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LOCALISATION OF LINEAR DIFFERENTIAL EQUATIONS IN THE UNIT DISC BY A CONFORMAL MAP

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Abstract

We obtain lower bounds for the growth of solutions of higher order linear differential equations, with coefficients analytic in the unit disc of the complex plane, by localising the equations via conformal maps and applying known results for the unit disc. As an example, we study equations in which the coefficients have a certain explicit exponential growth at one point on the boundary of the unit disc and consider the iterated M-order of solutions.

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1. Introduction

We study the growth of solutions of the linear differential equation

$$f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_1(z)f' + a_0(z)f = 0, \tag{1.1}$$

where $a_0(z), a_1(z), \ldots, a_{k-1}(z)$ are analytic in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane $\mathbb{C}$, denoted by $a_0, a_1, \ldots, a_{k-1} \in \mathcal{H}(D)$ for short. Since all solutions are analytic, one natural measure of their growth is the n-order defined by

$$\sigma_{M,n}(f) = \limsup_{r\to 1^-} \frac{\log^+_{n+1} M(r, f)}{-\log(1-r)}, \quad f \in \mathcal{H}(D),\ n \in \mathbb{N}.$$

Here $\log^+ x = \max\{\log x, 0\}$, $\log^+_1 x = \log^+ x$, $\log^+_{n+1} = \log^+ \log^+_n x$ and $M(r, f)$ is the maximum modulus of f on the circle of radius r centred at the origin.

It is known that the growth of the coefficients restricts the growth of the solutions and *vice versa*, since all solutions f satisfy $\sigma_{M,n+1}(f) \le \alpha$ if and only if $\sigma_{M,n}(a_j) \le \alpha$ for all $j = 0, 1, \ldots, k-1$ [11, Theorem 1.1]. On the other hand, all nontrivial solutions are of maximal growth at least when a_0 dominates the other coefficients in the whole disc in some suitable way. One sufficient condition is that $\sigma_{M,n}(a_j) < \sigma_{M,n}(a_0)$

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for all $j = 1, 2, \ldots, k-1$ [11, Theorem 1.2]. A refined condition is given in [10, Theorem 3], namely $(\sigma_{M,n}(a_j), \tau_{M,n}(a_j)) \prec (\sigma_{M,n}(a_0), \tau_{M,n}(a_0))$ for $j = 1, 2, \ldots, k-1$. Here $\tau_{M,n}$ is the n-type defined by

$$\tau_{M,n}(f) = \limsup_{r\to 1^-} (1-r)^{\sigma_{M,n}(f)} \log_n^+ M(r, f), \quad f \in \mathcal{H}(D),\ n \in \mathbb{N},$$

and we write $(a, b) \prec (c, d)$ if either $a < c$ or $a = c$ and $b < d$, for $a, b, c, d \in \mathbb{R} \cup \{\infty\}$.

If a_0 dominates the other coefficients near a point on the boundary of the unit disc, and we consider the equation there locally, it is possible to obtain a lower bound for the growth of all nontrivial solutions. Of course, this local study can only give a lower bound and the upper bound depends on the behaviour of the coefficients in the whole disc. This idea is valid for several measures of growth and, in particular, we can study the n-order of growth. Earlier results concerning this kind of question can be found in [9, 10].

Localisation is a standard technique found in the literature. If $f \in \mathcal{H}(D)$, $\Omega \subset D$ is a simply connected domain and $\phi : D \to \Omega$ is analytic and conformal, then we can study f in Ω by studying the function $f \circ \phi$ in D. In particular, we can apply known results to $f \circ \phi$. The localisation domain Ω and the mapping ϕ must be chosen in a suitable way, depending on the expected properties of f. For example, when considering the behaviour of f near the boundary of D, Ω should touch the boundary in some suitable way. Also, the geometric and analytic properties of ϕ must be appropriate.

The simplest localisation mapping is an affine map, in which the image of D is a horocycle. For example, all solutions of

$$f'' + \exp\Big(\frac{1}{1+z}\Big) f' + \exp\Big(\frac{1}{1-z}\Big) f = 0$$

satisfy $\sigma_{M,2}(f) = 1$. The inequality $\sigma_{M,2}(f) \le 1$ follows from [11, Theorem 1.1] and the converse inequality is seen by studying $g = f \circ \phi$, where $\phi : D \to D$ is given by $\phi(z) = \frac{1}{2}(1+z)$, and applying [11, Theorem 1.2]. For a more general result, see Theorem 1.1. Here ϕ' is a constant and $\phi(D)$ is a horocycle touching ∂D tangentially.

Another example of localisation is [6, Proof of Theorem 4], where the authors use a localisation map $\psi : D \to D$,

$$\psi(z) = e^{i\theta} \frac{\varphi(\zeta) - 1}{\varphi(\zeta) + 1}, \quad \varphi(z) = e^{-i\pi\delta/2} \Big(\frac{1+z}{1-z}\Big)^{1-\delta} - i\alpha,$$

where $\theta \in [0, 2\pi]$, $\alpha \in (0, \infty)$ and $\delta \in (0, \frac{2}{5})$. The Schwarzian derivative of ψ has sufficiently smooth behaviour for calculations. On the other hand, the boundary curve $\partial\psi(D)$ consists of two circular arcs, one of which is a part of the unit circle. Thus, $\psi(D)$ has a fairly simple crescent shape.

