SIMPLE MODULES OVER THE
COORDINATE RING OF QUANTUM AFFINE SPACE

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The simple modules of \( \mathcal{O}_q(C^n) \), the coordinate ring of quantum affine space, are classified in the case when \( q \) is a root of unity.

The coordinate ring of quantum affine space, \( \mathcal{O}_q(C^n) \), is the algebra generated by \( x_1, \ldots, x_n \) satisfying the relations \( x_j x_i = q x_i x_j, \) \( i < j, \) \( 0 \neq q \in \mathbb{C} \). Thus \( \mathcal{O}_q(C^n) \) is the iterated skew polynomial ring \( \mathbb{C}[x_1][x_2; \tau_2] \cdots [x_n; \tau_n] \), where automorphisms \( \tau_k \) are defined by \( \tau_k(x_i) = q x_i, i < k \). Therefore it is a Noetherian domain of Gelfand-Kirillov dimension \( n \), and has a \( \mathbb{C} \)-basis given by the monomials \( X^I \) where \( I = (i_1, \ldots, i_n) \) is a multi-index with each \( i_j \geq 0 \).

The quantum matrices \( \mathcal{O}_q M_n(\mathbb{C}) \) and the quantised universal enveloping algebra \( U_q(sl(n, \mathbb{C})) \) act on \( \mathcal{O}_q(C^n) \). The reader is referred to the articles [1] and [5] for further background and actions on \( \mathcal{O}_q(C^n) \).

The prime ideals and the primitive ideals of \( \mathcal{O}_q(C^n) \) are classified in [4] and [5, Section 3], in the case when \( q \) is not a root of unity. In this note, we prove that there is a surjective map \( \Psi \) from \( \mathbb{C}^n \) onto the set of all the simple modules of \( \mathcal{O}_q(C^n) \), in the case when \( q \) is a primitive \( m \)-th root of unity, such that

\[
\dim_{\mathbb{C}} \Psi(\alpha) = m^{[p/2]},
\]

where \( p \) is the number of nonzero \( \alpha_i \) in \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and \( [x] \) denotes the greatest integer \( \leq x \). Henceforce, assume throughout that \( q \) is a primitive \( m \)-th root of unity unless stated otherwise.

**Proposition 1.** Let \( R \) be an algebra over a field \( k \), \( Z \) a finitely generated subalgebra contained in the center of \( R \) and let \( R \) be finitely generated as a \( Z \)-module. For a simple right \( R \)-module \( M \), the following hold.

(i) \( R \) is Noetherian.

(ii) \( \dim_k (M) \) is finite.

(iii) \( \text{ann}_R(M) \) is a maximal ideal of \( R \).

(iv) \( \text{ann}_R(M) \cap Z \) is a maximal ideal of \( Z \).

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PROOF: (i) Since $Z$ is Noetherian by Hilbert's basis theorem and $R$ is finitely generated as a $Z$-module, $R$ is also Noetherian.

(ii) [3, 9.5.5 (ii)].

(iii) and (iv) Look at the following monomorphisms:

$$Z/(\text{ann}_R(M) \cap Z) \xrightarrow{\alpha} R/\text{ann}_R(M), \quad R/\text{ann}_R(M) \xrightarrow{\beta} \text{End}_k(M)$$

where $\alpha$ is induced from the inclusion map from $Z$ into $R$ and $\beta$ is induced from right module structure map on $M$. Since $\text{End}_k(M)$ is finite dimensional by (ii), $R/\text{ann}_R(M)$ is Artinian and prime. Hence it is simple. Moreover, $Z/(\text{ann}_R(M) \cap Z)$ is integral domain and Artinian. Therefore, $\text{ann}_R(M) \cap Z$ and $\text{ann}_R(M)$ are maximal ideals of $Z$ and $R$, respectively.

COROLLARY 2. Every simple right $O_q(C^n)$-module is finite dimensional.

PROOF: The subalgebra $Z$ of $O_q(C^n)$ generated by $x_i^m$, $i = 1, \ldots, n$, is contained in the center of $O_q(C^n)$ and $O_q(C^n)$ is finitely generated as a $Z$-module. This completes the proof by Proposition 1 (ii).

