sigma} g(z_3, t_3)^{-1} \bar{\sigma} g(z_4, t_4)^{-1} \bar{\sigma} (t \sigma)^{*}.$$

Thus, recalling that

$$g(t_1 Z, t_2 Z) = \overline{t_1 t_2 (t_3)^{-1} = g(t_1, t_2) g(z_3, t_3)^{-1} \bar{\sigma} t_3},$$

it follows that $\theta (g_1 g_2) = \theta (g_1) \theta (g_2)$. 

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Hence \( \theta \) is a ring homomorphism and, since \( \theta \) is clearly both one-one and onto, the required isomorphism is established.

**COROLLARY 2.2.** Let \( G \) be a group, \( Z \) a central normal subgroup of \( G \) and \( R \) a ring. Then there exists a twisted group ring of \( G/Z \) over \( R(Z) \) with twist \( \delta \), such that

\[
R(G) \cong R(Z)\delta(G/Z).
\]

Thus twisted group rings occur in a fairly natural way and we have a useful method of expressing a group ring in terms of a subgroup and a factor group.

For Lemma 2.5 we shall require the following result. We denote the set of positive integers by \( \mathbb{P} \).

**LEMMA 2.3.** Let \( R \) be a semi-simple, artinian ring and let \( n \in \mathbb{P} \). Let \( W = \{w \in R^*: w^n = 1\} \). Then \( W \) is finite.

**Proof.** Let \( S \) be the centre of \( R \). Then, since \( R \) is semi-simple artinian, there exist fields \( F_1, F_2, \ldots, F_r \) (say) such that \( S = F_1 \oplus F_2 \oplus \cdots \oplus F_r \). For \( w \in W \), let \( (w_1, w_2, \ldots, w_r) \) be the image of \( w \) in \( F_1 \oplus F_2 \oplus \cdots \oplus F_r \). Then \( w^n = 1 \) implies that \( w_i^n = 1 \) (\( i = 1, 2, \ldots, r \)). Hence \( W = W_1 \oplus W_2 \oplus \cdots \oplus W_r \), where \( W_i \) is the set of \( n \)th roots of unity in \( F_i \). But the set of \( n \)th roots of unity in a field is finite. Hence \( W \) is finite.

**COROLLARY 2.4.** Let \( R \) be a semi-prime left Goldie ring and let \( n \in \mathbb{P} \). Let \( W = \{w \in R^*: w^n = 1\} \). Then \( W \) is finite.

**Proof.** Let \( Q \) be the semi-simple, artinian quotient ring of \( R \). Then \( W \subseteq \{w \in Q^*: w^n = 1\} \) which, by the lemma, is finite.

**DEFINITION.** Let \( R^\prime(G) \) be a twisted group ring and let \( H \subseteq G \). Define

\[
\tilde{C}_G(H) = \{g \in G: \tilde{g}h = h\tilde{g} \text{ for all } h \in H\}
\]

\[
= \{g \in C_G(H): \gamma(g, h) = \gamma(h, g) \text{ for all } h \in H\}.
\]

It is readily verified that \( \tilde{C}_G(H) \) is a subgroup of \( G \).

**LEMMA 2.5.** Let \( R \) be a semi-prime left Goldie ring and let \( R^\prime(G) \) be a twisted group ring. Let \( H \) be a subgroup of \( G \). Then (i) \( \tilde{C}_G(H) \leq C_G(H) \) and (ii) if, further, \( |H| < \infty \), then

\[
|C_G(H): \tilde{C}_G(H)| < \infty.
\]

**Proof.** Let \( g_1, g_2 \in C_G(H), h \in H \). Then

\[
\frac{\gamma(g_1, h)\gamma(g_2, h)}{\gamma(h, g_1)\gamma(h, g_2)} = \frac{\gamma(g_1, h)\gamma(g_2, h)\gamma(hg_1, g_2)}{\gamma(h, g_1)\gamma(h, g_2)\gamma(hg_1, g_2)}
\]
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Now define \( \theta_h : C_\alpha(H) \rightarrow R^* \) by

\[
\theta_h(g) = \gamma(g, h)^{-1} \quad (g \in C_\alpha(H)).
\]

