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SPACES OF CONTINUOUS VECTOR FUNCTIONS AS DUALS

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ABSTRACT. A well known result due to Dixmier and Grothendieck for spaces of continuous scalar-valued functions $C(X)$, X compact Hausdorff, is that $C(X)$ is a Banach dual if, and only if, X is hyperstonean. Moreover, for hyperstonean X, the predual of $C(X)$ is strongly unique. Here we obtain a formulation of this result for spaces of continuous vector-valued functions. It is shown that if E is a Hilbert space and $C(X, (E, \sigma^*))$ denotes the space of continuous functions on X to E when E is provided with its weak * (= weak) topology, then $C(X, (E, \sigma^*))$ is a Banach dual if, and only if, X is hyperstonean. Moreover, for hyperstonean X, the predual of $C(X, (E, \sigma^*))$ is strongly unique.

0. **Introduction.** Throughout this article the letters E, U, V will stand for Banach spaces while X and Y will denote compact Hausdorff spaces. $C(X, E)$ denotes the space of continuous functions on X to E provided with the supremum norm. And, for a dual space E^*, we will denote by $C(X, (E^*, \sigma^*))$ the Banach space of continuous functions F on X to E^* when the latter space is provided with its weak * topology, again normed by $\|F\|_\infty = \sup_{x \in X} \|F(x)\|$. If E is the one-dimensional field of scalars then we write $C(X)$ for $C(X, E)$.

The notation $U \cong V$ is used to indicate that the Banach spaces U and V are isometric. The interaction between elements of a Banach space and those of its dual is denoted by $\langle \cdot, \cdot \rangle$. If S is a subset of the Banach space E, then $S^\perp$ denotes the subspace of E^* given by $S^\perp = \{e^* \in E^* : \langle e, e^* \rangle = 0 \text{ all } e \in S\}$. And if $S \subseteq E^*$ then we denote by ${}^\perp S$ the set $\{e \in E : \langle e, e^* \rangle = 0 \text{ all } e^* \in S\}$. For any subset $S \subseteq E$, $\overline{sp}(S)$ will denote the closed linear span of S.

Given a positive measure space (Ω, Σ, μ) and $1 \leqq p \leqq \infty$, the Bochner space $L^p(\Omega, \Sigma, \mu, E)$ will be denoted by $L^p(\mu, E)$ when there is no danger of confusing the underlying measurable space involved. We refer to [6] for the definitions and properties of these spaces. Facts about vector measures used in this paper can be found in [6] and [7]. We will, in particular, rely upon I. Singer's characterization of $C(X, E)^*$ as the space of all regular Borel vector measures on X to E^* with finite variation $|m|$, [14], or [7, p. 387]. Throughout the article, scalar measures are denoted by μ while vector measures are denoted by m and n.

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If X is an extremally disconnected compact Hausdorff space we will call a nonnegative, extended real-valued Borel measure μ on X a *category measure* if

(i) every nonempty clopen set has positive measure,

(ii) every nowhere dense Borel set has measure zero, and

(iii) every nonempty clopen set contains a nonempty clopen set with finite measure.

(In [1] and [3] measures having these properties are referred to as "perfect".) An extremally disconnected compact Hausdorff space on which a category measure is defined will be called *hyperstonean*. This is equivalent to the definition of hyperstonean space obtained via the use of normal measures, [13, p. 95] and [1, p. 26]. Since for hyperstonean X every Borel set B has a unique representation $B = C \Delta D$ with C clopen and D nowhere dense, [1, pp. 1-2] and [8, p. 160], it follows that the null sets for a category measure are precisely the nowhere dense Borel sets. Given a hyperstonean space X with category measure μ, property (iii), together with an application of Zorn's lemma, can be used to show that X is the Stone-Čech compactification of the disjoint union of clopen subsets X_γ, $X = \beta(\cup_{\gamma \in \Gamma} X_\gamma)$, with $\mu(X_\gamma) < \infty$ for all γ, and for all Borel subsets B of X, $\mu(B) = \sum_{\gamma \in \Gamma} \mu(B \cap X_\gamma)$.

