

A CONVEXITY RESULT FOR WEAK DIFFERENTIAL INEQUALITIES

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Introduction. In this note we present a natural “weak” form of a certain convexity estimate for evolution inequalities as given in Agmon-Nirenberg’s paper [1], p. 139 (see also A. Friedman [2], Theorem 4.2 and 4.3). Our proof will follow that given in [1] and [2] with the natural modifications due to the enlargement of the class of solutions which are taken into account.

1. Let us consider a Hilbert space H , and B ; $\mathcal{D}(B) \subseteq H \rightarrow H$ be a self-adjoint—generally unbounded—operator in H with domain $\mathcal{D}(B)$.

A class of test-functions $K_B[a, b]$ associated to B and to a given interval $[a, b]$ is defined as follows:

A function $\varphi(t)$, $a \leq t \leq b \rightarrow H$ belongs to $K_B[a, b]$ if and only if it is: once continuously differentiable in H ; has a compact support in the open interval (a, b) ; belongs to $\mathcal{D}(B)$ for any $t \in (a, b)$; $(B\varphi)(t)$ is H -continuous in $[a, b]$.

Now, if $u(t)$ is a function, $a \leq t \leq b \rightarrow H$ which belongs to $\mathcal{D}(B)$ for any $t \in [a, b]$, continuously differentiable in H with $(Bu)(t) - H$ continuous in $[a, b]$, then the function $f(t) = u'(t) - Bu(t)$ is also H -continuous.

If we assume that an inequality of the form

$$(1.1) \quad \|u'(t) - Bu(t)\|_H = \|f(t)\|_H \leq \phi(t) \|u(t)\|_H, \quad t \in [a, b]$$

is satisfied, where $\phi(t)$ is a given non-negative scalar function defined for $t \in [a, b]$ then we say that $u(t)$ is a “strong” solution of an abstract differential inequality or of an “evolution inequality”.

Let us take now the equality $u'(t) - Bu(t) = f(t)$ and multiply scalarly with an arbitrary function $\varphi(t) \in K_B[a, b]$. We get then

$$(1.2) \quad \langle u'(t), \varphi(t) \rangle_H - \langle Bu(t), \varphi(t) \rangle_H = \langle f(t), \varphi(t) \rangle_H$$

or also

$$\frac{d}{dt} \langle u(t), \varphi(t) \rangle_H - \langle u(t), \varphi'(t) \rangle_H - \langle u(t), B\varphi(t) \rangle_H = \langle f(t), \varphi(t) \rangle_H, \quad t \in [a, b]$$

If we integrate this last equality between a and b , we obtain, because $\varphi(t)$ is

(Received by the editors December 23, 1974.)

⁽¹⁾This research is supported by a grant of the National Research Council, Canada.

null near a and b , the equality

$$(1.3) \quad -\int_a^b \langle u(t), \varphi'(t) \rangle_H dt = \int_a^b \langle u(t), B\varphi(t) \rangle_H dt + \int_a^b \langle f(t), \varphi(t) \rangle_H dt, \forall \varphi \in K_B[a, b]$$

We see that this last expression can be written with a general H -continuous function $u(t)$ and this leads us to the following

DEFINITION. A H -continuous function $u(t)$ verifies a weak evolution inequality (1.1) if there exists a H -continuous function $f(t)$ defined on $[a, b]$, such that (1.3) holds for all test-functions and also that the estimate

$$(1.4) \quad \|f(t)\|_H \leq \phi(t) \|u(t)\|_H, t \in [a, b]$$

is satisfied, where $\phi(t)$ is an everywhere defined non-negative scalar function on $[a, b]$.

In the present paper we prove the following

THEOREM. Let us assume that the H -continuous function $u(t)$ verifies the weak evolution inequality (1.1) with a function $\phi(t)$ which is integrable on $[a, b]$ and if $\int_a^b \phi(t) dt \leq 1/2\sqrt{2}$ then the estimate

$$(1.5) \quad \|u(t)\| \leq 2\sqrt{2} \|u(a)\|^{(b-t)/(b-a)} \|u(b)\|^{(t-a)/(b-a)}, a \leq t \leq b$$

is also satisfied.

