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A COMMENT ON CERTAIN p-SHIFT ALGEBRAS

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Abstract

Let $G = \bigoplus_{i=0}^{\infty} \mathbb{Z}_p$, where p is a prime, let s be the shift mapping the i th summand of G to the $(i+1)$ st and let ω be a 2 cocycle on G with values in S^1, for which $\omega(s(g), s(h)) = \omega(g, h)$. If $\omega(e_j, e_k) = \omega(e_k, e_j)$ whenever $|j-k|$ is sufficiently large, where e_i is the generator of the i th summand of G, then it is shown that the twisted group C^*-algebra $C^*(G, \omega)$ is isomorphic to the UHF algebra $\mathrm{UHF}(p^{\infty})$. An immediate consequence, by results of Bures and Yin, is the existence of infinitely many non-conjugate shifts on $\mathrm{UHF}(p^{\infty})$.

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In his seminal study [4] of unital $*$-endomorphisms of C^*-algebras, Powers introduced a class of AF-algebras known as binary shift algebras which were subsequently shown in [1] to be twisted group C^*-algebras. More precisely, if ω is a *multiplier* on a discrete group G (i.e. a 2-cocycle of G with values in the circle S^1) then the (reduced) twisted group C^*-algebra $C^*(G, \omega)$ of G is the C^*-algebra generated by the left regular projective representation on $l^2(G)$. A shift on G is an endomorphism for which $\bigcap_{k=1}^{\infty} s^k(G) = \{e\}$, where e is the identity of G, and, when the multiplier ω is compatible with s in the sense that $\omega(s(g), s(h)) = \omega(g, h)$ for all h, $g \in G$, then $C^*(G, \omega)$ is known as a *group shift algebra.* The *binary shift algebras* introduced by Powers are the group shift algebras for which G is equal to an infinite direct sum $\bigoplus_{i=0}^{\infty} \mathbb{Z}_2$ of copies of $\mathbb{Z}_2$ and s is the shift mapping the i th summand of G to the $(i+1)$ st summand. If, instead, G is equal to an infinite direct sum $\bigoplus_{i=0}^{\infty} \mathbb{Z}_p$ of copies of $\mathbb{Z}_p$, where $p \in \mathbb{N}$, then $C^*(G, \omega)$ is known as

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a *p-shift algebra*. Each p-shift algebra is approximately finite dimensional, with the approximating finite dimensional subalgebras being given by $A_n = C^*(G_n, \omega)$, where $G_n = \bigoplus_{i=0}^n \mathbb{Z}_p$.

It is shown in [1] that every shift-invariant multiplier ω on $G = \bigoplus_{i=0}^\infty \mathbb{Z}_p$ is specified by an infinite sequence $\{a(n): n \in \mathbb{Z}\}$ of elements of $\mathbb{Z}_p$ with $a(-n) = -a(n)$ for each $n \in \mathbb{Z}$; the sequence is determined by $\rho(e_j, e_k) = \gamma^{a(k-j)}$, where $\gamma = \exp(2\pi i/p)$, e_i is the element of G corresponding to the generator 1 of the ith summand $\mathbb{Z}_p$ and ρ is the bicharacter of G associated with ω, defined by $\rho(g, h) = \omega(g, h)\overline{\omega(h, g)}$. Using a technique from [2] it will be shown in this note that when p is a prime and ω is specified by a sequence $\{a(n): n \in \mathbb{Z}\}$ with finite support, then the p-shift algebra $C^*(G, \omega)$ is isomorphic to the UHF algebra UHF(p^∞). An immediate consequence, by results from [1], is the existence of infinitely many non-conjugate shifts on the UHF algebra UHF(p^∞).

The key lemma is based directly on [2, Lemma 2.3].

LEMMA. *Let p be a prime and let $f(t) \in \mathbb{Z}_p[t]$ with $f(t) = \sum_{i=0}^n x(i)t^i$ and $x(0) = -x(n) \neq 0$. Then there exist $g(t) \in \mathbb{Z}_p[t]$ and $m \in \mathbb{N}$ such that $g(t)f(t) = x(n)(t^m - 1)$.*

PROOF. Since $x(n) \neq 0$ we can replace $f(t)$ by $x(n)^{-1}f(t)$ and hence assume that $x(n) = 1$. We inductively define $s_k \in \mathbb{Z}_p$ and $p_k(t) \in \mathbb{Z}_p[t]$ by $s_0 = 0$, $p_0(t) = f(t)$ and, for $k \geq 1$, $s_k = s_{k-1} + i_k$ and $p_k(t) = x_{k-1}(i_k)t^{s_k}p_0(t) + p_{k-1}(t)$, where $p_{k-1}(t) = (\sum_{i=1}^n x_{k-1}(i)t^{i+s_{k-1}}) - 1$ for some $x_{k-1}(1), \dots, x_{k-1}(n)$ with $x_{k-1}(n) \neq 0$ and where $i_k = \min\{i \geq 1: x_{k-1}(i) \neq 0\}$. This definition relies on $p_{k-1}(t)$ being of the stated form; to establish this inductively assume that p_{k-1} is as claimed and note that the lowest non-constant term in $p_{k-1}(t)$ is $x_{k-1}(i_k)t^{i_k+s_{k-1}} = x_{k-1}(i_k)t^{s_k}$, so that the degrees of the non-constant terms of $p_k(t)$ lie between $s_k + 1$ and $s_k + n$. Also the coefficient $x_k(n)$ of t^{n+s_k} is equal to $x_{k-1}(i_k)$, which is non-zero by the definition of i_k. Hence the definition is consistent.