We will say that a Banach dual U^* has *strongly unique predual* U if, given any isometry T of U^* onto a Banach dual V^* with predual V, then the adjoint mapping T^* carries the canonical image $J(V)$ of V in V^{**} onto the canonical image $J_0(U)$ of U in U^{**}. (One easily verifies that, T being a surjective isometry, it is enough to require that $T^* \circ J(V)$ is contained in $J_0(U)$ – in other words that T is $\sigma(U^*, U) - \sigma(V^*, V)$ continuous.) Now it is a well known result due to Dixmier [8] or [13, p. 95] that, when X is hyperstonean, $C(X)$ is a dual space. And Grothendieck has provided a strong converse. If $C(X)$ is a dual then X is hyperstonean; moreover, for hyperstonean X the predual of $C(X)$ is strongly unique [11] or [13, p. 96]. The goal of this article is to provide an analogue of these results for spaces of continuous vector functions.

It is a result of Cembranos [4] that if X is any infinite compact Hausdorff space and E is infinite dimensional, then $C(X, E)$ contains a complemented copy of c_0, and hence $C(X, E)$ is not even isomorphic to a dual space. However, when one deals with vector-valued functions, the space $C(X, (E^*, \sigma^*))$ with hyperstonean X arises repeatedly as a Banach dual. In [2] it is shown that, if E^* has the Radon-Nikodym property, then for any compact Hausdorff space Y the bidual of $C(Y, E)$ is of the form $C(X, (E^{**}, \sigma^*))$ for a certain hyperstonean space X related to Y. More generally, in [3] it is shown that the space $C(X, (E^*, \sigma^*))$ with X hyperstonean arises as the dual of a space of vector measures, and that it is always a dual space – specifically, it is the dual of $L^1(\mu, E)$ for μ a category measure on X. In this paper we obtain vector analogues of the Dixmier-Grothendieck results for the space $C(X, (E, \sigma^*))$ when E is a Hilbert space. We wish to prove the following:

THEOREM. *Let X be a compact Hausdorff space and E a Hilbert space. Then* (a) *$C(X, (E, \sigma^*))$ is a Banach dual if, and only if, X is hyperstonean. Furthermore,* (b) *if X is hyperstonean then the predual of $C(X, (E, \sigma^*))$ is strongly unique.*

1. **Proof of (a).** As previously mentioned, the "if" part of the assertion is known, and holds for any Banach dual E [3, Theorem 1]. We need to establish the "only if" portion. For this we will need the following:

PROPOSITION. *Let E be a Hilbert space and let m and n be finite regular Borel measures on X to E whose respective values are taken in two closed orthogonal subspaces of E. Then $||m||^2 + ||n||^2 \leqq ||m + n||^2$.*

PROOF. Suppose that m takes its values in M and n its values in N where M and N are closed orthogonal subspaces of E. We may clearly assume that at least one of m and n is distinct from the zero measure. Choose a sequence $\{F_k\} \subseteq C(X, E)$, with $||F_k||_\infty \leqq 1$ for all k, such that the F_k take their values in M and $\int F_k dm \to ||m||$ as $k \to \infty$. Then choose a sequence $\{G_k\} \subseteq C(X, E)$ taking values in N such that $||G_k||_\infty \leqq 1$ for all k and $\int G_k dn \to ||n||$. Define $H_k = [1/(||m||^2 + ||n||^2)^{1/2}](||m||F_k + ||n||G_k)$. Then $||H_k||_\infty \leqq 1$ for all k and we thus have

$$\begin{aligned} ||m + n|| &\geqq \left|\int H_k d(m + n)\right| \\ &= [1/(||m||^2 + ||n||^2)^{1/2}]\left[||m|| \int F_k dm + ||n|| \int G_k dn\right] \\ &\to (||m||^2 + ||n||^2)^{1/2} \text{ as } k \to \infty. \end{aligned}$$

In what follows we assume that V is a Banach space such that there exists an isometry T mapping $C(X, (E, \sigma^*))$ onto V^*. J denotes the canonical injection of V into V^{**}.