2. Proof of the theorem (I). To start the proof, which follows the main lines in [1], [2] with the appropriate modifications for the “weak” case, we let $\{E_\lambda\}_{-\infty}^\infty$ to be the spectral family of the self-adjoint operator B , so that $Bx = \int_{-\infty}^\infty \lambda dE_\lambda x$, $\forall x \in \mathcal{D}(B)$, in the well-known sense (see [3] for the spectral theorem).

Let then E be the projection operator defined by $Ex = \int_0^\infty dE_\lambda x$, $x \in H$, so that $E = I - E_0$. Define then two continuous H -valued functions $u_1(t)$, $u_2(t)$ through the relations $u_1(t) = (Eu)(t)$, $u_2(t) = (I - E)u(t) = E_0 u(t)$ (here I is the identity operator in H). In the same way, consider the H -continuous functions:

$$f_1(t) = (Ef)(t), \quad f_2(t) = (I - E)f(t)$$

where $f(t) = u'(t) - Bu(t)$ in the above defined weak sense (as in 1.3). It will follow that $u_1'(t) - Bu_1(t) = f_1(t)$ and $u_2'(t) - Bu_2(t) = f_2(t)$ in the same weak sense. More precisely, the following is true:

LEMMA 1. *The relations*

$$(1.6) \quad -\int_a^b \langle u_j(t), \varphi'(t) \rangle_H dt = \int_a^b \langle u_j(t), (B\varphi)(t) \rangle_H dt + \int_a^b \langle f_j(t), \varphi(t) \rangle_H dt$$

are verified for $j = 1, 2$ and for every test-function $\varphi(t) \in K_B[a, b]$.

In order to prove this Lemma it is obviously sufficient to consider just $j = 1$ or $j = 2$. If, say $j = 1$, we have the following

If $\varphi \in K_B[a, b]$ then $E\varphi \in K_B[a, b]$ too.

In fact, the strong H -derivative $dE\varphi/dt$ exists and equals $E d\varphi/dt$, so it is also strongly continuous; also $E\varphi = \theta$ where $\varphi = \theta$, hence $E\varphi$ has compact support in (a, b) ; furthermore, the range of $E\varphi$ is in the domain of B when $t \in (a, b)$: in fact, it is known that $h \in H$ belongs to $\mathcal{D}(B)$ if and only if

$$\int_{-\infty}^{\infty} |\lambda|^2 d\langle E_\lambda h, h \rangle = \int_{-\infty}^{\infty} |\lambda|^2 d\|E_\lambda h\|^2 < \infty$$

Now, if $h \in \mathcal{D}(B)$ then $Eh \in \mathcal{D}(B)$ because

$$\begin{aligned} \int_{-\infty}^{\infty} |\lambda|^2 d\|E_\lambda Eh\|^2 &= \int_0^{\infty} \lambda^2 d\|(E_\lambda - E_0)h\|^2 \\ &= \int_0^{\infty} \lambda^2 d\|E_\lambda h\|^2 < \infty \end{aligned}$$

Hence, $(E\varphi)(t) \in \mathcal{D}(B)$ for any $t \in [a, b]$; we need also that $B(E\varphi)$ is H -continuous as is for $B\varphi$. But $BE\varphi = EB\varphi$ (as B commutes with any of E_λ). So, if $B\varphi$ is continuous, $BE\varphi$ is too.

At this stage we write

$$\begin{aligned} &\int_a^b \langle u_1(t), (B\varphi)(t) \rangle dt + \int_a^b \langle f_1(t), \varphi(t) \rangle dt \\ &= \int_a^b \langle Eu(t), B\varphi(t) \rangle dt + \int_a^b \langle Ef(t), \varphi(t) \rangle dt \\ &= \int_a^b \langle u(t), B(E\varphi)(t) \rangle dt + \int_a^b \langle f(t), (E\varphi)(t) \rangle dt \\ &= - \int_a^b \langle u(t), (E\varphi)'(t) \rangle dt = - \int_a^b \langle u(t), E\varphi'(t) \rangle dt \\ &= - \int_a^b \langle Eu(t), \varphi'(t) \rangle dt = - \int_a^b \langle u_1(t), \varphi'(t) \rangle dt \end{aligned}$$

which gives Lemma for $j = 1$.