The explicit expression of the localisation map may not be needed. For a simply connected localisation domain, the existence of the mapping can be deduced from the Riemann mapping theorem and the smoothness of the mapping and the growth of its derivatives can be estimated by the geometric properties of the boundary curve of the image. For example, in [5, Proof of Theorem 3], the authors use a localisation map $\phi_{\delta,\rho} : D \to \Omega_{\delta,\rho}$, for which the boundary of the simply connected convex domain $\Omega_{\delta,\rho} \subseteq D$ consists of four circular arcs, one being a part of the unit circle. Since the boundary curve is smooth, the authors can deduce that $(\log \phi'_{\delta,\rho})'$ and $\phi''_{\delta,\rho}$ belong to the Hardy space H^p for all $p \in (0, \infty)$ and deduce that $\phi'_{\delta,\rho}$ is continuous on $\overline{D}$. With these estimates, the proof can proceed. See [5] for details and definitions.

The purpose of this paper is to explain how a localisation method can be used to study the growth of solutions of (1.1) when information on the coefficients is available near some boundary point only. To illustrate the method concretely, we consider the growth of solutions, in terms of the n-order, of the equation

$$g^{(k)} + \sum_{j=0}^{k-1} B_j(z) \exp_{n_j}\Big(\frac{d_j}{(z_0 - z)^{q_j}}\Big) g^{(j)} = 0, \tag{1.2}$$

where $B_j \in \mathcal{H}(D \cup \{z_0\})$, $d_j, q_j \in \mathbb{C}$ and $n_j \in \mathbb{N}$ for $j = 0, 1, \dots, k-1$. Here, we write $\exp_1(x) = \exp(x)$ and $\exp_{n+1}(x) = \exp(\exp_n(x))$. Throughout the paper, for a nonzero complex number $z \in \mathbb{C}$ and a noninteger power $p \in \mathbb{C}$, we define z^p by taking the principal branch. Hence, here $(z_0 - z)^q$ is well defined, since $z_0 - z$ is nonvanishing in D. We assume that $\mathrm{Re}(q_j) > 0$, since otherwise

$$z \mapsto \exp_{n_j}\Big(\frac{d_j}{(z_0 - z)^{q_j}}\Big)$$

is bounded in D, a case of no interest. By making the change of variable $z \to z_0 z$ and denoting $b_j = d_j/z_0^{q_j}$, $f(z) = g(z_0 z)$ and $A_j(z) = B_j(z_0 z) z_0^{k-j}$, (1.2) reduces to

$$f^{(k)} + \sum_{j=0}^{k-1} A_j(z) \exp_{n_j}\Big(\frac{b_j}{(1 - z)^{q_j}}\Big) f^{(j)} = 0, \tag{1.3}$$

where $A_j \in \mathcal{H}(D \cup \{1\})$, $b_j, q_j \in \mathbb{C}$ and $n_j \in \mathbb{N}$ for $j = 0, 1, \dots, k-1$.

The results of this paper improve the results in [9] concerning the growth of solutions of (1.2) and the proofs are simpler than the original ones. Our method is elementary and therefore of interest, even though the results concerning (1.2) can be deduced from [10, Theorem 2].

The study [9] was motivated by certain results concerning the differential equation

$$f'' + A(z)e^{az} f' + B(z)e^{bz} f = 0, \tag{1.4}$$

where $A(z)$ and $B(z)$ are entire functions and $a, b \in \mathbb{C}$; see [1–3, 7]. See also [4, 8, 11, 13] for methods based on the dominance of some coefficient. The techniques of [9] were inherited from the plane case and are analogous to those used in [2]. For example, if in (1.4) we have $ab \neq 0$ and either $\arg a \neq \arg b$ or $a/b \in (0, 1)$, then all nontrivial solutions f are of infinite order on the plane [2, Theorem 2]. Analogously, if in the equation

$$f'' + B_1(z) \exp\Big(\frac{b_1}{(z_0 - z)^q}\Big) f' + B_0(z) \exp\Big(\frac{b_0}{(z_0 - z)^q}\Big) f = 0,$$

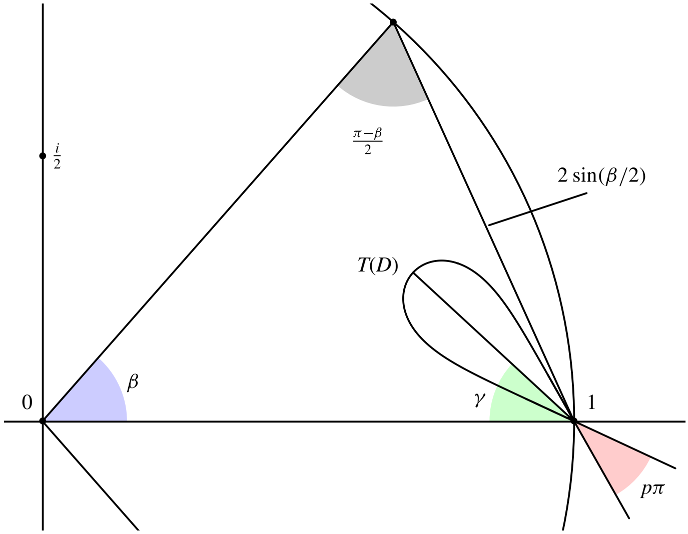

FIGURE 1. Domain $T(D)$ with parameters $\beta = 0.85$ and $\gamma = -0.75$. In this case, $p = \beta(\pi-\beta)/\pi^2 \approx 0.197$ and $2\sin(\beta/2) \approx 0.825$.