For convenience, set $i' = n + 1 - i$ for $1 \leq i < (n + 1)/2$. If $n = 2k + 1$ or $n = 2k$ then, for nonzero $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, let $M(\alpha) = M(\alpha_1, \ldots, \alpha_n)$ be the $\mathbb{C}$-vector space with basis $e(\alpha_1, \ldots, \alpha_k)$, $0 \leq \alpha_i \leq m - 1$. Then $M(\alpha)$ has a right $O_q(C^n)$-module structure defined as follows:

Case 1. $n = 2k + 1$:

$$e(\alpha_1, \ldots, \alpha_k)x_i = \alpha_i q^{-(\alpha_1 + \cdots + \alpha_{i-1})}e(\alpha_1, \ldots, \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, \ldots, \alpha_k),$$

$$1 \leq i \leq k$$

$$e(\alpha_1, \ldots, \alpha_k)x_i' = \alpha_i^{-1} q^{-(\alpha_1 + \cdots + \alpha_{i+1})}e(\alpha_1, \ldots, \alpha_{i-1}, \alpha_i + (-1), \alpha_{i+1}, \ldots, \alpha_k),$$

$$1 \leq i \leq k$$

$$e(\alpha_1, \ldots, \alpha_k)x_{k+1} = \alpha_{k+1} q^{-(\alpha_1 + \cdots + \alpha_k)}e(\alpha_1, \ldots, \alpha_k);$$

Case 2. $n = 2k$:

$$e(\alpha_1, \ldots, \alpha_k)x_i = \alpha_i q^{-(\alpha_1 + \cdots + \alpha_{i-1})}e(\alpha_1, \ldots, \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, \ldots, \alpha_k),$$

$$1 \leq i \leq k$$

$$e(\alpha_1, \ldots, \alpha_k)x_i' = \alpha_i^{-1} q^{-(\alpha_1 + \cdots + \alpha_{i+1})}e(\alpha_1, \ldots, \alpha_{i-1}, \alpha_i + (-1), \alpha_{i+1}, \ldots, \alpha_k),$$

$$1 \leq i \leq k - 1$$

$$e(\alpha_1, \ldots, \alpha_k)x_{k+1} = \alpha_{k+1} q^{-(\alpha_1 + \cdots + \alpha_k)}e(\alpha_1, \ldots, \alpha_k);$$

where $+$ is addition in the additive group $\mathbb{Z}_m$. To confirm the well-definedness of these rules, it suffices to check that

$$e(\alpha_1, \ldots, \alpha_k)x_jx_i = qe(\alpha_1, \ldots, \alpha_k)x_ix_j, \quad 1 \leq i < j \leq n.$$
These are all routinely verified.

**Proposition 3.** The right $O_q(C^n)$-module $M(\alpha)$ is simple.

**Proof:** If $n = 2k + 1$ and $e(a_1, \ldots, a_k) \neq e(a'_1, \ldots, a'_k)$, choose an index $i$ such that $a_i = a'_i$, $a_{i-1} = a'_{i-1}$, $a_i \neq a'_i$. The vectors $e(a_1, \ldots, a_k)$ and $e(a'_1, \ldots, a'_k)$ are eigenvectors of $x_i x_{i'}$ associated with the distinct eigenvalues $\alpha_i q^{-2(a_1 + \cdots + a_{i-1}) - a_i}$ and $\alpha'_i q^{-2(a'_1 + \cdots + a'_{i-1}) - a'_i}$, respectively.

If $n = 2k$, let $e(a_1, \ldots, a_k)$, $e(a'_1, \ldots, a'_k)$ and $i$ be as in the case $n = 2k + 1$. If $i < k$ then $e(a_1, \ldots, a_k)$ and $e(a'_1, \ldots, a'_k)$ are eigenvectors of $x_i x_{i'}$ associated with the distinct eigenvalues $\alpha_i q^{-2(a_1 + \cdots + a_{i-1}) - a_i}$ and $\alpha'_i q^{-2(a'_1 + \cdots + a'_{i-1}) - a'_i}$, respectively, and if $i = k$ then $e(a_1, \ldots, a_k)$ and $e(a'_1, \ldots, a'_k)$ are eigenvectors of $x_{k+1}$ associated with the distinct eigenvalues $\alpha_{k+1} q^{-2(a_1 + \cdots + a_k)}$ and $\alpha'_{k+1} q^{-2(a'_1 + \cdots + a'_k)}$, respectively.

Hence every nonzero submodule of $M(\alpha)$ contains a vector $e(a_1, \ldots, a_k)$, thus $M(\alpha)$ is simple by the action of $x_i$, $1 \leq i \leq k$.

**Proposition 4.** Let a simple right $O_q(C^n)$-module $N$ be $x_i$-torsion free for each $i = 0, \ldots, n$. Then $N$ is isomorphic to $M(\alpha)$ for some $\alpha = (\alpha_i) \in (C^*)^n$.