3. Proof of the Theorem (II). Let us consider now a sequence of scalar-valued functions $\{\alpha_n(t)\}_{n=1}^{\infty}$ which are non-negative C^1 -functions, vanishing for $|t| \geq 1/n$, with $\int_{-1/n}^{1/n} \alpha_n(\tau) d\tau = 1$ and then form the convolution

$$(u_1 * \alpha_n)(t) = \int_{|t-\tau| \leq 1/n} u_1(\tau) \alpha_n(t-\tau) d\tau$$

which is well-defined for $a + 1/n \leq t \leq b - 1/n$, and is continuously differentiable

there. As proved in our paper [4], after use of (1.6) we find that $(u_1 * \alpha_n)(t) \in \mathcal{D}(B)$ for $t \in [a + 1/n, b - 1/n]$, and in the same interval it is

$$(u_1 * \alpha_n)'(t) = B(u_1 * \alpha_n)(t) + (f_1 * \alpha_n)(t)$$

where

$$(f_1 * \alpha_n)(t) = \int_{|t-\tau| \leq 1/n} f_1(\tau) \alpha_n(t-\tau) d\tau$$

Now we see that

$$(u_1 * \alpha_n)(t) = \int_{|t-\tau| \leq 1/n} (Eu)(\tau) \alpha_n(t-\tau) d\tau = E(u * \alpha_n)(t), \forall t \in \left[a + \frac{1}{n}, b - \frac{1}{n} \right]$$

Hence, $(u_1 * \alpha_n)(t) \in E(H) \forall t \in [a + 1/n, b - 1/n]$, and then, remarking that $B \geq 0$ on $E(H)$, it is: $\langle B(u_1 * \alpha_n)(t), (u_1 * \alpha_n)(t) \rangle_H \geq 0 \forall t$ in this interval.

Now we see that, on $[a + 1/n, b - 1/n]$

$$\begin{aligned} \frac{d}{dt} \langle u_1 * \alpha_n, u_1 * \alpha_n \rangle &= 2 \operatorname{Re} \langle B(u_1 * \alpha_n), (u_1 * \alpha_n) \rangle \\ &+ 2 \operatorname{Re} \langle f_1 * \alpha_n, u_1 * \alpha_n \rangle \geq 2 \operatorname{Re} \langle f_1 * \alpha_n, u_1 * \alpha_n \rangle \end{aligned}$$

If we integrate between $t \in (a - 1/n, b - 1/n)$ and $b - 1/n$, we get

$$\begin{aligned} (1.7) \quad \|(u_1 * \alpha_n)(b - 1/n)\|^2 - \|(u_1 * \alpha_n)(t)\|^2 \\ \geq 2 \operatorname{Re} \int_t^{b-1/n} \langle f_1 * \alpha_n, u_1 * \alpha_n \rangle ds \end{aligned}$$

$$a + \frac{1}{n} < t < b - \frac{1}{n}.$$

Now we can prove

LEMMA 2. *The estimate*

$$\|u_1(b)\|^2 - \|u_1(t)\|^2 \geq 2 \operatorname{Re} \int_t^b \langle f_1(\tau), u_1(\tau) \rangle d\tau$$

is valid, $\forall t \in (a, b)$.

First we prove that $\lim_{n \rightarrow \infty} (u_1 * \alpha_n)(b - 1/n) = u_1(b)$. In fact

$$(u_1 * \alpha_n)\left(b - \frac{1}{n}\right) = \int_{b-2/n}^b u_1(\tau) \alpha_n\left(b - \frac{1}{n} - \tau\right) d\tau,$$

and

$$u_1(b) = \int_{b-2/n}^b u_1(b) \alpha_n\left(b - \frac{1}{n} - \tau\right) d\tau$$

