Can. J. Math., Vol. XXII, No. 4, 1970, pp. 746–752

LINEAR TRANSFORMATIONS ON MATRICES: THE INVARIANCE OF THE THIRD ELEMENTARY SYMMETRIC FUNCTION

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1. Introduction. Let T be a linear transformation on M_n the set of all $n \times n$ matrices over the field of complex numbers, $\mathscr{C}$. Let $A \in M_n$ have eigenvalues $\lambda_1, \ldots, \lambda_n$ and let $E_r(A)$ denote the rth elementary symmetric function of the eigenvalues of A:

$$E_r(A) = \sum_{1 \leqq i_1 < \ldots < i_r \leqq n} \prod_{j=1}^{r} \lambda_{i_j} = E_r(\lambda_1, \ldots, \lambda_n).$$

Equivalently, $E_r(A)$ is the sum of all the principal $r \times r$ subdeterminants of A. T is said to *preserve* E_r if $E_r[T(A)] = E_r(A)$ for all $A \in M_n$. Marcus and Purves [**3**, Theorem 3.1] showed that for $r \geqq 4$, if T preserves E_r then T is essentially a similarity transformation; that is, either $T: A \to UAV$ for all $A \in M_n$ or $T: A \to UA^tV$ for all $A \in M_n$, where $UV = e^{i\theta}I_n$, $r\theta \equiv 0 \pmod{2\pi}$. They also showed that not all linear transformations which preserve E_2 are essentially similarity transformations. However, their results did not include a definite theorem on E_3 preservers.

In this paper we shall prove the following.

THEOREM 1.1. *If T preserves E_3, then there exist non-singular matrices U and V in M_n such that either*

(i) *$T: A \to UAV$ for all $A \in M_n$,*

or

(ii) *$T: A \to UA^tV$ for all $A \in M_n$,*

where

(iii) *$UV = e^{i\theta}I_n$ and $3\theta \equiv 0 \pmod{2\pi}$.*

2. Preliminary lemmas. The main burden of the proof of Theorem 1.1 lies in showing that if T preserves E_3, then T maps rank one matrices into rank one matrices; for then a theorem of Marcus and Moyls [**2**, Theorem 1] shows that T has the structure indicated in either (i) or (ii). Obviously, if the rank $\rho(A) = 1$, then $E_r(xA + B)$, considered as a polynomial in x, has degree $\leqq 1$. Marcus and Purves [**3**, Lemma 3.1] showed that if T preserves E_r for some $r \geqq 2$, then T is non-singular. It follows that for such T,

Received May 26, 1969 and in revised form, September 11, 1969.

(2.1) $\deg E_r(xA + B) \leqq 1$ for all $B \in M_n$ if and only if

$$\deg E_r[xT(A) + B] \leqq 1 \text{ for all } B \in M_n.$$

They also showed that, for $r \geqq 4$,

(2.2) $\deg E_r(xA + B) \leqq 1$ for all $B \in M_n$ if and only if $\rho(A) = 1$.

With A replaced by $T(A)$ this leads to the desired result. For $r = 3$, their proof of (2.2) does not appear to work. However, (2.2) does turn out to be true if A has a non-zero eigenvalue (Lemma 2.2). If such is the case, $T(A)$ also has a non-zero eigenvalue (Lemma 2.3). With these lemmas, along with a continuity argument (Theorem 2.6), we show that T preserves rank one matrices.

We need two results of Marcus and Purves [**3**, Lemmas 3.2 and 3.3] which we state in the following.

LEMMA 2.1. *If $A \in M_n$ and $A \neq 0$, then:*

(i) $\deg \det(xA + B) \leqq 1$ *for all $B \in M_n$ if and only if $\rho(A) = 1$;*

(ii) *if $3 \leqq r < n$, then $\deg E_r(xA + B) \leqq 1$ for all $B \in M_n$ implies that A has at most one non-zero eigenvalue.*

LEMMA 2.2. *If $A \in M_n$ and A has a non-zero eigenvalue, then*

$$\deg E_3(xA + B) \leqq 1 \text{ for all } B \in M_n$$

if and only if $\rho(A) = 1$.

Proof. If $\rho(A) = 1$, then clearly $\deg E_3(xA + B) \leqq 1$ for all $B \in M_n$. Suppose that $\deg E_3(xA + B) \leqq 1$ for all $B \in M_n$. By Lemma 2.1 (ii), A has at most one non-zero eigenvalue, and hence exactly one, λ_1.