Then, by the above argument, \( \theta_h \) is a group homomorphism, \( \text{Ker} \ \theta_h = \{ g \in C_\alpha(H) : \gamma(g, h) = \gamma(h, g) \} \) and hence

\[
\overline{C_\alpha(H)} = \bigcap_{h \in H} \text{Ker} \ \theta_h.
\]

It follows that \( \overline{C_\alpha(H)} \triangleleft C_\alpha(H) \).

Now suppose that \( |H| = n \) and let \( h \in H, \ g \in C_\alpha(H) \). Then \( (h \bar{g})^n = ah^n\bar{g}^n \) for some \( a \in R \). But \( h^n = e \) therefore \( \bar{h}^n \in R \) and so \( (h \bar{g})^n = b\bar{g}^n \) for some \( b \in R \). Thus

\[
(h \bar{g})^n = \bar{a}(h \bar{g})^n\bar{g}^{-1} = \bar{g}(h \bar{g})^n = [\gamma(g, h)^{-1} \bar{h} \bar{g}]^n.
\]

Therefore

\[
[\gamma(g, h)^{-1} \bar{h} \bar{g}]^n = 1 \quad \text{and so}
\]

\[
C_\alpha(H) / \text{Ker} \ \theta_h \cong \text{subgroup of group of } n \text{th roots of unity in } R^*.
\]

Hence, by Corollary 2.4, \( |C_\alpha(H) : \text{Ker} \ \theta_h| < \infty \). Further, since \( |H| < \infty, |C_\alpha(H) : \overline{C_\alpha(H)}| < \infty \) and the result is proved.

We now give a lemma concerning rings of quotients.

**Lemma 2.6.** (i) Let \( H \triangleleft G \) such that \( R'(G) \) has a left quotient ring and let \( T \) be the set of regular elements in \( R'(H) \). Then \( T \) is a left Ore set in \( R'(G) \).

(ii) If \( R \) has a left quotient ring \( Q \), then \( Q'(G) \) is well-defined and is the left quotient ring of \( R'(G) \) with respect to the set of regular elements of \( R \).

**Proof.** (i) Adapt [12, Lemma 2.6].

(ii) This is clear.

We shall wish to know when \( R'(G) \) is semi-prime. We denote by \( PR'(G) \) the prime radical of \( R'(G) \). In the ‘untwisted’ situation we have the following theorem due to D. Passman [6, p. 162, see also 7] and I. Connell [6, Appendices 2 and 3].

**Theorem A.** The group ring \( R(G) \) is semi-prime if and only if \( R \) is semi-prime and the order of each finite normal subgroup of \( G \) is regular in \( R \).

In [8, Theorem 3.7] Passman proves the following extension of this.
THEOREM B. Let $K$ be an algebraically closed field of characteristic $p > 0$ and $K^\gamma(G)$ a twisted group ring. Then $K^\gamma(G)$ is semi-prime if and only if $G$ has no finite normal subgroups of order divisible by $p$.

Let $K$ be any field of characteristic $p > 0$, $F$ its algebraic closure and $K^\gamma(G)$ a twisted group ring. Then $F^\gamma(G)$ is well-defined and, arguing as in [1, Proposition 9], it can be shown that

$$PK^\gamma(G) = K^\gamma(G) \cap PF^\gamma(G).$$

It is immediate from this and Theorem B that, if $G$ has no finite normal subgroups of order divisible by $p$ then, $K^\gamma(G)$ is semi-prime and we generalise this below in Theorem 2.7. The converse of this, however, is not true. We recall a counter example discussed in [9]. Let $K$ be a field over which the polynomials $x^{p^n} - a$ are irreducible for some $a \in K$ and where $p = \text{char } K$. Let $G = \mathbb{Z}_p$. Then we may construct a twisted group ring $K^\gamma(G)$ which is a field and hence semi-prime. The orders of finite normal subgroups of $G$, however, are powers of $p$.