We let e be an element of E with $||e|| = 1$ and let $S(e)$ denote the subspace of $C(X, (E, \sigma^*))$ defined by $S(e) = \{f \cdot e : f \in C(X)\}$. If we can show that $T(S(e))$ is weak * closed in V^* then $T(S(e))$ is dual space [12, p. 212], and, since $C(X)$ is obviously isometric to $T(S(e))$, the fact that X is hyperstonean would thus follow from what is known about spaces of continuous scalar-valued functions.

Hence suppose, to the contrary, that $T(S(e))$ is not closed in the weak * topology of V^*. Then by the Krein-Šmulian theorem [9, p. 429] there would be a net $\{f_\alpha\} \subseteq C(X)$ with $||f_\alpha||_\infty \leqq 1$ for all α such that $T(f_\alpha \cdot e)$ tends weak * to an element $v^* \in V^*$ with $v^* \notin T(S(e))$. Thus $\langle f_\alpha \cdot e, T^* \circ J(v)\rangle = \langle v, T(f_\alpha \cdot e)\rangle \to \langle v, v^*\rangle = \langle T^{-1}(v^*), T^* \circ J(v)\rangle$ for all $v \in V$.

Now $T^{-1}(v^*)$ is an element $F \in C(X, (E, \sigma^*))$ with $||F||_\infty \leqq 1$ and $F \notin S(e)$ so that there exist an element $\phi \in E$ with $||\phi|| = 1$ and an element $x \in X$ such that $\langle e, \phi\rangle = 0$ and $\langle F(x), \phi\rangle \neq 0$. Define the element $g \in C(X)$ by $g(x) = \langle F(x), \phi\rangle$ and let $G = F - g \cdot \phi$. Then there is a $v^{**} \in V^{**}$ with

$||v^{**}|| = 1$ such that $|\langle g \cdot \phi, T^*(v^{**}) \rangle| = ||g||_\infty$ and $\langle G, T^*(v^{**}) \rangle = 0$. (Just pick any $v^{**} \in V^{**}$ such that $T^*(v^{**})$ is equal to the vector measure $\phi \cdot \mu_x$, where $x \in X$ is such that $|g(x)| = ||g||_\infty$.)

Next define the positive numbers δ and ϵ by

(1) $$\delta = (1 - ||g||_\infty^2/4)^{1/2}$$

and

(2) $$\epsilon = \max\left\{\delta, \frac{3}{4}\right\}.$$

Since the image under J of the unit ball in V is weak * dense in the unit ball of V^{**}, we can find a $v \in V$ with $||v|| \leqq 1$ such that

(3) $$|\langle g \cdot \phi, T^* \circ J(v) \rangle| > \epsilon \cdot ||g||_\infty$$

and

(4) $$|\langle G, T^* \circ J(v) \rangle| < ||g||_\infty/4.$$

Then

(5) $$|\langle F, T^* \circ J(v) \rangle| \geqq |\langle g \cdot \phi, T^* \circ J(v) \rangle| - |\langle G, T^* \circ J(v) \rangle| > ||g||_\infty/2$$

by (3), (4) and (2).

Now as $T^* \circ J(v)$ is an element of $C(X, (E, \sigma^*))^*$, its restriction to $C(X, E)$ is represented by a regular Borel vector measure m_0 on X to E with $||m_0|| \leqq ||T^* \circ J(v)|| \leqq 1$. Let P be the orthogonal projection of E onto $\overline{sp}(\{\phi\})$ and define the vector measures m and n by $m = Pm_0$ and $n = (I - P)m_0$. Then let $\bar{m}_0$ denote any Hahn-Banach extension of m_0 to an element of $C(X, (E, \sigma^*))^*$ and let $\phi = T^* \circ J(v) - \bar{m}_0$, so that $T^* \circ J(v) = \bar{m}_0 + \phi$ with $\phi \in C(X, E)^\perp$.