Let S: $B \rightarrow PBP^{-1}$ for all $B \in M_n$, where PAP^{-1} is the Jordan normal form of A:

$$PAP^{-1} = \begin{bmatrix} \lambda_1 & 0 & & & & \\ & 0 & \epsilon_2 & & \text{\Large 0} & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & \text{\Large 0} & & & \ddots & \epsilon_{n-1} \\ & & & & & 0 \end{bmatrix},$$

where $\epsilon_i = 1$ or $\epsilon_i = 0$ for all i, $i = 1, \ldots, n - 1$.

Suppose that $\epsilon_i = 1$ for some i, $i = 2, \ldots, n - 1$. Then, $E_3(xS(A) + B) = \lambda_1 \epsilon_i x^2$ for $B = E_{i+1,i}$, where $E_{i,j}$ denotes the matrix with a "1" in the (i, j) position and zeros in all other positions. Since E_3 is invariant under similarity transformations, $\deg E_3[xS(A) + B] = \deg E_3[xA + S^{-1}(B)] \leqq 1$. Hence $\epsilon_i = 0$ for all i, $i = 2, \ldots, n - 1$, and thus $\rho(A) = 1$.

LEMMA 2.3. *If T preserves E_3, $A \in M_n$, $n \geqq 4$, $\rho(A) = 1$, and A has a non-zero eigenvalue, then $T(A)$ has a non-zero eigenvalue.*

Proof. Suppose that all eigenvalues of $T(A)$ are zero. Since E_3 is invariant under similarity transformations, and since a matrix is similar to its Jordan

https://doi.org/10.4153/CJM-1970-084-x Published online by Cambridge University Press

normal form, we may assume that A and $T(A)$ are in Jordan normal form: $A = \lambda_1 E_{11}$, and

$$T(A) = \begin{bmatrix} 0 & \epsilon_1 & & & \\ & 0 & \epsilon_2 & & \mathbf{0} \\ & & \ddots & \ddots & \\ & \mathbf{0} & & \ddots & \epsilon_{n-1} \\ & & & & 0 \end{bmatrix},$$

where $\epsilon_i = 0$ or $\epsilon_i = 1$, $i = 1, \ldots, n-1$.

If $\epsilon_i \neq 0$, for some i, let $B = E_{i+2,i}$. Then, $E_3[xT(A) + B] = \epsilon_i \epsilon_{i+1} x^2$. However, by (2.1), $\deg E_3[xT(A) + B] \leqq 1$. Hence $\epsilon_{i+1} = 0$. Similarly, $\epsilon_{i-1} = 0$ if $\epsilon_i \neq 0$, $i > 1$. Hence $\rho[T(A)] \leqq n/2$.

We now close up the ones on the superdiagonal so that they alternate with zeros. Let P_{ij} represent the permutation matrix which by multiplication on the left (right) interchanges the ith and jth rows (columns). Thus $P_{ij}^{-1} = P_{ij}$. Suppose that in $T(A)$, $\epsilon_{i-2} = \epsilon_{i-1} = 0$ and $\epsilon_i \neq 0$. (Note that ϵ_{i+1} must then be zero.) Then

$$P_{i,i+1} P_{i-1,i} T(A) P_{i-1,i} P_{i,i+1} = \begin{bmatrix} 0 & \epsilon_1' & & & \\ & 0 & \epsilon_2' & & \mathbf{0} \\ & & \ddots & \ddots & \\ & \mathbf{0} & & \ddots & \epsilon_{n-1}' \\ & & & & 0 \end{bmatrix},$$

where $\epsilon_j' = \epsilon_j$ if $j \neq i, i-1$, and $\epsilon_{i-1}' \neq 0$ and $\epsilon_i' = 0$. Continuing in this way we obtain a permutation matrix P such that

$$PT(A)P^{-1} = \begin{bmatrix} 0 & \mu_1 & & & \\ & 0 & \mu_2 & & \mathbf{0} \\ & & \ddots & \ddots & \\ & \mathbf{0} & & \ddots & \mu_{n-1} \\ & & & & 0 \end{bmatrix}$$

has the property that for some α, $\mu_{2i-1} = 1$ and $\mu_{2i} = 0$ for $1 \leqq i \leqq \alpha$, and $\mu_j = 0$ for $j \geqq 2\alpha$. (Note that $PT(A)P^{-1}$ cannot be 0 since T is non-singular.)