THEOREM 2.7. Let $R$ be a semi-prime ring and one of the following: (i) commutative, (ii) a semi-direct product of simple rings, (iii) left Goldie. Let $G$ be a group such that the order of each finite normal subgroup is regular in $R$ and let $R^\gamma(G)$ be a twisted group ring. Then $R^\gamma(G)$ is semi-prime.

Proof. (i) As in [1, proof of Theorem 5, p. 668].
(ii) As in [1, proof of Proposition 10, pp. 669 and 670].
(iii) Let $Q$ be the semi-simple artinian left quotient ring of $R$. Then, by (ii), $Q^\gamma(G)$ is semi-prime and hence $R^\gamma(G)$ is semi-prime.

3. Twisted polynomial rings.

DEFINITION. Let $R$ be a ring and $\theta: R \to R$ an automorphism of $R$. Let $\langle x \rangle$ be an infinite cyclic group. We define $R_\theta(x)$ to be the free left $R$-module with basis $\langle x \rangle$ and, for $r \in R$, we define multiplication on $R_\theta(x)$ by

$$xr = \theta(r)x$$
$$x^{-1}r = \theta^{-1}(r)x^{-1},$$

extending by linearity to the whole of $R_\theta(x)$. With this definition of multiplication $R_\theta(x)$ is an associative ring.

Thus $R_\theta(x)$ is a ring of polynomials in $x$ and $x^{-1}$ with coefficients from $R$. The subring of $R_\theta(x)$ containing only the polynomials in non-negative powers of $x$, denoted by $R_\theta[x]$, is called a twisted polynomial ring.

A. Horn in [4, §2] has proved the following.

THEOREM 3.1. Let $R$ be a noetherian ring. Then $R_\theta[x]$ has an artinian left quotient ring if and only if $R$ has an artinian left quotient ring.
From this we may deduce the following corollary.

**Corollary 3.2.** Let $R$ have an artinian left quotient ring. Then $R_\rho(x)$ has an artinian left quotient ring.

**Proof.** Let $Q$ be the left quotient ring of $R$. Then, by the theorem, $Q_\rho[x]$ has an artinian left quotient ring $\tilde{Q}$. Since $x^i$ is regular in $Q_\rho[x]$, $x^{-i} \in \tilde{Q}$ ($i \in \mathbb{P}$) and hence

$$R_\rho(x) \subseteq Q_\rho(x) \subseteq \tilde{Q}.$$ 

It is now clear from Lemma 1.2 that $\tilde{Q}$ is the artinian left quotient ring of $R_\rho(x)$.

4. Quotient rings of $R'(G)$. In this section we obtain sufficient conditions for $R'(G)$ to have a semi-simple artinian quotient ring, similar to but less stringent than those obtained by P. Smith in [12, Theorem 2.18] for $R(G)$. By Goldie's Theorem, if $R'(G)$ is to have a semi-simple artinian left quotient ring, then it must itself be a semi-prime left Goldie ring and therefore must have both a.c.c. on left annihilators and finite left Goldie rank.

**Lemma 4.1.** Let $R'(G)$ be semi-prime and let $H \triangleleft G$ be such that (i) $|G:H| < \infty$ and (ii) $R'(H)$ is semi-prime left Goldie. Then $R'(G)$ is semi-prime left Goldie.

**Proof.** By Lemma 2.6, the set $T$ of regular elements of $R'(H)$ is a left Ore set in $R'(G)$. Let $S = [R'(G)]_T$. Then $S$ is semi-prime and $S = \sum_{c \in C} Q_c$, where $Q$ is the left quotient ring of $R'(H)$ and $C$ is a set of coset representatives for $H$ in $G$. But $C$ is finite; therefore $S$ is an artinian $Q$-module and hence an artinian ring. It follows from Lemma 1.2 that $S$ is the left quotient ring of $R'(G)$ and so, by Goldie's Theorem, $R'(G)$ is a semi-prime left Goldie ring.

