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References

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93.02 Bounds for the roots of polynomial equations

Nick Lord [1] has presented ‘golden bounds’ for the roots of the quadratic equation

\[ ax^2 + bx + c = 0, \]

namely,

\[ | \text{root} | < \frac{1 + \sqrt{5} \cdot \max(|a|, |b|, |c|)}{2 |a|}, \]

and the more useful result for practical checking by students

\[ | \text{root} | < \frac{2 \cdot \max(|a|, |b|, |c|)}{|a|}. \]

A similar result, which can be generalised to polynomials of degree \( n \), can be obtained as follows.

Write the quadratic as

\[ x^2 + \alpha x + \beta = 0, \]

where \( \alpha = \frac{|b|}{|a|} \) and \( \beta = \frac{|c|}{|a|} \). Then the quadratic can be rewritten as

\[ x = -\alpha - \frac{\beta}{x}. \]

Applying the triangle inequality we obtain

\[ |x| < |\alpha| + \left| \frac{\beta}{x} \right| = |\alpha| + \frac{|\beta|}{|x|}. \]

Clearly,

either \( |x| < 1 \) or \( |x| < |\alpha| + |\beta| \).
giving the simple bound

$$|x| \leq \max (1, |\alpha| + |\beta|).$$

Likewise, for the polynomial equation

$$x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0,$$

we can establish in the same way the bound

$$|x| \leq \max (1, |a_1| + |a_2| + \ldots + |a_n|).$$

However, by resorting to matrix theory, we can obtain an even more practical and simple bound. We note that the roots of the quadratic equation

$$x^2 + \alpha x + \beta = 0$$

are exactly the eigenvalues of the companion matrix

$$\begin{bmatrix}
-\alpha & 1 \\
-\beta & 0
\end{bmatrix}.$$

A theorem of Gerschgorin, which can be found in [2] and in most texts on numerical analysis, informs us that the bound for the largest eigenvalue is

$$| \text{root} | \leq \max (1 + |\alpha|, |\beta|) = \max (1 + \frac{|b|}{|a|}, \frac{|c|}{|a|}).$$

The same reasoning for the polynomial equation of degree \(n\) would give the result

$$| \text{root} | \leq \max (1 + |a_1|, |a_2|, \ldots, |a_n|).$$

Applying this last result to the polynomial equation \(x^2 - 3x - 2 = 0\), for example, produces \(|\text{root}| \leq \max (4,2) = 4\), as compared with a bound of \(3(1 + \sqrt{5})/2 < 4.9\) using Lord’s golden bound.

References


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93.03 Westlund’s criterion in full detail

Westlund’s criterion for the irreducibility of a polynomial can be stated as follows:

**Theorem 1:** ([1], reprinted in [2]).

Let

$$f_1(x) \equiv (x - a_1)(x - a_2)\ldots (x - a_n) - 1$$

and

$$f_2(x) \equiv (x - a_1)(x - a_2)\ldots (x - a_n) + 1$$