where $B_j \in \mathcal{H}(D \cup \{z_0\})$, $b_j \in \mathbb{C}\backslash\{0\}$, $q \in (1,\infty)$, we have in addition $\arg b_1 \neq \arg b_0$ or $b_1/b_0 \in (0,1)$, then all nontrivial solutions f satisfy $\sigma_{M,1}(f) = \infty$ [9, Theorem 1.11].

To define the localisation map employed here, let $T : D \to D$ be given by

$$T(z) = T_{\beta,\gamma}(z) = 1 - \sin(\beta/2)e^{i\gamma}\Big(\frac{1-z}{2}\Big)^p, \tag{1.5}$$

where $\beta \in (0,\pi/2]$, $\gamma \in (-\pi/2,\pi/2)$ are such that $|\gamma| \leq (\pi-\beta)^2/2\pi \in (0,\pi/2)$ and $p = p(\beta) = \beta(\pi-\beta)/\pi^2 \in (0,1/4]$. Here $T(D)$ is a tear-shaped region having a vertex of angle $p\pi$ touching ∂D at $z = 1$ (see Figure 1). The domain $T(D)$ has the symmetry axis $T((-1,1))$ which meets the real axis at angle γ. As β decreases, $T(D)$ becomes thinner, $T((-1,1))$ becomes shorter and the angle γ can be set larger. If f satisfies (1.3) and we set $g = f \circ T$, then g has to satisfy a differential equation whose coefficients correspond to those of (1.3) (see Lemma 2.1 and its proof). By applying either [11, Theorem 1.2] or [10, Theorem 3] to this differential equation, we obtain a lower bound for the n-order of g, which in turn gives a lower bound for the n-order of f by Lemma 2.2.

We do not obtain new upper bounds for the growth of solutions of (1.2). In fact, it is not possible to obtain such bounds for the growth of solutions of (1.2) without imposing conditions on the functions B_j. If for example $\sigma_{M,n}(B_m) = \alpha > 0$ for some $m \in \{0,1,\ldots,k-1\}$ and $n \in \mathbb{N}$ with $n > n_m$, then no cancellation can occur, the coefficient

$$a_m(z) = B_m(z)\exp_{n_m}\Big(\frac{d_m}{(z_0-z)^{q_m}}\Big)$$

satisfies $\sigma_{M,n}(a_m) \geq \sigma_{M,n}(B_m) = \alpha$ and there exists at least one solution f such that $\sigma_{M,n+1}(f) \geq \alpha$ by [11, Theorem 1.1].

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The first result in this paper concerns the case when only a_0 in (1.1) is unbounded near a boundary point of the unit disc. In the remainder of the paper, the argument of a complex number $z \neq 0$ takes values $\arg(z) \in (-\pi, \pi]$.

THEOREM 1.1. *Consider the differential equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) \exp_n\Big(\frac{b}{(1-z)^q}\Big) f = 0,$$

where $k, n \in \mathbb{N}$, $A_j \in \mathcal{H}(D \cup \{1\})$ *for* $j = 0, 1, \ldots, k-1$, $A_0 \not\equiv 0$, $b, q \in \mathbb{C}\backslash\{0\}$ *and* $\mathrm{Re}(q) > 0$. *Suppose that* $\mathrm{Im}(q) \neq 0$ *or* $|\arg(b)| < \frac{1}{2}\pi(\mathrm{Re}(q)+1)$. *Then all nontrivial solutions* f *satisfy* $\sigma_{M,n+1}(f) \geq \mathrm{Re}(q)$.

If $\mathrm{Re}(q) > 1$ in Theorem 1.1, then the condition $|\arg(b)| < \frac{1}{2}\pi(\mathrm{Re}(q)+1)$ is trivially satisfied. Moreover, we get [9, Theorem 1.6] as a special case, by setting $k = 2$, $n = 1$, $q \in (1, \infty)$ and making a change of variables $z = w/z_0$, $b = d/z_0^q$ for $z_0 \in \partial D$.

If $q \in (0, 1]$ in Theorem 1.1, then the condition $|\arg(b)| < \frac{1}{2}\pi(\mathrm{Re}(q)+1)$ cannot be removed. For example, if $|\arg(-b)| \leq \frac{1}{2}(1-q)\pi$ for $q \in [0, 1]$, then $z \mapsto \exp(b(1-z)^{-q})$ is bounded on D and the solutions of $f'' + \exp(b(1-z)^{-q}) f = 0$ are bounded by [12, Corollary 3.16]. In particular, by setting $k = 2$, $A_1 \equiv 0$, $b = -1$ and $q = n = 1$, we obtain the equation

$$f'' + A_0(z) \exp\Big(\frac{-1}{1-z}\Big) f = 0,$$

where $A_0 \in \mathcal{H}(D \cup \{1\})$. Since $A_0(z)\exp(-(1-z)^{-1})$ remains bounded as $z \to 1$ in D, nothing can be said about the growth of solutions f without placing conditions on A_0. This is the reason why the method of [9] cannot work in general for $0 < q \leq 1$; see the discussion in [9, Remark 3.1].