because

$$\int_{|\tau| < 1/n} \alpha_n(\tau) d\tau = 1$$

Then

$$\begin{aligned} \|(u_1 * \alpha_n)\left(b - \frac{1}{n}\right) - u_1(b)\| &\leq \int_{b-2/n}^b \|u_1(\tau) - u_1(b)\| \alpha_n\left(b - \frac{1}{n} - \tau\right) d\tau \\ &\leq \sup_{b-2/n \leq \tau \leq b} \|u_1(\tau) - u_1(b)\| \int_{b-2/n}^b \alpha_n\left(b - \frac{1}{n} - \tau\right) d\tau \\ &= \sup_{b-2/n \leq \tau \leq b} \|u_1(\tau) - u_1(b)\|, \forall n = 1, 2, \dots \end{aligned}$$

and this $\rightarrow 0$ as $n \rightarrow \infty$ by continuity of $u_1(\tau)$ for $\tau = b$.

Hence, we have also:

$$\lim_{n \rightarrow \infty} \left\| (u_1 * \alpha_n)\left(b - \frac{1}{n}\right) \right\| = \|u_1(b)\|.$$

But the estimate

$$\left\| (u_1 * \alpha_n)\left(b - \frac{1}{n}\right) \right\| \leq \sup_{[a, b]} \|u_1(\tau)\|$$

is also valid, hence we get too:

$$\lim_{n \rightarrow \infty} \left\| (u_1 * \alpha_n)\left(b - \frac{1}{n}\right) \right\|^2 = \|u_1(b)\|^2.$$

Furthermore:

$$\lim_{n \rightarrow \infty} \|(u_1 * \alpha_n)(t)\|^2 = \|u_1(t)\|^2$$

for $t \in (a, b)$ and

$$\lim_{n \rightarrow \infty} \int_t^{b-1/n} \langle f_1 * \alpha_n, u_1 * \alpha_n \rangle ds = \int_t^b \langle f_1(s), u_1(s) \rangle ds.$$

This last limit holds because of the following: consider the difference

$$\int_t^b \langle f_1(s), u_1(s) \rangle ds - \int_t^{b-1/n} \langle f_1 * \alpha_n, u_1 * \alpha_n \rangle ds$$

Now, denote

$$\langle f_1(s), u_1(s) \rangle = \phi_1(s), \langle (f_1 * \alpha_n)(s), (u_1 * \alpha_n)(s) \rangle = \phi_n(s)$$

we see that $\phi_1(s)$ is continuous on $t \leq s \leq b$, and $\phi_n(s)$ are continuous on $t \leq s \leq b - 1/n$.

Then our expression equals

$$\int_t^b \phi_1(s) ds - \int_t^{b-1/n} \phi_n(s) ds.$$

Let us extend $\phi_n(s)$ as:

$$\tilde{\phi}_n(s) = \begin{cases} \phi_n(s), & t \leq s < b - 1/n \\ 0, & b - 1/n \leq s \leq b \end{cases}$$

It follows

$$\int_t^{b-1/n} \phi_n(s) ds = \int_t^b \tilde{\phi}_n(s) ds,$$

so that

$$\lim_{n \rightarrow \infty} \int_t^b [\phi_1(s) - \tilde{\phi}_n(s)] ds,$$

must be null.

We can apply here Lebesgue's theorem:

(i) $\tilde{\phi}_n(s) \rightarrow \phi_1(s)$ almost-everywhere on $[t, b]$.

In fact, for any $s \in [t, b)$, $\tilde{\phi}_n(s) = \phi_n(s)$ when n is big enough, such that $b - 1/n > s$: furthermore $\phi_n(s) \rightarrow \phi_1(s)$ for any $a < s < b$ because $(f_1 * \alpha_n)(s) \rightarrow f_1(s)$, $(u_1 * \alpha_n)(s) \rightarrow u_1(s)$; hence, $\tilde{\phi}_n(s) \rightarrow \phi_1(s)$ for any $s > a$, with possible exception of $s = b$. ($\phi_1(b)$ need not be null, whereas $\tilde{\phi}_n(s)$ are all null for $s = b$.)