**Proof:** Let $n = 2k + 1$. Since each $x_i x_{i'}$, $i = 1, \ldots, k$ and $x_{k+1}$ commutes and $N$ is finite dimensional, there is a common eigenvector $v$ of $x_i x_{i'}$, $i = 1, \ldots, k$ and $x_{k+1}$. Put $v x_i^m = \nu_i v$, $v x_{i'}^m = \alpha_i v$, $v x_{k+1} = \alpha_{k+1} v$, $i = 1, \ldots, k$. For each $i = 1, \ldots, k$, let $\alpha_i$ be an $m$-th root of $\nu_i$. Notice that the $\alpha_i$ are all nonzero and

$$v x_k^{a_k} \cdots x_1^{a_1} x_i = \begin{cases} \alpha_i q^{-2(a_1 + \cdots + a_i + 1)} v x_k^{a_k} \cdots x_i^{a_i-1} \cdots x_1, & a_i > 0 \\ \nu_i^{-1} \alpha_i q^{-2(a_1 + \cdots + a_i + 1)} v x_k^{a_k} \cdots x_i^{m-1} \cdots x_1, & a_i = 0. \end{cases}$$

Define a linear transformation

$$\psi : M(\alpha) \to N, \quad \psi(e(a_1, \ldots, a_k)) = \alpha_1^{-a_1} \cdots \alpha_k^{-a_k} v x_k^{a_k} \cdots x_1^{a_1}.$$ 

It is routinely verified that

$$\psi(e(a_1, \ldots, a_k) x_i) = \psi(e(a_1, \ldots, a_k)) x_i, \quad i = 1, \ldots, n.$$ 

Hence $\psi$ is an $O_q(C^n)$-homomorphism, and thus it is an isomorphism because $M(\alpha)$ and $N$ are both simple. The proof of the case when $n = 2k$ is similar.

**Theorem 5.** There is a surjective map $\Psi$ from $C^n$ onto the set of all the simple right $O_q(C^n)$-modules such that $\dim \Psi(\alpha) = m[p/2]$, where $p$ is the number of nonzero $\alpha_i$ in $\alpha = (\alpha_i)$ and $[x]$ is the greatest integer $\leq x$.

**Proof:** Let $M$ be a simple $O_q(C^n)$-module and let $Z$ be the subalgebra generated by $x_i^n$, $i = 1, \ldots, n$. Then $\text{ann}(M) \cap Z$ is a maximal ideal of $Z$ by Proposition 1 (iv).
Hence $x_i^n - \lambda_i \in \text{ann}(M)$ for some $\lambda_i \in \mathbb{C}$, $i = 1, \ldots, n$. If $\lambda_i = 0$ then $x_i \in \text{ann}(M)$ since $x_i$ is a normal element of $O_q(C^n)$ and $\text{ann}(M)$ is prime. For convenience, assume that $\lambda_1, \ldots, \lambda_p$ are all nonzero and $\lambda_{p+1} = \cdots = \lambda_n = 0$. Thus $M$ is a simple $O_q(C^p)$-module and is $x_i$-torsion free for each $i = 1, \ldots, p$, since $\text{ann}(M)$ contains $x_{p+1}, \ldots, x_n$ and $O_q(C^n)/(x_{p+1}, \ldots, x_n)$ is isomorphic to $O_q(C^p)$. Hence the result follows from Proposition 4.

REMARK 1. The map $\Psi$ of Theorem 5 is not injective.

**Proof:** Let $n = 2k + 1$ and $\alpha, \beta$ be two elements in $(C^*)^n$ such that $\alpha_i = \beta_i$, $i \neq k + 1, k + 2$ and $\alpha_{k+1} = q^{-1}\beta_{k+1}$, $\alpha_{k+2} = q^{-1}\beta_{k+2}$. Then, it is easy to see that the map $\psi : M(\alpha) \rightarrow M(\beta)$ given by $\psi(e(a_1, \ldots, a_k)) = e(a_1, \ldots, a_{k-1}, a_k + 1)$ is an isomorphism.

REMARK 2. All primitive ideals of $O_q(C^n)$ are annihilators of $\Psi(\alpha), \alpha \in C^n$.

REMARK 3. If $q$ is not a root of unity then every finite dimensional simple $O_q(C^n)$-module is one-dimensional by [2, 1.3]. The classification of the one-dimensional simple modules is a fairly easy exercise.

REMARK 4. Smith classified all simple modules of $O_q(C^2)$ in [6, pp. 123].

**References**


