Canad. Math. Bull. Vol. **18** (4), 1975

CONTRACTIONS WITH FIXED POINTS AND CONDITIONAL EXPECTATION

BY
A. N. AL-HUSSAINI

1. **Introduction.** Let (Ω, α, μ) be a σ-finite measure space. By $L_p(\Omega, \alpha, \mu)$ or L_p for short we denote the usual Banach space of pth power μ-integrable functions on Ω if $1 \leq p < +\infty$ and μ-essentially bounded functions on Ω, if $p = +\infty$. In section (2) we characterize conditional expectation, by a method different than those used previously. Modulus of a given contraction is discussed in section (3). If the given contraction has a fixed point, then its modulus has a simple form (theorem 3.2). In section (4) we use results from section (3) to relate projections conditional expectation. Finally in section (5) we give a version of Chacon-Ornestein ratio ergodic theorem. 1_A will denote the indicator function of A i.e. $1_A = 1$ on A, $1_A = 0$ off A.

2. **Conditional expectation.** Let (Ω, α, μ) be an arbitrary measure space. For a given sub-σ-algebra $\beta \subset \alpha$, the conditional expectation $E\{f \mid \beta\}$ of f given β is a function measurable relative to β, such that

$$(*) \qquad \int_B E\{f \mid \beta\}\, d\mu = \int_B f\, d\mu, \quad \text{all} \quad B \in \beta$$

If $\mu(\Omega) = 1$, then a linear operator T on L_1 is a conditional expectation relative to some sub-σ-algebra $\beta \subset \alpha$ if and only if $\|T\| \leq 1$, $T^2 = T$ and $T1 = 1$ ([6], [2]). The condition $T1 = 1$ does not make sense if $\mu(\Omega) = +\infty$. As it will turn out, our conditions for a σ-finite μ include the case when μ is finite. If T is a linear operator on L_1, we denote its adjoint by T^* i.e.

$$(**) \qquad \int Tfg\, d\mu = \int f T^* g\, d\mu, \qquad f \in L_1, \qquad g \in L_\infty$$

THEOREM 2.1. *A linear operator T on L_1 is a conditional expectation relative to some sub-σ-algebra $\beta \subset \alpha$ if and only if* (1) $\|T\| \leq 1$, (2) $T^2 = T$, (3) $Tf = f$ *some* $0 < f \in L_1$, (4) $T = T^*$ *on* $L_1 \cap L_\infty$.

Proof. We give the if part of the proof only. Due to existence of one-dimensional projections, the condition (4) cannot be removed. By (1) and (3) of the hypothesis $T^*1 \cdot f \leq f$ and $\int T^*1 \cdot f = \int f$. Hence $T^*1 = 1$, which together with (1) would imply that T^* is positive. The rest of the proof depends on relating $(*)$

Received by the editors November 6, 1974 and, in revised form, February 12, 1975.

https://doi.org/10.4153/CMB-1975-086-3 Published online by Cambridge University Press

to (**). Let $\beta=\{B:T^*1_B=1_B\}$. β is a sub-σ-algebra of α as it can easily be verified by using additivity and positivity of T^*. Conditions (3) and (4) imply that β is σ-finite. To complete the proof let $\mathscr{E}$, $\mathscr{T}$ be the class of all conditional expectations and the class of linear operators on L_1 satisfying the hypothesis of the theorem, respectively.

Define $\varphi:\mathscr{T}\to\mathscr{E}$, by $\varphi(T)=C$ where $Cg=E\{g \mid \beta\}$, $g\in L_1$, and $\beta=\{B:T^*1_B=1_B\}$. By the only if part $\mathscr{E}\subset\mathscr{T}$. φ is one-one by (2) and (4). The proof is complete.

COROLLARY 2.1. *A linear operator T on L_1 is a conditional expectation relative to some σ-finite sub-σ-algebra if and only if* (1) $\|T\|\le 1$, (2) $T^2=T$, (3) $T^*Tg=Tg$ *for some* $Tg>0$.