(ii) $\tilde{\phi}_n(s)$ are uniformly bounded on $[t, b]$. In fact

$$\begin{aligned} \sup_{t \leq s \leq b} |\tilde{\phi}_n(s)| &\leq \sup_{t \leq s \leq b-1/n} |\phi_n(s)| \leq \sup_{t \leq s \leq b-1/n} \|(f_1 * \alpha_n)(s)\| \|(u_1 * \alpha_n)(s)\| \\ &\leq \sup_{a \leq s \leq b} \|f_1(s)\| \sup_{a \leq s \leq b} \|u_1(s)\| \end{aligned}$$

REMARK. We can also avoid Lebesgue's theorem as follows: take an arbitrary $\delta > 0$. Then

$$\begin{aligned} \int_t^b [\phi_1(s) - \tilde{\phi}_n(s)] ds &= \int_t^{b-\delta} [\phi_1(s) - \tilde{\phi}_n(s)] ds \\ &\quad + \int_{b-\delta}^b [\phi_1(s) - \tilde{\phi}_n(s)] ds \\ &= \int_t^{b-\delta} [\phi_1(s) - \phi_n(s)] ds + \int_{b-\delta}^b [\phi_1(s) - \tilde{\phi}_n(s)] ds, \end{aligned}$$

$$\text{for } \delta > \frac{1}{n}.$$

The second integral estimates by $C \cdot \delta$, $C = 2 \sup_{a \leq s \leq b} \|u_1(s)\| \|f_1(s)\|$.

Then given $\varepsilon > 0$, take first $\delta(\varepsilon)$ such that $C\delta < \varepsilon/2$. Then, because $\phi_n(s) \rightarrow \phi_1(s)$ uniformly on $[t, b - \delta]$, there exist an integer $N(\varepsilon)$ such that $\delta > 1/N$ and $n > N \Rightarrow$

$$\left\| \int_t^{b-\delta} [\phi_1(s) - \phi_n(s)] ds \right\| < \frac{\varepsilon}{2}$$

so that $n > N \Rightarrow$

$$\left\| \int_t^b [\phi_1(s) - \tilde{\phi}_n(s)] ds \right\| < \varepsilon$$

Hence, from (1.7), Lemma 2 follows for $a < t < b$. However, the Lemma is true also for $t = a$, or $t = b$, as follows by continuity.

In exactly same way we see that the following is true.

LEMMA 3. *The estimate*

$$\|u_2(t)\|^2 - \|u_2(a)\|^2 \leq 2 \operatorname{Re} \int_a^t \langle f_2(s), u_2(s) \rangle ds$$

holds, $\forall t \in [a, b]$.

4. Proof of Theorem (III).

We see firstly that

$$\begin{aligned} \left| 2 \operatorname{Re} \int_t^b \langle f_1(s), u_1(s) \rangle ds \right| &\leq 2 \left| \int_t^b \langle f_1(s), u_1(s) \rangle ds \right| \leq 2 \int_t^b \|f_1(s)\| \|u_1(s)\| ds \\ &\leq 2 \int_t^b \|f(s)\| \|u(s)\| ds. \end{aligned}$$

It follows:

$$-2 \int_t^b \|f(s)\| \|u(s)\| ds \leq 2 \operatorname{Re} \int_t^b \langle f_1(s), u_1(s) \rangle ds.$$

Hence, applying Lemma 2, we obtain

$$\|u_1(b)\|^2 - \|u_1(t)\|^2 \geq -2 \int_t^b \|f(s)\| \|u(s)\| ds.$$

Then it is:

$$\|u_1(t)\|^2 \leq \|u_1(b)\|^2 + 2 \int_t^b \|f(s)\| \|u(s)\| ds \leq \|u_1(b)\|^2 + 2M \int_t^b \|f(s)\| ds,$$

where $M = \sup_{a \leq s \leq b} \|u(s)\|$.