Next we consider a second-order equation with both coefficients possibly unbounded near the point $z = 1$, namely

$$f'' + A_1(z) \exp\Big(\frac{b_1}{(1-z)^{q_1}}\Big) f' + A_0(z) \exp\Big(\frac{b_0}{(1-z)^{q_0}}\Big) f = 0, \tag{1.6}$$

where $A_j \in \mathcal{H}(D \cup \{1\})$, $A_0 \not\equiv 0$, $b_j, q_j \in \mathbb{C}\backslash\{0\}$ for $j = 0, 1$ and $\mathrm{Re}(q_0) > 0$. The most interesting case is when $q_1 = q_0$. First, we consider $q_1 = q_0 \in (0, \infty)$, then $q_1 = q_0 \in \mathbb{C}\backslash\mathbb{R}$ and after that the case $q_1 \neq q_0$.

THEOREM 1.2. *Let* $q_1 = q_0 = q \in (2, \infty)$ *and* $\arg(b_1) \neq \arg(b_0)$ *in (1.6). Then all nontrivial solutions* f *satisfy* $\sigma_{M,2}(f) \geq q$.

The case $q \in (0, 2]$, which is not covered by Theorem 1.2, can be done with stronger assumptions, as in Theorem 2.3. For $q \in (2, \infty)$, Theorem 1.2 improves [9, Theorem 1.8], which states that for $q \in (1, \infty)$, we have $\sigma_{M,1}(f) = \infty$. Moreover, for $q \in (2, \infty)$, Theorem 2.3 improves [9, Theorem 1.11].

THEOREM 1.3. *Let* $q_1 = q_0 = q$, $\mathrm{Im}(q) \neq 0$, $\mathrm{Re}(q) > 0$ *and* $|b_1| < |b_0|$ *in (1.6). Then all nontrivial solutions* f *satisfy* $\sigma_{M,2}(f) \geq \mathrm{Re}(q)$.

THEOREM 1.4. *Let $q_1 \neq q_0$ in (1.6). Assume that either $q_0, q_1 \in (0, \infty)$ and*

$$\operatorname{Re}\left(\frac{b_1}{e^{i\gamma q_1}}\right) < 0 < \operatorname{Re}\left(\frac{b_0}{e^{i\gamma q_0}}\right) \quad \text{for some } \gamma \in (-\pi/2, \pi/2), \tag{1.7}$$

or $\operatorname{Im}(q_0) \neq 0$ *and* $\operatorname{Re}(q_1) < \operatorname{Re}(q_0)$. *Then all nontrivial solutions f of (1.6) satisfy* $\sigma_{M,2}(f) \geq \operatorname{Re}(q_0)$.

COROLLARY 1.5. *Let $q_0, q_1 \in (0, \infty)$, $q_1 \neq q_0$ in (1.6). Suppose that one of the following conditions is satisfied:*

(i) $\operatorname{Re}(b_1) < 0 < \operatorname{Re}(b_0)$;
(ii) $|\arg(b_0)| < \frac{1}{2}\pi(q_0 + 1)$ *and* $q_1 > 2q_0/(q_0 + 1 - (2/\pi)|\arg(b_0)|)$;
(iii) $|\arg(-b_1)| < \frac{1}{2}\pi(q_1 + 1)$ *and* $q_0 > 2q_1/(q_1 + 1 - (2/\pi)|\arg(-b_1)|)$;
(iv) $q_0 \in (1, 3]$ *and* $q_1 > 2q_0/(q_0 - 1)$;
(v) $q_0 \in [3, \infty)$ *and* $q_1 > q_0/(q_0 - 2)$;
(vi) $q_0, q_1 \in [3, \infty)$.

Then all nontrivial solutions f satisfy $\sigma_{M,2}(f) \geq q_0$.

Condition (1.7) follows from each of the conditions (i)–(vi) in Corollary 1.5 and is symmetric with respect to q_0 and q_1 in the following sense: if the assumption $q_0 = a$ and $q_1 = b$ gives (1.7) for all $b_0, b_1 \in \mathbb{C}\setminus\{0\}$, then the assumption $q_0 = b$ and $q_1 = a$ implies the same conclusion. On the other hand, we see that (1.7) fails in the following cases:

(a) $|\arg(b_0)| \geq \frac{1}{2}\pi(q_0 + 1)$ or $|\arg(-b_1)| \geq \frac{1}{2}\pi(q_1 + 1)$;
(b) $0 < q_0 < q_1 \leq 3$ and $b_0 = b_1 = -1$;
(c) $0 < q_1 < q_0 \leq 3$ and $b_0 = b_1 = 1$;
(d) $q_0 \in (2, \infty)$, $q_1 = q_0/(q_0 - 1)$, $b_0 = \exp(\frac{1}{2}i\pi(q_0 - 3))$ and $b_1 = \exp(\frac{1}{2}i\pi(1 - q_1))$;
(e) $q_0 = 2m + 1$, $q_1 = q_0/(q_0 - 2)$, $b_0 = (-1)^{m+1}$ and $b_1 = 1$ for some $m \in \mathbb{N} \cap [2, \infty)$.