Since $\langle g \cdot \phi, T^* \circ J(v) \rangle = \int (g \cdot \phi) dm$, it follows from (3) that $||m|| > \epsilon$. Hence, as $||m + n|| = ||m_0|| \leqq 1$, it is a consequence of (2), (1) and the Proposition that $||n|| < ||g||_\infty/2$. Thus for all α we have $\langle f_\alpha \cdot e, T^* \circ J(v) \rangle = \int (f_\alpha \cdot e) dn$ which has modulus less than $||g||_\infty/2$, whereas, by (5), $|\langle F, T^* \circ J(v) \rangle| > ||g||_\infty/2$. This contradicts our assumption that $\langle f_\alpha \cdot e, T^* \circ J(v) \rangle \to \langle F, T^* \circ J(v) \rangle$, and completes the proof that X is hyperstonean.

2. **Proof of (b).** The proof of part (b) will be established by means of a sequence of lemmas. Throughout, μ will denote a fixed category measure on X.

LEMMA 1. *Let E^* be any Banach dual with the Radon-Nikodym property. If $G \in C(X, (E^*, \sigma^*))$ then there exists an open dense set $O(= O_G)$ of X such that G is continuous from O to E^* when the latter space is given its norm topology.*

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PROOF. X is of the form $X = \beta(\cup_{\gamma \in \Gamma} X_\gamma)$, where the X_γ are pairwise disjoint clopen sets with $\mu(X_\gamma) < \infty$ for all γ and $\mu(B) = \sum_{\gamma \in \Gamma} \mu(B \cap X_\gamma)$ for all Borel sets B. We denote by μ_γ the restriction of μ to the Borel sets of X_γ, and by G_γ the restriction of G to X_γ.

As mentioned in the introduction, the dual of $L^1(\mu_\gamma, E)$ is $C(X_\gamma, (E^*, \sigma^*))$. Here the interaction between elements $F_0 \in L^1(\mu_\gamma, E)$ and $G_0 \in C(X_\gamma, (E^*, \sigma^*))$ is given by $\langle F_0, G_0 \rangle = \int \langle F_0(x), G_0(x) \rangle d\mu_\gamma(x)$, [3, Theorem 1]. And it is known that there exists an isometry of $L^\infty(\mu_\gamma, E^*)$ into $C(X_\gamma, (E^*, \sigma^*))$, [10, Proposition 2.4]. But since E^* has the Radon-Nikodym property it follows (as μ_γ is a finite measure) that $L^\infty(\mu_\gamma, E^*)$ is also the dual of $L^1(\mu_\gamma, E)$, [6, p. 98]. Thus the isometry of Proposition 2.4 in [10] is surjective. In particular, elements of $C(X_\gamma, (E^*, \sigma^*))$ are μ_γ-measurable. We note for future reference that, as a consequence, the restriction of a $G \in C(X, (E, \sigma^*))$ to a σ-finite subset of X is μ-measurable.

Thus as countably valued functions are dense in $L^\infty(\mu_\gamma, E^*)$ [6, p. 97], for each positive integer k we can find a countably valued measurable function $G_{\gamma,k}$ on X_γ such that ess sup$\|G_\gamma(x) - G_{\gamma,k}(x)\| < 1/k$. Moreover, since every measurable subset of X_γ differs from a clopen set by a set of measure zero [1, p. 1] we may assume that $G_{\gamma,k} = \sum_{j=1}^\infty e_{\gamma,k,j}\chi_{A_{\gamma,k,j}}$, where the $A_{\gamma,j,k}$ are pairwise disjoint clopen sets with $(\cup_{j=1}^\infty A_{\gamma,k,j})^- = X_\gamma$. Note that since $G_{\gamma,k}$ is norm-continuous on $\cup_{j=1}^\infty A_{\gamma,k,j}$ and since G_γ is weak * continuous, we must have $\|G_\gamma(x) - G_{\gamma,k}(x)\| \leqq 1/k$ for all $x \in \cup_{j=1}^\infty A_{\gamma,k,j}$. Also note that $C_{\gamma,k} = X_\gamma - \cup_{j=1}^\infty A_{\gamma,k,j}$ is nowhere dense, and thus $\mu(C_{\gamma,k}) = 0$.