Remark. The proof does not show that μ_1 can be made 1, although the argument for this is similar to (but not part of) the argument given above.

Let Q be the permutation matrix $\prod_{i=1}^{\beta} P_{2i,\gamma-2i}$, where $\beta = [(n+1)/4]$ (greatest integer) and $\gamma = n+2$, if n is odd; and $\beta = [n/4]$ and $\gamma = n+1$, if n is even.

https://doi.org/10.4153/CJM-1970-084-x Published online by Cambridge University Press

We observe that in $PT(A)P^{-1}$ the only columns with non-zero entries are even-numbered columns and hence can conclude that $QPT(A)P^{-1}Q^{-1}$ has the form $\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}$, where X is of order $(n/2) \times (n/2)$ if n is even, and of order $[(n+1)/2] \times [(n-1)/2]$ if n is odd, and has non-zero entries (in fact, ones) on only one diagonal.

Define S by $S(B) = QPT(B)P^{-1}Q^{-1}$. Then, S is non-singular, S preserves E_3, and all eigenvalues of $S(A)$ are zero.

Let $\mathscr{M}$ be the subspace of M_n generated by $\{E_{ij}: 1 \leqq i \leqq \alpha, \beta \leqq j \leqq n\}$, where $\alpha = (n+1)/2$ and $\beta = (n+3)/2$, if n is odd; and $\alpha = n/2$ and $\beta = (n+2)/2$, if n is even. That is, if $G \in \mathscr{M}$, then

$$G = \begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix},$$

where Y is a matrix of order $(n/2) \times (n/2)$ if n is even, and of order

$$[(n+1)/2] \times [(n-1)/2]$$

if n is odd.

Now, if $G \in \mathscr{M}$, then any principal 3×3 submatrix of G is either of the form

$$\begin{bmatrix} 0 & g_{ij} & g_{ik} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

or of the form

$$\begin{bmatrix} 0 & 0 & g_{ij} \\ 0 & 0 & g_{kj} \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $E_3(C)$ is the sum of the principal 3×3 subdeterminants of C, it follows that if $G \in \mathscr{M}$, then $\deg E_3(xG + B) \leqq 1$ for all $B \in M_n$. Now, $S(A) \in \mathscr{M}$. Hence, $\deg E_3(x[\lambda S(A) + G] + B) \leqq 1$ for all $\lambda \in \mathscr{C}$, for all $B \in M_n$, and for any $G \in \mathscr{M}$. Hence by (2.1), $\deg E_3(x[\lambda A + S^{-1}(G)] + B) \leqq 1$ for all $B \in M_n$, for all $\lambda \in \mathscr{C}$, and for any $G \in \mathscr{M}$. Now, for

$$\lambda \neq [-\operatorname{tr} S^{-1}(G)]/\lambda_1,$$

$\lambda A + S^{-1}(G)$ has a non-zero eigenvalue. Hence, by Lemma 2.4, $\rho[\lambda A + S^{-1}(G)] = 1$ for $\lambda \neq [-\operatorname{tr} S^{-1}(G)]/\lambda_1$.

Let G be any member of $\mathscr{M}$, and $S^{-1}(G) = (s_{ij})$. Then, $s_{ij} = 0$ if $i > 1$ and $j > 1$; for if not, say $s_{ij} \neq 0$, then $\det(\lambda A + S^{-1}(G))[1, i; 1, j] \neq 0$ for all but one value of λ, which contradicts the fact that $\rho[\lambda A + S^{-1}(G)] = 1$. If $s_{1j} \neq 0$ for some $j > 1$, and $s_{i1} \neq 0$ for some $i > 1$, then $\det(S^{-1}(G))[1, i; 1, j] \neq 0$. In a similar way we can argue that, if for some $G \in \mathscr{M}$, $S^{-1}(G)$ has a non-zero entry in the first row (column) which is not in the first column (row), then for every $H \in \mathscr{M}$, $S^{-1}(H)$ may have non-zero entries only in the first row (column). It now follows that $\dim S^{-1}(\mathscr{M}) \leqq n$, however, $\dim \mathscr{M} = n^2/4$ if n is even and $\dim \mathscr{M} = (n^2 - 1)/4$ if n is odd. In either case we conclude that $n \leqq 4$.