For $q_0 \in (1, \infty)$, it is not clear how q_1 satisfying $q_0/(q_0 - 1) < q_1 \leq q_0/(q_0 - 2)$ should be restricted to obtain (1.7) for all $b_0, b_1 \in \mathbb{C}\setminus\{0\}$. Numerical investigations suggest that conditions

$$q_1 > \frac{q_0}{q_0 - 1}, \qquad q_0 \in \bigcup_{m=2}^{\infty} (2m - 1, 2m)$$

and

$$q_1 > \frac{2m}{2m - 1}(1 - (q_0 - 2m)) + \frac{2m + 1}{(2m + 1) - 2}(q_0 - 2m), \qquad q_0 \in [2m, 2m + 1],$$

for $m \in \mathbb{N} \cap [2, \infty)$, could be sharp. The latter condition says that as q_0 increases from $2m$ to $2m + 1$, the lower bound of q_1 increases linearly.

Our method works also for nonhomogeneous equations, as part (ii) of Theorem 2.4 shows.

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2. Proofs of theorems

The following lemma allows us to study the differential equation (1.1) locally on a subset of the unit disc.

Lemma 2.1. *Let f be a solution of*

$$f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_1(z)f' + a_0(z)f = a_k(z),$$

where $a_0, a_1, \ldots, a_k \in \mathcal{H}(D)$. Let $T : D \to D$ be locally univalent and $g = f \circ T$. Then g is a solution of

$$g^{(k)} + c_{k-1}(z)g^{(k-1)} + \cdots + c_1(z)g' + c_0(z)g = c_k(z), \tag{2.1}$$

where $c_j \in \mathcal{H}(D)$. Moreover, if $T^{(s)}$ is nonvanishing and $\sigma_{M,n}((T^{(s)})^t) = 0$ for $n, s \in \mathbb{N}$ and $t \in \mathbb{Z}$, then

$$\sigma_{M,n}(c_j) \le \max_{m\ge j}\{\sigma_{M,n}(a_m \circ T)\} \tag{2.2}$$

and

$$\tau_{M,n}(c_j) \le \max\{\tau_{M,n}(a_N \circ T) : \sigma_{M,n}(a_N \circ T) = \max_{m\ge j}\{\sigma_{M,n}(a_m \circ T)\}\}, \tag{2.3}$$

for $j = 0, 1, \ldots, k-1$, whereas

$$\sigma_{M,n}(c_k) = \sigma_{M,n}(a_k \circ T) \quad \text{and} \quad \tau_{M,n}(c_k) = \tau_{M,n}(a_k \circ T). \tag{2.4}$$

Proof. By a straightforward calculation, g is a solution of (2.1), where

$$c_j = \frac{1}{P_{j,j}(T)}\Big[(a_j \circ T)(T')^k - P_{k,j}(T) - \sum_{m=j+1}^{k-1} c_m P_{m,j}(T)\Big], \quad j = 0, 1, \ldots, k-1, \tag{2.5}$$

$c_k = (a_k \circ T)P_{k,k}(T)$ and $P_{m,j}(T)$ is defined by $g^{(m)} = \sum_{j=1}^{m}(f^{(j)} \circ T)P_{m,j}(T)$. Hence, $P_{m,j}(T)$ is a polynomial in $T', T'', \ldots, T^{(m)}$ with integer coefficients. For $j = k-1$, the sum on the right-hand side of (2.5) is empty, and we can solve for c_{k-1}:

$$c_{k-1} = \frac{1}{P_{k-1,k-1}(T)}[(a_{k-1} \circ T)(T')^k - P_{k,k-1}(T)].$$

After this, we can inductively solve for $c_{k-2}, c_{k-3}, \ldots, c_0$. By the assumption, T is locally univalent, that is, T' has no zeros in D. Since $P_{j,j} = (T')^j$ is nonvanishing for $j = 0, 1, \ldots, k$, we see that $c_j \in \mathcal{H}(D)$ for all $j = 0, \ldots, k$.

Assume now that $\sigma_{M,n}((T^{(s)})^t) = 0$ for $s \in \mathbb{N}$ and $t \in \mathbb{Z}$. Since for $j = 0, 1, \ldots, k-1$ the coefficient c_j is a linear combination of the functions $a_j \circ T, a_{j+1} \circ T, \ldots, a_{k-1} \circ T$, the assertions (2.2) and (2.3) trivially hold. The assertion (2.4) is also evident. □

Clearly, T defined by (1.5) satisfies all the assumptions of Lemma 2.1. Hence, if we set $g = f \circ T$, then we can study the differential equation (1.1) for f by studying the differential equation (2.1) for g. In this case, if we can find a lower bound for the n-order of g, we have a lower bound for the n-order of f by the next lemma.