Now let $V_k = \cup_{\gamma \in \Gamma} \cup_{j=1}^\infty A_{\gamma,k,j}$ and define G_k on V_k by $G_k = G_{\gamma,k}$ on $\cup_{j=1}^\infty A_{\gamma,k,j}$. Then G_k is norm-continuous on V_k and $X - V_k$ is nowhere dense. It follows that the set $N = \cup_{k=1}^\infty (X - V_k)$ is nowhere dense. (Here again we use the fact that a set of first category in a hyperstonean space is nowhere dense [8, p. 160].) Thus $O = X - \bar{N}$ is an open dense subset of X on which G is the uniform limit of the norm-continuous functions $G_k|_O$.

Throughout the remainder of this section E will denote a Hilbert space while V, V^*, T and J will be as given in Section 1. J_0 denotes the canonical injection of $L^1(\mu, E)$ into $C(X, (E, \sigma^*))^*$.

LEMMA 2. *For $v \in V$, $e \in E$ and $f \in C(X)$ we have $\langle f \cdot e, T^* \circ J(v) \rangle = \int f d\mu_{e,v}$ for some normal regular Borel measure $\mu_{e,v}$ on X.*

PROOF. We first note that if U is any weak * closed subspace of V^*, then U is isometric to $(V/{}^\perp U)^*$ under the linear map $A: U \to (V/{}^\perp U)^*$ defined by $\langle [v], Au \rangle = \langle v, u \rangle$ for $u \in U$, $v \in V$. (Here, for $v \in V$, $[v]$ denotes the equivalence class of v in $V/{}^\perp U$.) For since U is weak * closed, $U = ({}^\perp U)^\perp$ by the bipolar theorem, and our assertion is thus contained in [15, p. 227, problem 5], or [5, p. 29, Lemma 1].

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We may clearly assume that $||e|| = 1$, and, as in the previous section, we let $S(e) = \{f \cdot e : f \in C(X)\}$. We have seen that $T(S(e))$ is a weak * closed subspace of V^*. By the first paragraph of this proof the map sending $u = T(f \cdot e)$ into $\langle [v], Au\rangle = \langle v, T(f \cdot e)\rangle$ is weak * continuous on the dual space $T(S(e))$. Since this dual is the isometric image of $C(X)$ under $f \to T(f \cdot e)$, and every isometry between $C(X)$ and a dual space is continuous with respect to the weak * topologies of these spaces by Grothendieck's result, it follows that the map $f \to T(f \cdot e) \to \langle v, T(f \cdot e)\rangle$ is weak * continuous. Thus, again by Grothendieck's theorem, $\langle f \cdot e, T^* \circ J(v)\rangle = \int f d\mu_{e,v}$, where $\mu_{e,v}$ is a normal regular Borel measure on X.

Henceforth $\{e_\alpha : \alpha \in A\}$ will denote a fixed orthonormal basis for E. For simplicity of notation given $\alpha, \alpha_j \in A$ we will denote by $\mu_{\alpha,v}$ the normal regular Borel measure determined via Lemma 2 by $\langle f \cdot e_\alpha, T^* \circ J(v)\rangle$, $f \in C(X)$, and by $\mu_{j,v}$ the measure determined by $\langle f \cdot e_{\alpha_j}, T^* \circ J(v)\rangle$.