Also, from Lemma 3, we get

$$\|u_2(t)\|^2 \leq \|u_2(a)\|^2 + 2M \int_a^t \|f(s)\| ds$$

and by addition

$$\|u_1(t)\|^2 + \|u_2(t)\|^2 \leq \|u_1(b)\|^2 + \|u_2(a)\|^2 + 2M \int_a^b \|f(s)\| ds$$

As $u_1(t) = Eu(t)$, $u_2(t) = (I - E)u(t)$, $\langle u_1(t), u_2(t) \rangle = \langle Eu(t), (I - E)u(t) \rangle = \langle u(t), (E - E)u(t) \rangle = 0$; so $\|u_1(t) + u_2(t)\|^2 = \langle u_1 + u_2, u_1 + u_2 \rangle = \|u_1(t)\|^2 + \|u_2(t)\|^2$. Hence

$$\|u(t)\|^2 \leq \|u_1(b)\|^2 + \|u_2(a)\|^2 + 2M \int_a^b \|f(s)\| ds$$

If we use inequality

$$2MN \leq \left(\frac{M}{\sqrt{2}}\right)^2 + (\sqrt{2}N)^2 \quad \text{where} \quad N = \int_a^b \|f(s)\| ds,$$

we have

$$\|u(t)\|^2 \leq \|u_1(b)\|^2 + \|u_2(a)\|^2 + \frac{M^2}{2} + 2 \left(\int_a^b \|f(s)\| ds \right)^2, \quad \forall t \in [a, b].$$

Hence

$$M^2 \leq \|u_1(b)\|^2 + \|u_2(a)\|^2 + \frac{M^2}{2} + 2 \left(\int_a^b \|f(s)\| ds \right)^2$$

and finally we have the estimate

$$(*) \quad M^2 \leq 2(\|u(a)\|^2 + \|u(b)\|^2) + 4 \left(\int_a^b \|f(s)\| ds \right)^2$$

Let us define now, for any real σ , the H -continuous function $w_\sigma(t) = e^{\sigma t}u(t)$, and let $B_\sigma = B + \sigma I$ which is again self-adjoint, with $\mathcal{D}(B_\sigma) = \mathcal{D}(B) \forall$ real σ . Then we have

LEMMA 4. *The relation $w'_\sigma(t) - B_\sigma w_\sigma(t) = e^{\sigma t}f(t)$ holds in the weak sense over (a, b) .*

So, we must prove that, $\forall \varphi \in K_{B_\sigma}[a, b] = K_B[a, b]$ is

$$-\int_a^b \langle w_\sigma(s), \varphi'(s) \rangle ds = \int_a^b \langle w_\sigma(s), B_\sigma \varphi(s) \rangle ds + \int_a^b \langle e^{\sigma s} f(s), \varphi(s) \rangle ds$$

or

$$-\int_a^b \langle u(s), e^{\sigma s} \varphi'(s) \rangle ds = \int_a^b \langle u(s), e^{\sigma s} B_\sigma \varphi(s) \rangle ds + \int_a^b \langle f(s), e^{\sigma s} \varphi(s) \rangle ds.$$

Now $e^{\sigma s} \varphi'(s) = (e^{\sigma s} \varphi(s))' - \sigma e^{\sigma s} \varphi(s)$. Also $\psi_\sigma(s) = e^{\sigma s} \varphi(s) \in K_B[a, b]$ as obviously. We have to prove

$$\begin{aligned} -\int_a^b \langle u(s), \psi'_\sigma(s) - \sigma \psi_\sigma(s) \rangle ds &= \int_a^b \langle u(s), B \psi_\sigma(s) + \sigma \psi_\sigma(s) \rangle ds \\ &\quad + \int_a^b \langle f(s), \psi_\sigma(s) \rangle ds, \end{aligned}$$

that is to prove

$$-\int_a^b \langle u(s), \psi'_\sigma(s) \rangle ds = \int_a^b \langle u(s), B\psi_\sigma(s) \rangle ds + \int_a^b \langle f(s), \psi_\sigma(s) \rangle ds$$

which is true because $u' - Bu = f$ in weak sense on (a, b) .