**Lemma 4.2.** Let $R'(G)$ have a left quotient ring and let $H \triangleleft G$ be such that

(i) $R'(H)$ is semi-prime left Goldie, and

(ii) $G/H$ is ordered.

Then $R'(G)$ is semi-prime left Goldie.

**Proof.** We prove that every essential left ideal in $R'(G)$ contains a regular element. Let $E$ be an essential left ideal in $R'(G)$ and let

$$E_0 = \{a \in R'(H): g_0a + g_1a_1 + \ldots + g_na_n \in E \text{ for some } n \text{ and } a_i \in R'(H) \text{ and where } g_0H < g_1H < \ldots < g_nH \text{ in } G/H\}.$$ 

Then $E_0$ is a left ideal in $R'(H)$. Let $a \in R'(H)$, $a \neq 0$. Then there exists $\alpha = k_1b_1 + k_2b_2 + \ldots + k_mb_m \in R'(G)$, $b_i \in R'(H)$, $k_1H < k_2H < \ldots < k_mH$ in $G/H$, such that $\alpha a \neq 0$ and $aa \in E$. Therefore $b_1a \neq 0$ and $b_ia \in E_0$ for some $1 \leq i \leq m$ and it follows that $E_0$ is essential in $R'(H)$. But $R'(H)$ is semi-prime left Goldie; therefore $E_0$ contains a regular element of
$R'(H)$. That is, there exists $x \in E$ with $x = \bar{g}_0 c + \bar{g}_1 c_1 + \ldots + \bar{g}_n c_n$, where $c_i \in R'(H)$, $c$ is regular in $R'(H)$ and $g_0 H < g_1 H < \ldots < g_n H$ in $G/H$. It is readily verified that $x$ is regular in $R'(G)$.

Now since every essential left ideal of $R'(G)$ contains a regular element, $Q$, the left quotient ring of $R'(G)$, contains no proper essential left ideals and is therefore a semi-simple artinian ring [2, p. 234 and p. 219].

**Corollary 4.3.** Let $R'(G)$ be a twisted group ring and $H \triangleleft G$ be such that $G/H$ is infinite cyclic and $R'(H)$ is semi-prime left Goldie. Then $R'(G)$ is semi-prime left Goldie.

**Proof.** $G/H = \langle gH \rangle$ for some $g \in G \setminus H$. Define $\theta: R'(H) \rightarrow R'(H)$ by $\theta(a) = \bar{g}a\bar{g}^{-1}$ ($a \in R'(H)$). Then, since $H \triangleleft G$, $\theta$ is an automorphism of $R'(H)$ and, in the notation of §3, with $\bar{g} = x$, $R'(G) = R'(H)_{\theta(\bar{g})}$. Now it follows from Corollary 3.2 that $R'(G)$ has an artinian left quotient ring and so, $G/H$ being an ordered group, $R'(G)$ is semi-prime left Goldie.

**Lemma 4.4.** Let $R'(G)$ be a twisted group ring and let $H \triangleleft G$ be such that (i) $R'(H)$ is semi-prime left Goldie, and (ii) $G/H$ is torsion-free abelian. Then $R'(G)$ is semi-prime left Goldie.

**Proof.** $G/H$ is an ordered group. Thus, from Lemma 4.2, it will be sufficient to prove that $R'(G)$ has a left quotient ring. To do so it is enough to show that $R'(G_1)$ has a left quotient ring for every subgroup $G_1$ such that $G_1/T$ is finitely generated. But $G_1/H$ is a direct sum of a finite number of infinite cyclic groups and the required result follows by induction from Corollary 4.3.

