ON POLYNOMIAL EXPANSIONS
OF ANALYTIC FUNCTIONS

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1. Introduction

A set of polynomials \( p_0(z), p_1(z), \cdots \) is said to form a basic set if every polynomial can be expressed in one and only one way as a finite linear combination of them.

Given any family \( F \) of polynomials we shall let \( U(n) \) denote the number of polynomials in \( F \) of degree less than \( n \). It is clear that any linearly independent set of polynomials satisfying the condition \( U(n) = n \) is a basic set. Such a basic set is called a simple set.

Suppose that \( \{ p_i(z) \} \ i = 0, 1, 2, \cdots \) is a simple set of polynomials. We may write

\[
(1) \quad p_i(z) = \sum_{j=0}^{i} p_{ij} z^j, \quad \text{where} \quad p_{ii} = 1 \quad (i = 0, 1, 2, \cdots)
\]

\[
(2) \quad z^i = \sum_{j=0}^{i} \pi_{ij} p_j(z), \quad (i = 0, 1, 2, \cdots),
\]

where \( \pi_{ii} = 1 \).

Let us define the operator \( \Pi_i \) as

\[
\Pi_i = \sum_{k=i}^{\infty} \frac{\pi_{ki}}{k!} D^k,
\]

where \( D \) denotes the differential operator.

The purpose of this paper is to generalize the following result ([1], Theorem 2).

**Theorem 1.** Let

\[
p_i(z) = \sum_{i=0}^{i} p_{ij} z^j, \quad \text{where} \quad p_{ii} = 1 \quad (i = 0, 1, 2, \cdots)
\]

be a simple set of polynomials whose coefficients satisfy the inequality

\[1\] The author is indebted to Professor E. G. Straus, who suggested some of the ideas in this paper.
and let $f(z)$ be analytic in $|z| < R$, where $R > 1 + M$. Then the basic series
\[ \sum_{i=0}^{\infty} p_i(z)(\Pi_i f)(0) \]
converges absolutely to $f(z)$ in $|z| < R$, where $(\Pi_i f)(0)$ is defined as 
$[\Pi_i f(z)]_{z=0}$.

More specifically, let \( \{p^k_i(z)\} \ (k = 0, 1, 2, \cdots, n; \ i = 0, 1, 2, \cdots) \) be a
finite family of simple sets of polynomials such that $p^0_i(z) = z^i$ and for $j \leq 1$ define $p^k_{ij}$ by
\begin{equation}
(3) \quad p^k_{i+1} = \sum_{j=0}^{i} p^k_{ij} p^k_j(z)
\end{equation}

$k = 0, 1, 2, \cdots, n-1; \ i = 0, 1, 2, \cdots$ where $p^k_{i0} = 1$. For $j > i$ define $p^k_{ij} = 0$.

Let $\pi^n_{ij}$ be defined by
\[ z^i = \sum_{j=0}^{i} \pi^n_{ij} p^n_j(z) \quad (i = 0, 1, 2, \cdots), \]
so that $\pi^n_{i0} = 1$. Furthermore, let $\Pi^n_i$ denote the operator
\[ \sum_{k=1}^{\infty} \frac{\pi^n_{ki}}{k} D^k. \]

We shall show that if $|p^k_{ij}| \leq M$ and if $f(z)$ is analytic in $|z| < R$
where $R > 1 + M$, then the basic series
\[ \sum_{i=0}^{\infty} p_i(z)(\Pi^n_i f)(0) \]
converges absolutely to $f(z)$ in $|z| < R$.

We shall show further that the boundedness condition of theorem 1
is not a necessary condition and that for certain simple sets of polynomials
the uniform boundedness of the zeros of the polynomials is a necessary and
sufficient condition for the theorem to hold.