LEMMA 3. (a) *Given $v \in V$ then $\mu_{\alpha,v} = 0$ except for those α belonging to a countable subset K_v of A.*

(b) *If $K_v = \{e_{\alpha_j} : j = 1, 2, \ldots\}$ then the vector measures m_N defined by $m_N = \sum_{j=1}^N e_{\alpha_j} \cdot \mu_{j,v}$ constitute a Cauchy sequence in $C(X, E)^*$ and thus converge to an $m_v \in C(X, E)^*$ with $m_v \ll \mu$.*

PROOF. (a): Let k be any fixed positive integer and suppose that there are n indices $\alpha_1, \ldots, \alpha_n \in A$ with $||\mu_{j,v}|| > 1/k$, $1 \leqq j \leqq n$. For each such j choose $f_j \in C(X)$ with $||f_j||_\infty = 1$ and $\int f_j d\mu_{j,v}$ a real number greater than $1/k$. Then $||(f_1 \cdot e_{\alpha_1} + \ldots + f_n \cdot e_{\alpha_n})/\sqrt{n}||_\infty \leqq 1$ so that

$$||v|| = ||T^* \circ J(v)|| \geqq \langle (f_1 e_{\alpha_1} + \ldots + f_n e_{\alpha_n})/\sqrt{n}, T^* \circ J(v)\rangle > \sqrt{n}/k$$

and hence $n < ||v||^2 \cdot k^2$ from which (a) follows.

(b): Suppose, to the contrary, that $\{m_N : N = 1, 2, \ldots\}$ is not a Cauchy sequence. Then there is an $\epsilon > 0$ such that for each positive integer M there exists N greater than M with $||m_N - m_M|| > 2\epsilon$. Choose $N_1 > 0$ such that $||m_{N_1}|| > \epsilon$ and suppose that $N_2 < N_3 < \ldots < N_p$ have been chosen with $||m_{N_k} - m_{N_{k-1}}|| > \epsilon$ for $k = 2, \ldots, p$. For simplicity of notation we write e_j for e_{α_j} and set $N_0 = 0$. Then for each k, $0 \leqq k \leqq p - 1$ take $H_{k+1} \in C(X, E)$ such that the range of H_{k+1} lies in $\overline{sp}(\{e_{N_k+1}, \ldots, e_{N_{k+1}}\})$, $||H_{k+1}||_\infty \leqq 1$, and such that $\langle H_1, m_{N_1}\rangle$ and $\langle H_k, m_{N_k} - m_{N_{k-1}}\rangle$, $2 \leqq k \leqq p$, are each real numbers greater than ϵ. Thus

$$\left\|(1/\sqrt{p}) \sum_{k=1}^{P} H_k\right\|_\infty \leqq 1 \text{ but } \left\langle (1/\sqrt{p}) \sum_{k=1}^{P} H_k, T^* \circ J(v)\right\rangle$$

$$= (1/\sqrt{p})[\langle H_1, m_{N_1}\rangle + \langle H_2, m_{N_2} - m_{N_1}\rangle + \ldots$$

$$\ldots + \langle H_p, m_{N_p} - m_{N_{p-1}}\rangle] > \sqrt{p} \cdot \epsilon$$

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which, for sufficiently large p, will be greater than $||T^* \circ J(v)|| = ||v||$. This contradiction shows that the m_N do indeed form a Cauchy sequence in $C(X, E)^*$ and hence converge to an $m_v \in C(X, E)^*$. Since m_N is absolutely continuous with respect to μ for each N, so then is m_v. This completes the proof.

Now given $v \in V$ the restriction of $T^* \circ J(v)$ to $C(X, E)$ is represented by a regular Borel vector measure n_v on X to E with $||n_v|| \leqq ||T^* \circ J(v)|| = ||v||$. Moreover, for all $e \in E$ and $f \in C(X)$ it is clear that

$$\langle f \cdot e, n_v \rangle = \langle f \cdot e, T^* \circ J(v) \rangle = \langle f \cdot e, m_v \rangle.$$