**Theorem 4.5.** Let $G$ be a poly- (torsion-free abelian or finite) group and let $S$ be the set of all periodic elements of $G$. Let $R$ be semi-prime left Goldie and let $S$ be finite with $|S|$ regular in $R$. Then $R'(G)$ is semi-prime left Goldie.

**Proof.** By Theorem 2.7, $R'(G)$ is semi-prime and so the result follows by induction from Lemmas 4.1, 4.4.

**Examples of poly- (torsion-free abelian or finite) groups.**

(i) Nilpotent groups with finite set of periodic elements. (A torsion-free nilpotent group has central series with factors all torsion-free abelian [5, Theorem 1.2].)

(ii) Soluble groups with derived series whose factors have only a finite number of periodic elements.

(iii) $FC$-soluble groups [10, pp. 121, 129] with series

$$\{e\} = H_0 \vartriangleleft H_1 \vartriangleleft \ldots \vartriangleleft H_n = G$$

such that $H_{i+1}/H_i$ is an $FC$-group whose torsion subgroup [10, p. 121, Theorem 4.32] is finite ($i = 0, 1, \ldots, n-1$).

((i) and (ii) are particular examples of (iii).)
5. Ore groups.

**Definition.** A ring $R$ is called a *left Ore domain* if

(i) $R$ contains no proper zero divisors, and
(ii) $R$ satisfies the left Ore condition.

We shall be interested in the class of groups such that, given $G$ torsion-free and an Ore domain $R$, then $R^\gamma(G)$ is an Ore domain. We therefore make the following definition.

**Definition.** Let $\mathcal{G}$ be the class of groups such that

(i) $G \in \mathcal{G}$, $H \leq G \Rightarrow H \in \mathcal{G}$,
(ii) $G \in \mathcal{G}$, $H < G$, $|H| < \infty \Rightarrow G/H \in \mathcal{G}$,
(iii) if $G \in \mathcal{G}$ is torsion-free, $D$ is a division ring and $D^\gamma(G)$ a twisted group ring, then $D^\gamma(G)$ is an Ore domain.

If $G \in \mathcal{G}$ we call $G$ an *Ore group*. Every periodic group is an Ore group. Also abelian groups, nilpotent groups and $FC$-groups are Ore groups.

**Theorem 5.1.** Let $G$ be a group such that any twisted group ring $D^\gamma(G)$, where $D$ is a division ring, is semi-prime left Goldie. Let $R$ be a semi-prime left Goldie ring. Then $R^\gamma(G)$ is semi-prime left Goldie.

**Proof.** Let $Q$ be the semi-simple artinian quotient ring of $R$. By Lemmas 2.6 and 1.3, (iv), it is sufficient to prove that $Q^\gamma(G)$ is semi-prime left Goldie. Then

$$Q = M_{n_1}(D_1) \oplus M_{n_2}(D_2) \oplus \ldots \oplus M_{n_r}(D_r)$$

for some integers $n_1, \ldots, n_r$ and division rings $D_1, D_2, \ldots, D_r$. Also there exist orthogonal central idempotents $e_1, e_2, \ldots, e_r \in Q$ such that $M_{n_i}(D_i) = Qe_i$ $(i = 1, 2, \ldots, r)$. Let $g, h \in G$. Since $\gamma(g, h)$ is a central unit of $R$, $\gamma(g, h)e_i$ is a central unit of $D_i$ $(i = 1, 2, \ldots, r)$ and thus, defining $\gamma(g, h) = \gamma(g, h)e_i$, we have defined twisted group rings $D_i^\gamma(G)$ $(i = 1, 2, \ldots, r)$. It follows that

$$Q^\gamma(G) = M_{n_1}(D_1^\gamma(G)) \oplus M_{n_2}(D_2^\gamma(G)) \oplus \ldots \oplus M_{n_r}(D_r^\gamma(G)).$$

Hence it is sufficient to prove that each $M_{n_i}(D_i^\gamma(G))$ is semi-prime left Goldie. But $D_i^\gamma(G)$ has a semi-simple artinian quotient ring $Q_i$, by the hypotheses of the theorem; hence [11, Theorem 3.1] $M_{n_i}(Q_i)$ is the semi-simple artinian quotient ring of $M_{n_i}(D_i^\gamma(G))$.