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LEMMA 2.2. *Let $f \in \mathcal{H}(D)$ and $g = f \circ T$, where T is defined by (1.5). Then we have $\sigma_{M,n}(f) \geq \sigma_{M,n}(g)/p$ for $n \in \mathbb{N}$.*

PROOF. If $|1 - z| \leq \sin(\beta/2)$ and $|\arg(1 - z)| \leq (\pi - \beta)/2$, then the law of cosines gives

$$|1 - z| \leq \frac{2}{\sin(\beta/2)}(1 - |z|)$$

and, therefore, by the definition of T,

$$|1 - T(z)| \leq \frac{2}{\sin(\beta/2)}(1 - |T(z)|), \quad z \in D.$$

Now, for $r \in [0, 1)$ and $\theta \in [0, 2\pi)$ such that $|T(re^{i\theta})| = M(r, T)$,

$$\begin{aligned} 1 - M(r, T) &\leq 1 - |T(r)| \leq |1 - T(r)| \leq |1 - T(re^{i\theta})| \\ &\leq \frac{2}{\sin(\beta/2)}(1 - |T(re^{i\theta})|) = \frac{2}{\sin(\beta/2)}(1 - M(r, T)). \end{aligned} \tag{2.6}$$

Since

$$|1 - T(r)| = \frac{\sin(\beta/2)}{2^p}(1 - r)^p,$$

inequality (2.6) gives

$$\lim_{r\to1^-} \frac{\log(1 - M(r, T))}{p\log(1 - r)} = 1. \tag{2.7}$$

Now, by (2.7),

$$\frac{\sigma_{M,n}(g)}{p} = \limsup_{r\to1^-} \frac{\log^+_{n+1} M(r, g)}{-p\log(1 - r)} \leq \limsup_{r\to1^-} \frac{\log^+_{n+1} M(M(r, T), f)}{-\log(1 - M(r, T))} = \sigma_{M,n}(f),$$

the last inequality holding since $M(r, T)$ is an increasing continuous function of r and $M(r, T) \to 1^-$ as $r \to 1^-$. □

PROOF OF THEOREM 1.1. Let $q = x + iy$ for $x \in (0, \infty)$ and $y \in \mathbb{R}$, and let $g = f \circ T$, where T is defined by (1.5). Use the differential equation for f in the claim to obtain the differential equation (2.1) for g. In this differential equation, $c_k \equiv 0$ and $\sigma_{M,n}(c_j) = 0$ for $j = 1, 2, \ldots, k - 1$. Moreover, $\sigma_{M,n}(c_0) = px$. To show this, we start by observing that

$$\frac{b}{(1 - T(z))^q} = \frac{b2^{pq}}{(\sin(\beta/2))^q e^{i\gamma q}} \frac{1}{(1 - z)^{pq}} = \frac{b2^{pq}e^{-ipy\log(1-z)}}{(\sin(\beta/2))^q e^{i\gamma q}} \frac{1}{(1 - z)^{px}}.$$

First, assume that $y \neq 0$. Now, for some sequence of points $r_n \in (0, 1)$, $r_n \to 1^-$ as $n \to \infty$, the value of $\log(1 - r_n)$ is such that

$$\frac{b2^{pq}e^{-ipy\log(1-r_n)}}{(\sin(\beta/2))^q e^{i\gamma q}} = \left|\frac{b2^{pq}}{(\sin(\beta/2))^q e^{i\gamma q}}\right| = C \in (0, \infty).$$

Hence, for this sequence $\{r_n\}_{n\in\mathbb{N}}$,

$$\frac{b}{(1 - T(r_n))^q} = \frac{C}{(1 - r_n)^{px}}, \quad n \in \mathbb{N},$$

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giving

$$\left|\exp_n\left(\frac{b}{(1-T(r_n))^q}\right)\right| = \exp_n\left(\frac{C}{(1-r_n)^{px}}\right), \quad n \in \mathbb{N},$$

and we see that $\sigma_{M,n}(c_0) = px$.

Second, assume that $y = 0$, that is, $q = x \in (0, \infty)$, and $|\arg(b)| < \frac{1}{2}\pi(x + 1)$. Now there exist $\gamma \in (-\pi/2, \pi/2)$ such that

$$\left|\arg\left(\frac{b}{e^{i\gamma x}}\right)\right| < \frac{\pi}{2} \quad \text{that is, } \mathrm{Re}\left(\frac{b}{e^{i\gamma x}}\right) > 0$$

and $\beta \in (0, \pi/2]$ such that $|\gamma| \leq (\pi - \beta)^2/2\pi$, giving $T = T_{\beta,\gamma} : D \to D$. Now there exists a sequence of points $r_n \in (0, 1)$, $r_n \to 1^-$ as $n \to \infty$, such that

$$\frac{b}{(1-T(r_n))^x} = \frac{b2^{px}}{(\sin(\beta/2))^x e^{i\gamma x}} \frac{1}{(1-r_n)^{px}} = \frac{2^{px}\mathrm{Re}(be^{-i\gamma x})}{(\sin(\beta/2))^x} \frac{1}{(1-r_n)^{px}} + i2\pi m_n,$$

for some integers m_n such that either $m_n = 0$ for all $n \in \mathbb{N}$ or $|m_n| \to \infty$ as $n \to \infty$. Thus, also in this case, $\sigma_{M,n}(c_0) = px$. Now, by Lemma 2.2 and [11, Theorem 1.1], we have $\sigma_{M,n+1}(f) \geq \sigma_{M,n+1}(g)/p \geq \sigma_{M,n}(c_0)/p = x$, given that $f \not\equiv 0$. □

Theorem 1.2 is a special case of Theorem 2.3, since, for $q_1 = q_0 = q$, (1.6) is a special case of (2.8) and, if $q \in (2, \infty)$, then one of the conditions (i)–(iii) in Theorem 2.3 is satisfied.