It thus follows that n_v and m_v agree on $C(X) \otimes E$ which is dense in $C(X, E)$ [7, p. 375], and so $n_v = m_v$. The elements of $C(X, (E, \sigma^*))$ are integrable with respect to m_v, for they are μ-measurable on μ-σ-finite sets as mentioned in the proof of Lemma 1, and as $|m_v|$ is finite, the μ-continuous measure m_v has μ-σ-finite support. Therefore $F \to \int F dm_v$ defines a continuous linear functional on $C(X, (E, \sigma^*))$. Then $\phi_v = T^* \circ J(v) - m_v \in C(X, (E, \sigma^*))^*$ with $\phi_v \in C(X, E)^{\perp}$ and we have $T^* \circ J(v) = m_v + \phi_v$. Whenever we write, for $v \in V$, $T^* \circ J(v) = m_v + \phi_v$ it will be understood that m_v is the vector measure which is determined by Lemma 3 and is the restriction of $T^* \circ J(v)$ to $C(X, E)$, and that $\phi_v \in C(X, E)^{\perp}$.

LEMMA 4. *For $v \in V$ we have $T^* \circ J(v) = G_v d\mu$ for some $G_v \in L^1(\mu, E)$ with $||G_v||_1 = ||v||$. Consequently $V \cong L^1(\mu, E)$ under the mapping $J_0^{-1} \circ T^* \circ J$.*

PROOF. We have established that for $v \in V$ one has $T^* \circ J(v) = m_v + \phi_v$, and we want to show that $\phi_v = 0$. For if this is established we would have $T^* \circ J(v) = m_v$, and, since E has the Radon-Nikodym property, [6, p. 218], this latter element is of the form $G_v d\mu$ for some $G_v \in L^1(\mu, E)$ with $||G_v||_1 = ||m_v|| = ||v||$. We would thus have established that $T^* \circ J$ embeds V isometrically into $J_0(L^1(\mu, E))$, which, as previously noted, shows that $T^* \circ J$ maps V onto $J_0(L^1(\mu, E))$.

Thus, to show that $\phi_v = 0$ for each $v \in V$, take any $F \in C(X, (E, \sigma^*))$ and define $v_F^* \in V^*$ by $\langle v, v_F^* \rangle = \langle F, m_v \rangle$, $v \in V$. Then since v_F^* is a continuous linear functional on V there exists an $H_F \in C(X, (E, \sigma^*))$ with $||H_F||_\infty = ||v_F^*|| \leqq ||F||_\infty$ and $\langle v, v_F^* \rangle = \langle v, T(H_F) \rangle$. If we can show that $F = H_F$ we would have, for $v \in V$,

$$\langle F, m_v \rangle = \langle v, v_F^* \rangle = \langle v, T(H_F) \rangle = \langle v, T(F) \rangle$$

$$= \langle F, T^* \circ J(v) \rangle = \langle F, m_v \rangle + \langle F, \phi_v \rangle,$$

so that $\langle F, \phi_v \rangle = 0$. Since this would be true for all $F \in C(X, (E, \sigma^*))$, it would follow that $\phi_v = 0$.

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Thus suppose, to the contrary, that $F \neq H_F$ and let $\delta = ||F - H_F||_\infty$. Then choose $\epsilon > 0$ such that

$$\delta(1 - \epsilon) > \delta/2, \tag{6}$$

and

$$5\epsilon \cdot ||F||_\infty < \delta/2. \tag{7}$$

We know that $F - H_F$ is norm-continuous on an open dense subset O_1 of X and we have $\sup_{x \in O_1} ||F(x) - H_F(x)|| > \delta(1 - \epsilon)$. (Note that $||F(\cdot)||$ is lower semicontinuous on X.) Take a clopen subset C of O_1 such that $\sup_{x \in C} ||F(x) - H_F(x)|| > \delta(1 - \epsilon)$. Then $\chi_C(F - H_F) \in C(X, E)$ and $||\chi_C(F - H_F)||_\infty > \delta(1 - \epsilon)$.