**Corollary 5.2.** Let $R$ be a semi-prime left Goldie ring and $G$ a torsion-free Ore group. Then $R^\gamma(G)$ is semi-prime left Goldie.

Before the main theorem of this section we require the following lemma, the proof of which is routine.

**Lemma 5.3.** Let $G$ be a group and let $S$ be the set of all periodic elements of $G$. Then

(i) $C_G(S) \leq G$,
(ii) $|S| < \infty \Rightarrow S \leq G$,
(iii) $|S| < \infty \Rightarrow |G: C_G(S)| < \infty$.
THEOREM 5.4. Let $R$ be a semi-prime left Goldie ring and let $G$ be an Ore group such that the set $S$ of all periodic elements of $G$ is finite with $|S|$ regular in $R$. Then $R^\ast(G)$ is semi-prime left Goldie.

Proof. Let $\mathcal{C}_G(S) = \{g \in C_G(S): \gamma(g, s) = \gamma(s, g) \text{ for all } s \in S\}$. By Lemma 2.5, $|C_G(S): C_G(S)| < \infty$. Hence, since $|G: C_G(S)| < \infty$, $|G: C_G(S)| < \infty$. Also, by Theorem 2.7, $PR^\ast(G) = 0$ and so, by Lemma 4.1, it is sufficient to prove that $R^\ast(\mathcal{C}_G(S))$ is semi-prime left Goldie. Let $C = \mathcal{C}_G(S) \cap S$. Then $C$ is a central subgroup of $\mathcal{C}_G(S)$ and, since $C \subseteq S$, $\bar{c} \bar{c} = \bar{c} \bar{c}$ for all $g \in \mathcal{C}_G(S), c \in C$. Therefore, by Theorem 2.1, we may construct a twisted group ring of $\mathcal{C}_G(S)/C$ over $R^\ast(C)$ with twist $\delta$ (say) such that

$$R^\ast(\mathcal{C}_G(S)) \cong [R^\ast(C)]^\Phi(\mathcal{C}_G(S)/C).$$

But, since $|C| < \infty$ and $|C|$ is regular in $R$, $R^\ast(C)$ is semi-prime left Goldie (Lemma 4.1). Also, since $G$ is an Ore group, $\mathcal{C}_G(S)$ is an Ore group. Then, since $C$ is the set of periodic elements of $\mathcal{C}_G(S)$ and $C$ is finite, $\mathcal{C}_G(S)/C$ is a torsion-free Ore group. It now follows from Corollary 5.2 that $[R^\ast(C)]^\Phi(\mathcal{C}_G(S)/C)$ is a semi-prime left Goldie ring. That is, $R^\ast(\mathcal{C}_G(S))$ is semi-prime left Goldie and hence $R^\ast(G)$ is also semi-prime left Goldie.

DEFINITIONS. If $\mathcal{X}$ is a class of groups, $L\mathcal{X}$ is the class of locally $\mathcal{X}$-groups consisting of all groups $G$ such that every finite subset of $G$ is contained in a $\mathcal{X}$-subgroup. $\mathcal{X}$ is called a local class if $L\mathcal{X} = \mathcal{X}$. [10, part 1 p. 5, part 2 p. 93].

THEOREM 5.5. The class $\mathcal{C}$ of Ore groups is a local class.