Theorem 2.3. *Consider the differential equation*

$$f^{(k)} + \sum_{j=0}^{k-1} A_j(z) \exp\left(\frac{b_j}{(1-z)^q}\right) f^{(j)} = 0, \tag{2.8}$$

where $k \in \mathbb{N}$, $A_j \in \mathcal{H}(D \cup \{1\})$, $q \in (0, \infty)$ and $b_j \in \mathbb{C}$ for $j = 0, 1, \ldots, k-1$. Let $A_0 \not\equiv 0$ and $b_0 \neq 0$. Assume that $b_j/b_0 \in [0, 1)$ for all $j = 0, 1, \ldots, k-1$ with at most one exception $b_j = b_m$ for which $\arg(b_m) \neq \arg(b_0)$. Suppose that one of the conditions:

(i) $\max(\mathrm{Re}(b_m), 0) < \mathrm{Re}(b_0)$;

(ii) $0 < \mathrm{Re}(b_0) \leq \mathrm{Re}(b_m)$, $\arg(b_m/b_0) \in (0, \pi)$ *and* $\arg(i/(b_m - b_0)) < \frac{1}{2}\pi q$;

(iii) $\mathrm{Re}(b_0) \leq 0$, $\arg(b_m/b_0) \in (0, \pi]$ *and* $\arg(b_0/i) < \frac{1}{2}\pi q$

holds or that one of the conditions holds when b_0 and b_m are replaced by $\overline{b_0}$ and $\overline{b_m}$, respectively. Then all nontrivial solutions f satisfy $\sigma_{M,2}(f) \geq \mathrm{Re}(q)$.

Proof. Let $g = f \circ T$, where T is defined by (1.5). Use the differential equation for f in the claim to obtain the differential equation (2.1), where $c_k \equiv 0$, for g. First, we treat the case

$$f'' + A_1(z) \exp\left(\frac{b_1}{(1-z)^q}\right) f' + A_0(z) \exp\left(\frac{b_0}{(1-z)^q}\right) f = 0,$$

where the assumptions in the claim are satisfied by $b_m = b_1$.

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Now the assumptions ensure the existence of $\gamma \in (-\pi/2, \pi/2)$ such that

$$\max\Big(\mathrm{Re}\Big(\frac{b_1}{e^{i\gamma q}}\Big), 0\Big) < \mathrm{Re}\Big(\frac{b_0}{e^{i\gamma q}}\Big).$$

Fix one such γ and choose $\beta \in (0, \pi/2]$ sufficiently small to obtain $|\gamma| \le (\pi - \beta)^2/2\pi$. With these parameters γ and β, we have $T = T_{\beta,\gamma} : D \to D$. By taking β even smaller, we find some $\varepsilon \in (0, 1)$ such that

$$\max\Big(\mathrm{Re}\Big(\frac{b_1}{e^{i\gamma q}} \frac{|1-z|^{pq}}{(1-z)^{pq}}\Big), 0\Big) < \varepsilon \mathrm{Re}\Big(\frac{b_0}{e^{i\gamma q}} \frac{|1-z|^{pq}}{(1-z)^{pq}}\Big), \quad z \in D.$$

Hence, in (2.1), $(\sigma_{M,1}(c_1), \tau_{M,1}(c_1)) \prec (\sigma_{M,1}(c_0), \tau_{M,1}(c_0))$. The assertion follows by [10, Theorem 3] and Lemma 2.2.

The general case is proved in a similar manner. In particular, for $j \neq m$, the coefficient c_j is small in the sense that $(\sigma_{M,1}(c_j), \tau_{M,1}(c_j)) \prec (\sigma_{M,1}(c_0), \tau_{M,1}(c_0))$. □

Theorem 1.1 can be trivially generalised to obtain part (i) of Theorem 2.4. Part (ii) of Theorem 2.4 shows that our method works also for nonhomogeneous equations.

THEOREM 2.4. *Consider the differential equation*

$$f^{(k)} + \sum_{j=0}^{k-1} A_j(z) \exp_{n_j}\Big(\frac{b_j}{(1-z)^q}\Big) f^{(j)} = A_k(z) \exp_{n_k}\Big(\frac{b_k}{(1-z)^{q_k}}\Big), \tag{2.9}$$

where $k \in \mathbb{N}$, $A_j \in \mathcal{H}(D \cup \{1\})$, $q, q_k \in \mathbb{C}\setminus\{0\}$ and $b_j \in \mathbb{C}$ for $j = 0, 1, \ldots, k$. Then the following assertions hold.