Choose $v \in V$ with $||v|| \leqq 1$ such that $|\langle v, T(\chi_C(F - H_F))\rangle| = |\langle \chi_C(F - H_F), T^* \circ J(v)\rangle| > \delta(1 - \epsilon)$. If $T^* \circ J(v) = m_v + \phi_v$ then $\langle \chi_C(F - H_F), \phi_v\rangle = 0$ and $|m_v|(C) > 1 - \epsilon$, hence $|m_v|(X - C) < \epsilon$. We would next like to show that $||\phi_v||$ is small.

To this end take $G \in C(X, (E, \sigma^*))$ with $||G||_\infty \leqq 1$ such that $\langle G, \phi_v\rangle > ||\phi_v|| - \epsilon$. Now G is norm-continuous on an open dense subset $O_2 \subseteq X$ and since $|m_v|(X - O_2) = 0$, we can find a clopen set $D \subseteq O_2$ with $|m_v|(D) > 1 - \epsilon$, hence $|m_v|(X - D) < \epsilon$. Thus we can take an $F_0 \in C(X, E)$ such that the support of F_0 is contained in D, $||F_0||_\infty \leqq 1$, and $\langle F_0, m_v\rangle$ is real and greater than $1 - \epsilon$. Then $F_0 + G - \chi_D G \in C(X, (E, \sigma^*))$ with $||F_0 + G - \chi_D G||_\infty \leqq 1$. Hence, (noting that $\langle \chi_D G, \phi_v\rangle = 0$ as $\chi_D G$ is norm-continuous), we have

$$\begin{aligned} 1 &\geqq |\langle F_0 + G - \chi_D G, T^* \circ J(v)\rangle| = |\langle F_0 + G - \chi_D G, m_v + \phi_v\rangle| \\ &\geqq \langle F_0, m_v\rangle + \langle G, \phi_v\rangle - |\langle G - \chi_D G, m_v\rangle| \\ &> 1 - \epsilon + ||\phi_v|| - \epsilon - |m_v|(X - D) > 1 + ||\phi_v|| - 3\epsilon. \end{aligned}$$

Therefore $||\phi_v|| < 3\epsilon$.

We thus have

$$\begin{aligned} \int \chi_C F dm_v + \int_{X-C} F dm_v &= \langle F, m_v\rangle \\ &= \langle v, v_F^*\rangle = \langle v, T(H_F)\rangle = \langle H_F, T^* \circ J(v)\rangle \\ &= \int \chi_C H_F dm_v + \int_{X-C} H_F dm_v + \langle H_F, \phi_v\rangle. \end{aligned}$$

It follows that

$$\begin{aligned} \langle \chi_C(F - H_F), T^* \circ J(v)\rangle = \langle \chi_C(F - H_F), m_v\rangle &= \int \chi_C(F - H_F) dm_v \\ &= \int_{X-C} H_F dm_v + \langle H_F, \phi_v\rangle - \int_{X-C} F dm_v. \end{aligned}$$

But the modulus of the quantity on the left is greater than $\delta(1 - \epsilon) > \delta/2$ by

(6), whereas the modulus of the quantity on the right is less than $5\epsilon \cdot \|F\|_\infty < \delta/2$ by (7). This contradiction completes the proof.

3. **Remarks and Problems.** Obviously our theorem is false if we attempt to replace $C(X, (E, \sigma^*))$ by $C(X, (E^*, \sigma^*))$ for an arbitrary (even separable) Banach dual E^*. For if X is a one-point space then $C(X, (E^*, \sigma^*)) \cong E^*$. Thus if E^* fails to have a unique predual, e.g. if $E^* = \ell^1$, then the same may be true of $C(X, (E^*, \sigma^*))$. However one may ask whether we can replace Hilbert space $1E$ in our theorem by a suitable class of Banach duals E^* properly containing Hilbert space. Ideally, can one characterize the class of Banach duals E^* for which our theorem holds with E replaced by E^*? In particular, if E^* has the Radon-Nikodym property and strongly unique predual then, for X hyperstonean, is the predual of $C(X, (E^*, \sigma^*))$ also strongly unique?

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