Finally, we remark that for a suitably restricted class of entire functions
Whittaker [1 p. 11] needs no condition on the $p_{ij}$ to assure that a basic
series converges to $f(z)$. We are, however, throughout concerned with the
convergence of a basic series to $f(z)$ for arbitrary $f$, in which case it is
necessary to restrict the polynomials, though not necessarily as severely
as in theorem 1.
2. An extension of theorem 1

With $p_{ij}$ and $\tau_{ij}$ defined for $j \leq i$ by (1) and (2), and $p_{ij} = \tau_{ij} = 0$ for $j > i$, Whittaker [1, pp. 6, 15] shows that $(\tau_{ij})(p_{ij}) = I$ (the unit matrix) and that if also $|p_{ij}| \leq M$ (a constant) then

$$|\tau_{ij}| < (1+M)^{i-j} \quad (0 \leq j \leq i, i = 0, 1, 2, \cdots).$$

Lemma 1 below is a generalization of (4) and will be used together with Lemma 2 to prove Theorem 2. Before proceeding, however, we would like to make some comments about the notation $(p_{ij})^{-1}$ to be used in the sequel. Indeed $(\tau_{ij})$ is the unique left inverse of $(p_{ij})$ among row-finite matrices — conceivably $(p_{ij})$ could have some other (non-row-finite) left inverse. But, a lower triangular matrix $A$ with non-zero diagonal elements has a unique right inverse $A^{-1}$ (which is also lower triangular), and $A^{-1}$ is also a left inverse (and the only row-finite left inverse) — consult, for example, Cooke [3, p. 22].

**Lemma 1.** Let $T_0 = I$ and $T_m = P_{m-1} \cdots P_1 P_0$ ($m = 1, 2, \cdots, n$), where $P_k = (p_{ij}^k)$ is defined by (3) and satisfies, for some constant $M$,

$$|p_{ij}^k| \leq M \quad (i, j = 0, 1, 2, \cdots; k = 0, 1, \cdots, n-1).$$

Then $T_m^{-1} = (\tau_{ij}^m)$ has the property

$$|\tau_{ij}^m| < (i-j+1)^{m-1}(1+M)^{i-j} \quad (0 \leq j \leq i, i = 0, 1, 2, \cdots).$$

**Proof.** The result is trivial for $m = 0$, and reduces to (4) for $m = 1$. Suppose the inequality holds for some $m < n$.

Now $T_{m+1} = P_m T_m$, so that $T_m^{-1} = T_{m+1} P_m^{-1}$; using the inductive hypothesis on $T_{m+1}^{-1}$, and (4) on $P_m^{-1}$, we then obtain

$$|\tau_{ij}^{m+1}| < \sum_{k=0}^i (i-k+1)^{m-1}(1+M)^{i-k} \cdot (1+M)^{k-j}$$

$$\leq (i-j+1) \cdot (i-j+1)^{m-1}(1+M)^{i-j}$$

and the lemma follows.

**Lemma 2.** If (5) holds and $R > 1+M$ then

$$M_i^k(R) \leq (i+1)^k R^i \quad (k = 0, 1, \cdots, n; i = 0, 1, 2, \cdots),$$

where $M_i^k(R) = \max_{|z|=R} |p_{ij}^k(z)|$.

**Proof.** Since $p_{ij}^k(z) = z^i$ we have $M_i^k(R) = R^i$, so that the result holds for $k = 0$. Suppose it holds for some $k < n$. Then, by (3) and (5),
and the lemma follows.

Let \( f(z) = \sum_{i=0}^{\infty} a_i z^i \) be analytic in the region \(|z| < R\) with \( R > M+1\). We have

\[
z^i = \sum_{j=0}^{i} \pi_{i,j} p_j^n(z), \quad \pi_{i,i} = 1.
\]

Let

\[
E(z) = \sum_{j=0}^{\infty} p_j^n(z) \sum_{k=j}^{\infty} a_k \pi_{k,j} = \sum_{j=0}^{\infty} p_j^n(z) (I_k f)(0).
\]