Proof. Let $G \in L\mathcal{C}$. Let $S$ be a finite subset of $G$ and let $H = \langle S \rangle$. Since $G \in L\mathcal{C}$, there exists $K \in \mathcal{C}$ such that $S \subseteq K$. Then $H \leq K$ and so $H \in \mathcal{C}$. From this it is clear that $L\mathcal{C}$ satisfies (i) and (ii) of the definition of an Ore group. We must now prove that if $G \in L\mathcal{C}$ is torsion-free and $D$ is a division ring then $D^\ast(G)$ is an Ore domain. To prove this we show that

(a) $xy = 0$ if and only if $x = 0$ or $y = 0$ ($x, y \in D^\ast(G)$);

(b) given $x, y \in D^\ast(G)$, there exist $x', y' \in D^\ast(G)$ such that $x'x = y'y$.

Let $x, y \in D^\ast(G)$; then there exists a finitely generated subgroup $H$ such that $x, y \in D^\ast(H)$. Then $H \in \mathcal{C}$ so that $H$ is a torsion-free Ore group and $D^\ast(H)$ is an Ore domain. Now, since $x, y \in D^\ast(H)$, they satisfy conditions (a) and (b). Hence $D^\ast(G)$ is an Ore domain. We have shown that $L\mathcal{C}$ satisfies (i), (ii) and (iii) of the definition of $\mathcal{C}$. Hence $L\mathcal{C} \subseteq \mathcal{C}$ and so $L\mathcal{C} = \mathcal{C}$.

COROLLARY 5.6. Let $G$ be a locally nilpotent group (locally FC group); then $G$ is an Ore group.

THEOREM 5.7. Let $G$ be a locally nilpotent (locally FC) group. Then $R(G)$ is semi-prime left Goldie if and only if

(i) $R$ is semi-prime left Goldie, and

(ii) the subgroup $S$ of all periodic elements of $G$ is finite with $|S|$ regular in $R$. 

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**Proof.** That (i) and (ii) are sufficient for \( R(G) \) to be semi-prime left Goldie follows from Theorem 5.4.

Conversely, let \( R(G) \) be semi-prime left Goldie. It is not hard to show that \( R \) must be a left Goldie ring. Then, by Theorem A and the fact that the set of periodic elements of a locally nilpotent (locally FC) group is a locally finite subgroup, it follows that (i) and (ii) hold true.

**Theorem 5.8.** Let \( G \) be a group and let \( H \triangleleft G \) be such that \( H \) is periodic and \( G/H \) is an Ore group. Then \( G \) is an Ore group.

**Proof.** Let \( \mathcal{X} = \{ G : G \) has a periodic normal subgroup \( H \) with \( G/H \) an Ore group\}. Clearly \( \mathcal{G} \subseteq \mathcal{X} \). We shall prove that \( \mathcal{X} \) satisfies the definition of \( \mathcal{G} \) and hence that \( \mathcal{X} = \mathcal{G} \).

Let \( G \in \mathcal{X} \) with \( H \triangleleft G \) such that \( H \) is periodic and \( G/H \in \mathcal{G} \).

(i) If \( K \leq G \), then \( K \cap H \) is a periodic normal subgroup of \( K \). Also \( K/(K \cap H) \cong KH/H \leq G/H \in \mathcal{G} \). Hence \( K/(K \cap H) \in \mathcal{G} \) and it follows that \( K \in \mathcal{X} \).

(ii) Let \( K \triangleleft G \), \( |K| < \infty \). Now \( HK/K \cong H/(H \cap K) \) is a periodic normal subgroup of \( G/K \). Also \( (G/K)/(HK/K) \cong (G/H)/(HK/H) \) which belongs to \( \mathcal{G} \), since \( G/H \in \mathcal{G} \) and \( HK/H \cong K/(H \cap K) \) is a finite normal subgroup of \( G/H \). Hence \( G/K \in \mathcal{X} \).

(iii) If \( G \) is torsion-free, then \( H \) is trivial and hence \( G \in \mathcal{G} \).

We have shown that \( \mathcal{X} \) satisfies conditions (i), (ii) and (iii) of the definition of \( \mathcal{G} \). Hence \( \mathcal{X} = \mathcal{G} \).

**References**


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