Once this is established, we apply (*) to this slightly changed situation and obtain the estimate

$$\left(\sup_{a \leq t \leq b} \|e^{\sigma t} u(t)\| \right)^2 \leq 2(\|e^{\sigma a} u(a)\|^2 + \|e^{\sigma b} u(b)\|^2) + 4 \left(\int_a^b \|e^{\sigma s} f(s)\| ds \right)^2$$

We use now the main assumption (1.4)

$$\|f(s)\| \leq \phi(s) \|u(s)\|, \quad s \in [a, b].$$

Then

$$\begin{aligned} \int_a^b \|e^{\sigma s} f(s)\| ds &\leq \int_a^b e^{\sigma s} \phi(s) \|u(s)\| ds \leq \sup_{[a, b]} (e^{\sigma s} \|u(s)\|) \int_a^b \phi(s) ds \\ &\leq \frac{1}{2\sqrt{2}} \sup_{[a, b]} (e^{\sigma s} \|u(s)\|) \end{aligned}$$

and squaring get

$$\left(\int_a^b \|e^{\sigma s} f(s)\| ds \right)^2 \leq \frac{1}{8} \left(\sup_{[a, b]} (e^{\sigma s} \|u(s)\|) \right)^2$$

Hence

$$(\sup \|e^{\sigma s} u(s)\|)^2 \leq 2(\|e^{\sigma a} u(a)\|^2 + \|e^{\sigma b} u(b)\|^2) + \frac{1}{2} (\sup e^{\sigma s} \|u(s)\|)^2$$

and

$$\left(\sup_{[a, b]} \|e^{\sigma s} u(s)\| \right)^2 \leq 4(\|e^{\sigma a} u(a)\|^2 + \|e^{\sigma b} u(b)\|^2)$$

Then $\forall t \in [a, b]$, $\|e^{\sigma t} u(t)\| \leq \sup_{[a, b]} \|e^{\sigma s} u(s)\|$ and

$$\|e^{\sigma t} u(t)\|^2 \leq \left(\sup_{[a, b]} \|e^{\sigma s} u(s)\| \right)^2;$$

so

$$(**) \|e^{\sigma t} u(t)\|^2 \leq 4(\|e^{\sigma a} u(a)\|^2 + \|e^{\sigma b} u(b)\|^2), \quad \forall t \in [a, b]$$

We pass now to the final part of the proof for (1.5). First we consider the case when $u(a) = \theta$ or $u(b) = \theta$. If both are, from $(**)$ $\Rightarrow \|u(t)\| = 0 \quad \forall t \in [a, b]$, so (1.5) holds. If say, $u(a) = \theta$, from $(**)$ we get

$$\|e^{\sigma t} u(t)\| \leq 2 \|e^{\sigma b} u(b)\|, \quad \|u(t)\| \leq 2 e^{\sigma(b-1)} \|u(b)\|$$

As σ was chosen arbitrarily, we deduce, when $t < b$ and $\sigma \rightarrow -\infty$, that $\|u(t)\| = \theta$, $a \leq t < b$ and hence $u(b) = \theta$ also and (1.5) holds. The non-trivial case is when both $u(a)$ and $u(b)$ are $\neq \theta$. We can choose σ so that $\|e^{\sigma a} u(a)\| = \|e^{\sigma b} u(b)\|$

$$\left(e^{\sigma(b-a)} = \frac{\|u(a)\|}{\|u(b)\|} \right), \quad \sigma(b-a) = \log \frac{\|u(a)\|}{\|u(b)\|}, \quad \sigma = \log \left(\frac{\|u(a)\|}{\|u(b)\|} \right)^{\frac{1}{b-a}}.$$

In that case, $e^{\sigma t} = (\|u(a)\|/\|u(b)\|)^{t/(b-a)}$. Hence $(**)$ becomes

$$\left(\frac{\|u(a)\|}{\|u(b)\|} \right)^{2t/(b-a)} \|u(t)\|^2 \leq 8 \left(\|u(a)\|^2 \left(\frac{\|u(a)\|}{\|u(b)\|} \right)^{2a/(b-a)} \right) = 8 \frac{\|u(a)\|^{\frac{2b}{b-a}}}{\|u(b)\|^{\frac{2a}{b-a}}}$$

hence

$$\|u(t)\| \leq 2\sqrt{2} \|u(a)\|^{b-t/b-a} \|u(b)\|^{t-a/b-a}, \quad a \leq t \leq b$$

which proves our theorem.

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