(i) *Let $b_k = 0$, $A_0 \not\equiv 0$, $b_0 \neq 0$, $\mathrm{Re}(q) > 0$ and either $n_j < n_0$, or $n_j = n_0$ but $b_j/b_0 \in [0, 1)$, for $j = 1, 2, \ldots, k-1$. Suppose $\mathrm{Im}(q) \neq 0$ or $|\arg(b_0)| < \frac{1}{2}\pi(\mathrm{Re}(q) + 1)$. Then all nontrivial solutions f of (2.9) satisfy $\sigma_{M,n_0+1}(f) \geq \mathrm{Re}(q)$.*

(ii) *Let $A_k \not\equiv 0$ and $b_k \neq 0$. Assume that $n_j \le n_k - 1$ for $j = 1, 2, \ldots, k-1$ and $\mathrm{Re}(q) < \mathrm{Re}(q_k)$. Suppose that $\mathrm{Im}(q_k) \neq 0$ or $|\arg(b_k)| < \frac{1}{2}\pi(\mathrm{Re}(q_k) + 1)$. Then all solutions f of (2.9) satisfy $\sigma_{M,n_k}(f) \geq \mathrm{Re}(q_k)$.*

PROOF. Assertion (i) is clear. Let the assumptions in (ii) be satisfied. Let $g = f \circ T$, where T is defined by (1.5). Use the differential equation (2.9) for f to obtain the differential equation (2.1) for g. Fix one particular solution f_2 of (2.9) and let $g_2 = f_2 \circ T$. Now every solution g is of the form $g = g_1 + g_2$, where g_1 is a solution of the homogeneous equation. By the assumptions and the proof of Theorem 1.1, $\sigma_{M,n_k}(g_1) \le \mathrm{Re}(q)p < \mathrm{Re}(q_k)p$. On the other hand, the parameters of $T = T_{\beta,\gamma}$ can be chosen such that $\sigma_{M,n_k}(c_k) = \mathrm{Re}(q_k)p$, which gives $\sigma_{M,n_k}(g_2) = \sigma_{M,n_k}(c_k) = \mathrm{Re}(q_k)p$. Hence, $\sigma_{M,n_k}(g) = \mathrm{Re}(q_k)p$, since no cancellation can occur. By Lemma 2.2, $\sigma_{M,n_k}(f) \geq \sigma_{M,n_k}(g)/p = \mathrm{Re}(q_k)$. □

PROOF OF THEOREM 1.3. Let $q = x + iy$, $x \in (0, \infty)$ and $y \in \mathbb{R}$. Let $g = f \circ T$, where T is defined by (1.5). Use the differential equation for f in the claim to obtain the differential equation (2.1), with $c_k \equiv 0$, for g. By the assumptions and the proof of

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Theorem 1.1, we can choose the parameter γ of $T = T_{\beta,\gamma}$ such that the coefficients c_j in (2.1) satisfy $(\sigma_{M,1}(c_j), \tau_{M,1}(c_j)) \prec (\sigma_{M,1}(c_0), \tau_{M,1}(c_0))$ for all $j = 1, 2, \ldots, k-1$. Moreover, in this case $\sigma_{M,1}(c_0) = px$. Hence, all nontrivial solutions g of (2.1) satisfy $\sigma_{M,2}(g) \geq px$ by [10, Theorem 3]. By Lemma 2.2, all nontrivial solutions f of (1.6) satisfy $\sigma_{M,2}(f) \geq \sigma_{M,2}(g)/p \geq x = \mathrm{Re}(q)$. □

Proof of Theorem 1.4. If (1.7) is valid, then the assertion follows as in the proof of Theorem 2.3.

Assume that $\mathrm{Im}(q_0) \neq 0$ and $\mathrm{Re}(q_1) < \mathrm{Re}(q_0)$ and let $g = f \circ T$, where T is defined by (1.5). Use the differential equation for f in the claim to obtain the differential equation (2.1), with $c_k \equiv 0$, for g. Now, in (2.1), we have $c_k \equiv 0$, $\sigma_{M,1}(c_1) < \sigma_{M,1}(c_0)$ and in addition $\sigma_{M,1}(c_0) = \mathrm{Re}(q_0)p$. Now, by [11, Theorem 1.2] and Lemma 2.2, we deduce that $\sigma_{M,2}(f) \geq \sigma_{M,2}(g)/p = \mathrm{Re}(q_0)$ for every nontrivial solution f, as desired. □

Proof of Corollary 1.5. Trivially, (i) implies (1.7) of Theorem 1.4.

Assume that (ii) is true. Now, there exist $(\gamma_1, \gamma_2) \subset (-\pi/2, \pi/2)$ such that

$$|\arg(b_0 e^{-i\gamma q_0})| < \frac{\pi}{2}, \quad \gamma \in (\gamma_1, \gamma_2)$$

and

$$|\gamma_1 - \gamma_2| \geq \frac{\frac{1}{2}\pi q_0 + \frac{1}{2}\pi - |\arg(b_0)|}{q_0} = \frac{q_0 + 1 - (2/\pi)|\arg(b_0)|}{2q_0}\pi.$$

By the assumption,

$$q_1|\gamma_1 - \gamma_2| \geq q_1 \frac{q_0 - 1}{2q_0}\pi > \pi,$$

so that $|\arg(-b_1 e^{-i\gamma q_1})| < \pi/2$ for some $\gamma \in (\gamma_1, \gamma_2)$ and (1.7) is valid. Similarly (iii) gives (1.7).

Trivially, condition (iv) implies (ii). Condition (v) holds if and only if $q_1 \in (1, 3)$ and $q_0 > 2q_1/(q_1 - 1)$. Therefore, (v) implies (iii).

If condition (vi) holds, then either (iv) or (v) is valid. □

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