Suppose then that $n = 4$. Define $\mathscr{R}$ to be the subspace of M_4 generated by $\{E_{i1}: i = 1, \ldots, 4\}$, $\mathscr{V}$ to be the subspace of M_4 generated by

$$\{E_{1j}: j = 1, \ldots, 4\},$$

$\mathscr{M}$ to be the subspace of M_4 generated by $\{E_{13}, E_{14}, E_{23}, E_{24}\}$, and $\mathscr{S}$ to be the subspace of M_4 generated by $\{E_{ij}: i < j\}$.

We know that $S^{-1}(\mathscr{M}) \subseteq \mathscr{V}$ or $S^{-1}(\mathscr{M}) \subseteq \mathscr{R}$. We may assume that $S^{-1}(\mathscr{M}) \subseteq \mathscr{V}$ since the argument for $S^{-1}(\mathscr{M}) \subseteq \mathscr{R}$ is parallel. However, $\dim \mathscr{V} = \dim \mathscr{M}$; hence $S(\mathscr{V}) = \mathscr{M}$. We shall show that there exists a linear transformation S^* such that S^* preserves E_3 (hence is non-singular), and

$$S^*(\mathscr{R} + \mathscr{V}) \subseteq \mathscr{S}.$$

This will yield a contradiction since $\dim \mathscr{S} = 6$ and $\dim(\mathscr{R} + \mathscr{V}) = 7$.

Since $S(\mathscr{V}) = \mathscr{M}$, it follows that there exist coefficients $\alpha_2, \alpha_3, \alpha_4$ such that $\rho[S(A')] = 2$, where

$$A' = A + \sum_{i=2}^{4} \alpha_i E_{1i}.$$

Let $Q_1 = I_4 - (\alpha_2/\lambda_1)E_{12} - (\alpha_3/\lambda_1)E_{13} - (\alpha_4/\lambda_1)E_{14}$. Now, the Jordan normal form of A' is $Q_1^{-1}A'Q_1 = \lambda_1 E_{11} = A$. (Note that $Q_1^{-1}\mathscr{V} Q_1 = \mathscr{V}$.) Also, there exists non-singular matrices R and Z such that

$$\begin{bmatrix} R & 0 \\ 0 & Z \end{bmatrix} S(A') \begin{bmatrix} R & 0 \\ 0 & Z \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Define S' by $S'(B) = P_1[S(Q_1 B Q_1^{-1})]P_1^{-1}$, where

$$P_1 = \begin{bmatrix} R & 0 \\ 0 & Z \end{bmatrix}.$$

Thus S' preserves E_3 (hence is non-singular) and

$$S'(A) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Also, $S'(\mathscr{V}) = \mathscr{M}$ since $Q_1 \mathscr{V} Q_1^{-1} = \mathscr{V}$, $P_1 \mathscr{M} P_1^{-1} = \mathscr{M}$, and $S(\mathscr{V}) = \mathscr{M}$.

Let $G \in \mathscr{R}$, $G \neq \gamma E_{11}$; then $\deg E_3[xG + B] \leqq 1$ for all $B \in M_4$. By (2.1) it follows that $\deg E_3[xS'(G) + B] \leqq 1$ for all $B \in M_4$. We then have $\rho[S'(G)] \leqq n/2 = 2$ as in the first paragraph of this proof. In particular, $\rho[S'(xA + G)] \leqq 2$ for all $x \in \mathscr{C}$. Hence every 3×3 minor of $S'(xA + G)$ is zero. Suppose that

$$S'(G) = \begin{bmatrix} K & L \\ J & M \end{bmatrix},$$

where K, L, J, and M are 2×2 matrices. Then

$$S'(xA + G) = \begin{bmatrix} K & L + xI_2 \\ J & M \end{bmatrix}.$$

Since each minor of the form $\det S'(xA + G)[1, 2, i; j, 3, 4]$ ($j = 1, 2; i = 3, 4$) is zero for all $x \in \mathscr{C}$, it follows that $J = 0$. Now, since for some $x \in \mathscr{C}$, $\rho[S'(xA + G)] = 2$, if $S'(G)$ had a non-zero eigenvalue, it would follow that $\deg E_3[z(xA + G) + B] > 1$ by Lemma 2.2 and (2.1), which would contradict the fact that $\rho(xA + G) = 1$. Thus $S'(G)$ and hence K and M have non-zero eigenvalues. Let D_1 and D_2 be non-singular 2×2 matrices such that

$$D_1^{-1}KD_1 = \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad D_2^{-1}MD_2 = \begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix}$$

and define S^* by

$$S^*(B) = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}^{-1} S'(B) \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}.$$

Then

$$S^*(xA + G) = \begin{bmatrix} 0 & c & \multicolumn{2}{c}{N} \\ 0 & 0 & & \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$S^*(\mathscr{V}) = \mathscr{M}$, and S^* preserves E_3.