Proof. Let $f=Tg$. By (2) $Tf=f$ implying that T, T^* are positive as before. We will show that $T=T^*$ on $L_1\cap L_\infty$. Define $d\nu=f\,d\mu$, then T on $L_1(\Omega, \alpha, \nu)$ satisfies the hypothesis of the corollary and further that T^* is contraction in L_1. Therefore (by the Riez-Convexity theorem) $\|T\|_p\le 1$ for $1\le p\le +\infty$, and in particular for $p=2$. Thus $T=T^*$ on $L_1(\Omega, a, \nu)\cap L_\infty(\Omega, \alpha, \nu)$, and consequently $Th=E_\nu\{h \mid \beta\}$, where E_ν refers to conditional expectation relative to ν, from which we conclude that $Th=E\{h \mid \beta\}$ as f is β-measurable.

This corollary was proved in [1] by a different method and under further condition that T is positive, which is redundant.

COROLLARY 2.2 (R. G. Douglas). *Suppose $\mu(\Omega)=1$. A linear operator T on L_1 is a conditional expectation if and only if* (1) $\|T\|\le 1$, (2) $T^2=T$, (3) $T1=1$.

Proof. $T^*1=1$ using (1) and (3). The proof follows from the previous corollary by putting $1=g$.

3. **Modulus and consequences.** Throughout this section (Ω, α, μ) is a σ-finite measure space. Modulus of a linear operator T on L_1 is denoted by $|T|$. Its definition and some properties are given in the following theorem.

THEOREM 3.1. *For a linear operator T on L_1, there exists a linear operator $|T|$ the modulus of T, satisfying:*

(1) $\|\,|T|\,\|\le\|T\|$

(2) $|Tg|\le|T|\,|g|$ all $g\in L_1$

(3) $|T|\,h=\sup_{|g|\le h}|Tg|,\ 0\le h\in L_1$

Proof. See [4].

LEMMA 3.1. *If T is contraction on L_1 with $Tf=f$, then $|T|\,|f|=|f|$.*

Proof. By (2) of theorem 3.1 $|T|\,|f|\ge|Tf|=|f|$. However by (1) of the same theorem $\int|T|\,|f|\le\int|f|$. Therefore $|T|\,|f|=|f|$.

https://doi.org/10.4153/CMB-1975-086-3 Published online by Cambridge University Press

LEMMA 3.2. *If T is a contraction with $Tf=f$ for some $0\neq f\in L_1$, then $T^*(f/|f|=f/|f|$.*

Proof. $\int|f|-\int|f|=\int((f/|f|)-T^*(f/|f|))\cdot f$, since $Tf=f$. But $|T^*(f/|f|)|\leq |(f/|f|)|$ for T^* is contraction. Hence $T^*(f/|f|)=f/|f|$.

THEOREM 3.2 (representation). *If T is a linear contraction on L_1 with $Tf=f$ for some $0\neq f\in L_1$ then*

$$|T|\, g=\frac{f}{|f|}\, T\left(\frac{f}{|f|}\, g\right), \quad \text{or equivalently} \quad Tg=\frac{f}{|f|}\, |T|\, \left(\frac{f}{|f|}\, g\right)$$

Proof. We shall show that $|T|g=(f/|f|)T((f/|f|)g)$. Equivalence of this with $Tg=(f/|f|)|T|((f/|f|)g)$ follows by observing that $(f/|f|)=(|f|/f)$

We may and do assume that $g\geq 0$. Now $|T|g\geq(f/|f|)T((f/|f|)\cdot g)$ using (3) of theorem 3.1. By Lemma 3.2 and property (1) of $|T|$; we have:

$$\int |T|\, g\geq \int \frac{f}{|f|}\, T\left(\frac{f}{|f|}\cdot g\right)=\int g\geq \int |T|\, g$$

Hence

$$|T|\, g=\frac{f}{|f|}\, T\left(\frac{f}{|f|}\cdot g\right).$$

4. **Projections on L_1.** In this section we imploy the representation theorem of the previous section, to represent projections defined on $L_1(\Omega, \alpha, \mu)$ where μ is σ-finite. The representation we prove is different than those given in ([2], [5]).