We can now prove

**Theorem 2.** \( E(z) \) converges absolutely to \( f(z) \) in \(|z| < R\).

**Proof.** If the order of summation is reversed in the double series defining \( E(z) \), we obtain \( f(z) \). Consequently the theorem will be proved if we can show that, for \(|z| < R\) (and \( R > 1+M\)), the series

\[
S = \sum_{i=0}^{\infty} |a_i| \sum_{j=0}^{i} |\pi_{i,j}| |p_j^n(z)|
\]

converges. First choose \( R_0 \) such that \( M+1 < R_0 < R \); then, if \(|z| \leq R_0\), \(|p_j^n(z)|\) is majorized by \( M_j^n(R_0) \), and using Lemmas 1 and 2 we obtain

\[
S \leq \sum_{i=0}^{\infty} |a_i| \sum_{j=0}^{i} (i-j+1)^{n-1}(1+M)^{i-j} \cdot (i+1)^n R_0^j
\]

\[
\leq \sum_{i=0}^{\infty} |a_i| (i+1)^{2n} R_0^i.
\]

The last series converges, since if we choose \( R_1 \) in \( R_0 < R_1 < R \), we can make \((i+1)^{2n} R_0^i < R_1^i\) for all sufficiently large \( i\); and this proves the theorem.

We now show that the condition of theorem 1 that \(|p_{ij}| < M\) is not a necessary condition. Though the following lemma is not really essential to prove this fact, nevertheless it is of independent interest and is worth mentioning.

**Lemma 3.** Given a sequence of polynomials \( z^n - c_1 z^{n-1} - c_2 z^{n-2} \ldots - c_n \)
(n = 0, 1, 2, \cdots), where the coefficients are uniformly bounded, the zeros of these polynomials must be uniformly bounded.

**Proof.** If the coefficients are uniformly bounded by \( M \) but the conclusion is false, then for some \( n \), there is a zero \( z \) with \( |z| > M + 1 \). But then
\[
|z|^n \leq M(|z|^{n-1} + \cdots + 1) = M(|z|^{n-1})/(|z| - 1) < |z|^n - 1
\]
and this contradiction establishes the lemma.

**Lemma 4.** There exist simple sets of polynomials \( \{p_n(z)\} \) such that their zeros are not uniformly bounded and yet every analytic function \( f \) is representable in terms of these polynomials in its region of analyticity.

**Proof.** Let \( p_{2n}(z) = z^{2n} \) and \( p_{2n+1}(z) = z^{2n+1} - (2n+1)z^{2n} \). Clearly \( 2n+1 \) is a zero of \( p_{2n+1}(z) \), so that the zeros are unbounded. Suppose that
\[
\sum_{k=0}^{\infty} a_k z^k = \sum_{n=0}^{\infty} \left( a_{2n} z^{2n} + a_{2n+1} z^{2n+1} \right)
\]
These last two series clearly converge for \(|z| < R\) whenever \( \sum a_k z^k \) does so, and the lemma follows.

Thus it follows that

**Theorem 3.** There exist simple sets of polynomials \( \{p_n(z)\} \) such that their coefficients are not uniformly bounded and yet every analytic function \( f \) is representable in terms of these polynomials in its region of analyticity.

Now let \( \{z_n\} \) be a sequence of complex numbers such that the set consisting of its distinct elements has no limit point. We consider the simple set \( S \) of polynomials whose elements \( p_n(z) \) are given by
\[
p_0(z) = 1
\]
\[
p_1(z) = (z-z_1)
\]
\[
\vdots
\]
\[
p_n(z) = p_{n-1}(z)(z-z_n).
\]

**Theorem 4.** Let \( S \) be as above and \( f(z) \) be analytic for \(|z| < R\). Then \( f(z) \) can be expressed in \(|z| < R \) as
\[
a_0 + a_1(z-z_1) + a_2(z-z_1)(z-z_2) + a_3(z-z_1)(z-z_2)(z-z_3) + \cdots
\]
if and only if the \( z_i \) are bounded (i.e., \( \{z_n\} \) as a set is finite).

**Proof.** Sato [2] showed that for every bounded set of \( \{z_n\} \) (even if they have a limit point) such a representation is possible. On the other hand
assume that \( z_n \) is unbounded, then one can find an entire function \( f \) which vanishes at \( z_n \) with the appropriate multiplicity. Such an \( f \) cannot be represented by (6), since all \( a_i \) in the series would have to vanish.

References


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