Now, either $c \neq 0$ or $d \neq 0$; for if $c = d = 0$, then $S^{*-1}(\mathscr{M})$ strictly contains $\mathscr{V}$, a contradiction. Also, if for any $H \in \mathscr{R}$, $S'(H)$ has a non-zero (u, v) entry, for some $u = 3, 4$, $v = 3, 4$, and if for some $H' \in \mathscr{R}$, $S'(H')$ has a non-zero (u, v) entry, for some $u = 1, 2$, $v = 1, 2$, then we may take $G \in \mathscr{R}$ such that both $c \neq 0$ and $d \neq 0$.

Let $B \in \mathscr{R}$; then $S^*(B)$ has the form $\left[\begin{smallmatrix} K' & L' \\ 0 & M' \end{smallmatrix}\right]$, and K' and M' have no non-zero eigenvalues. Consider

$$K^*(x) = K' + \begin{bmatrix} 0 & xc \\ 0 & 0 \end{bmatrix}.$$

Now, $K^*(x)$ has no non-zero eigenvalues, and $\rho[K^*(x)] \leqq 1$. Since x can be taken to be an indeterminate and since $\rho[K^*(x)] \leqq 1$, it follows that $k_{21}' = 0$. Hence K' is an upper triangular matrix with zero diagonal; that is $k_{21}' = k_{11}' = k_{22}' = 0$. Similarly $m_{21}' = m_{11}' = m_{22}' = 0$. Hence $S^*(\mathscr{R} + \mathscr{V}) \subseteq \mathscr{S}$. We have arrived at our contradiction.

THEOREM 2.1. *If T preserves E_3 and $\rho(A) = 1$, then $\rho[T(A)] = 1$.*

Proof. If $n = 3$, the lemma is an immediate consequence of Lemma 2.1(i) and (2.1). Thus assume that $n \geqq 4$.

If $\rho(A) = 1$, assume that A is in Jordan normal form: $A = \lambda E_{11} + \epsilon E_{12}$, where $\lambda = 0$ and $\epsilon = 1$, or $\lambda \neq 0$ and $\epsilon = 0$. If $\lambda \neq 0$, let $A(t) = A$ for all $t \in \mathscr{C}$. On the other hand, if $\lambda = 0$, let $A(t) = tE_{11} + E_{12}$.

Now, for all $t \in \mathscr{C}$, $t \neq 0$, $T[A(t)]$ has a non-zero eigenvalue by Lemma 2.3. Since $\rho[A(t)] = 1$ for all $t \in \mathscr{C}$, $\deg E_3[xA(t) + B] \leqq 1$ for all $B \in M_n$, and by (2.1), $\deg E_3(xT[A(t)] + B) \leqq 1$ for all $B \in M_n$. Thus by Lemma 2.2, $\rho(T[A(t)]) = 1$ for all $t \neq 0$. By a continuity argument and the non-singularity of T, $\rho[T(A)] = 1$.

3. On the proof of Theorem 1.1. For the proof of Theorem 1.1, one must first show that T satisfies (i) or (ii). However, this is an immediate consequence of Theorem 2.1 and the result of Marcus and Moyls [**2**, Theorem 1] mentioned above.

Marcus and Purves [**3**, Theorem 3.1] proved that if T preserves E_r, $r \geqq 4$, then T has the structure given in Theorem 1.1 (i) or (ii), where (iii′) $UV = e^{i\theta}I_n$ and $r\theta \equiv 0 \pmod{2\pi}$. However, the proof given by Marcus and Purves for (iii′) assuming (i) or (ii) is valid for $r \geqq 3$. We thus omit the proof of (iii).

Acknowledgement. I would like to thank Professor B. N. Moyls for his suggestions during the preparation of this paper.

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