THEOREM 4.1. *Let T be a linear operator on L_1 satisfying* (1) $\|T\|\leq 1$ (2) $T^2=T$ (3) *$T=T^*$ on $L_1\cap L_\infty$, then there exists a unique $C\in\alpha$ such that:*

$$1_C T 1_C g=\frac{f}{|f|}\, E\left\{\frac{f}{|f|}\cdot g \mid \beta\right\},$$

where C=support of f, and β is a σ-finite sub-σ-algebra of C.

Proof. Let C be the largest support among the supports of all Tg, as g ranges over L_1. By [2] there is a $g\in L_1$ such that $f=Tg$ and C=support of f. Actually in [2] this is proved when μ is finite, but extension to the case when μ is σ-finite is easy. It is easy to check that $1_C T 1_C$ is contraction, idempotent and fixes $1_C\cdot f$. By theorem 3.2

$$1_C T 1_C g=\frac{f}{|f|}\, 1_C\, |T|\, 1_C\left(\frac{f}{|f|}\cdot g\right), \qquad \text{since } |T| \text{ fixes } |f|.$$

But $1_C\, |T|\, 1_C g=E\{g\mid B\}$ using theorem 2.1. Here $\beta=\{B: 1_C\, |T^*|\, 1_C 1_B=1_B\}$. The proof is complete. We must remark that condition (3) cannot be removed. See other representations in ([2], [5]).

5. **A version of Chacon-Ornstein theorem.** Let T be a positive contraction on $L_1(\Omega, \alpha, \mu)$ some σ-finite measure space (Ω, α, μ). The Chacon-Ornstein theorem [3] says:

$$\frac{\sum_0^n T^i h}{\sum_0^n T^i g}$$

converges almost everywhere to a finite limit as $n \to \infty$ on the set $\{\sum_0^{+\infty} T^i g > 0\}$ where $g \geq 0$. If T is a contraction and $Tf = f > 0$ then T is positive as is shown in the preceeding sections so that such a T will satisfy the Chacon-Ornstein theorem. However if T is a contraction and $Tf = f \neq 0$ then T is not necessarily positive, and the Chacon-Ornstein Theorem fails in this case. The version we have in mind is:

THEOREM 5.1. *If T is a linear operator on L_1 such that $Tf = f \neq 0$ then*

$$\frac{\sum_0^n (1_D T 1_D)^i h}{\sum_0^n (1_D T 1_D)^i g}$$

converges almost everywhere on $\{\sum_0^\infty (1_D T 1_D)^n g > 0\}$ where $g \geq 0$. Here $D = \{f > 0\}$ or $\{f < 0\}$.

Proof. $1_D T 1_D$ is positive by theorem 3.2.

6. REMARKS. Chacon's identification theorem ([5], pp. 104) could be utilized in characterizing conditional expectation as a linear operator (see [1]) for example. However our approach in section (2) would seem to be more direct and in a sense a head on.

Also one may give an alternative proof to theorem 5.1, and as follows: Assume $D = \{f > 0\}$. The case where $D = \{f < 0\}$ is handled by considering $-f$ instead of f. Now if $|g| \leq |f|$ on D then $|1_D T 1_D g| \leq |1_D T 1_D f|$. Using theorem 3.1. Setting $P_n = 1_D f$, $n = 1, 2, \ldots$. The proof follows from Lemma 4 of ([5], pp. 102).

REFERENCES

1. A. N. Al-Hussaini, *On characterization of conditional expectation*, Canad. Math. Bull. vol. **16** (2) 1973.
2. T. Ando, *Contractive projections in L_p space*, Pacific Journal of Math. vol. **17** (3) 1966.
3. R. V. Chacon and D. Ornstein, *A general ergodic theorem*, Illinois J. Math. (1960).
4. R. V. Chacon and U. Krengel, *Linear modulus of linear operators*, Proc. Amer. Math. Soc. (1964).
5. R. V. Chacon, *Convergence of operator averages*, Proceeding Tulane Symp. Ergodic theory (1962) pp. 89–120.
6. R. G. Douglas, *Contractive projections on an L_1-space*, Pacific J. Math. (1965).

UNIVERSITY OF ALBERTA AND UNIVERSITY OF ILLINOIS