There are only finitely many elements in $(\mathbb{Z}_p)^n$ so there exist k, l with $k < l$ and $x_k(i) = x_l(i)$ for each $1 \leq i \leq n$. Then $p_l(t) - p_k(t)t^{s_l - s_k} = t^{s_l - s_k} - 1$. This establishes the required result since, by construction, both $p_l(t)$ and $p_k(t)$ are multiples of $p_0(t) = f(t)$.

THEOREM. *Let p be a prime and let ω be a shift-compatible multiplier on $\bigoplus_{m=0}^\infty \mathbb{Z}_p$ specified by a sequence $\{a(n): n \in \mathbb{Z}\}$ for which $a(d) \neq 0$ and $a(n) = 0$ for $|n| > d$. Then $C^*(G, \omega)$ is isomorphic to* UHF(p^∞).

PROOF. It suffices to prove that an infinite number of the finite-dimensional subalgebras $A_n = C^*(G_n, \omega)$ have trivial centre since then, noting that $\{\delta_g : g \in G_n\}$ is a vector space basis for A_n with p^{n+1} elements, A_n is isomorphic to the full $p^{(n+1)/2} \times p^{(n+1)/2}$ matrix algebra. Thus, on passing to a subsequence, $C^*(G, \omega)$ is the inductive limit under unital isomorphisms of a sequence of increasing p-power matrix algebras.

It is well-known (as described in [3, Theorem 3.6.3]) that the dimension of the centre of A_n is equal to the cardinality of $H_n = \{h \in G_n : \rho(g, h) = 1$ for all $g \in G_n\}$, so it suffices to prove that H_n is trivial for infinitely many n. Let $f(t) = \sum_{i=0}^{2d} a(i-d)t^i$, let m be as in the lemma, let $k \in \mathbb{N}$ and let $h = \sum_{i=0}^{km-1} h(i)e_i$ belong to H_{km-1}. Then, from the definition of H_{km-1} applied to $g = e_j$,

$$\sum_{i=-d}^{d} h(i+j)a(i) = 0 \quad \text{in } \mathbb{Z}_p \text{ for each } 0 \le j \le km-1, \tag{1}$$

where $h(r)$ is defined to be zero if $r > km-1$ or $r < 0$.

By the lemma there exists a polynomial $g(t) = \sum_{i=0}^{m-2d} b(i)t^i$ such that

$$f(t)g(t) = \left(\sum_{i=0}^{2d} a(i-d)t^i\right)\left(\sum_{i=0}^{m-2d} b(i)t^i\right) = a(d)(t^m - 1).$$

Hence, equating coefficients of t^r, we obtain

$$\sum_{i=-d}^{d} b(r-d-i)a(i) = \begin{cases} 0 & \text{if } 1 \le r \le m-1, \\ a(d) & \text{if } r = m, \\ a(-d) & \text{if } r = 0, \end{cases} \tag{2}$$

(where $b(j)$ is taken to be zero if $j < 0$ and if $j > m-2d$).

For each $0 \le s \le (k-1)m + 2d - 1$, equations (1) yield

$$\begin{aligned} 0 &= \sum_{j=s}^{s+m-2d} b(j-s) \sum_{i=-d}^{d} h(i+j)a(i) \\ &= \sum_{r=0}^{m} h(r-d+s) \sum_{i=-d}^{d} b(r-d-i)a(i) \end{aligned}$$

and then equations (2) yield

$$0 = [h(m-d+s) - h(s-d)]a(d).$$

Hence $h(s-d) = h(m+s-d)$ for each $0 \le s \le (k-1)m + 2d - 1$ and thus, in particular, $h(km-q) = h((k-1)m-q) = \cdots = h(-q) = 0$ for each $1 \le q \le d$. Therefore $h(t) = 0$ for each $t \ge km-d$ and then

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successive consideration of equation (1) for $j = km - 1, km - 2, \ldots, d$ yields $h(km - d - 1) = 0$, $h(km - d - 2) = 0, \ldots, h(0) = 0$. Hence $h = \sum_{i=0}^{km-1} h(i)e_i = 0$ and so $H_{km-1} = \{0\}$ for each $k \in \mathbb{N}$, as required.

COROLLARY. *The* UHF *algebra* UHF(p^∞) *admits infinitely many non-conjugate shifts.*

PROOF. Recall that $C^*(G, \omega)$ is the C^*-algebra on $l^2(G)$ generated by the left regular projective representation of G. From [1, Proposition 1.1] any shift of $C^*(G, \omega)$ extends to a shift of the weak closure $W^*(G, \omega)$. If $G = \bigoplus_{i=0}^{\infty} \mathbb{Z}_p$ and ω is described by a sequence with finite support, so that $C^*(G, \omega)$ is isomorphic to UHF(p^∞), then, by [1, Proposition 1.5], $W^*(G, \omega)$ is the hyperfinite type II_1 factor. Furthermore, since the left projective representation of G is cyclic with cyclic vector δ_0 and since the corresponding linear functional on UHF(p^∞) $= C^*(G, \omega)$ is the unique trace, $W^*(G, \omega)$ can be identified with the closure of UHF(p^∞) in the GNS representation associated with the trace. Hence, if two shifts of UHF(p^∞) are conjugate then the automorphism implementing the conjugacy extends to an automorphism of $W^*(G, \omega)$, which will implement a conjugacy between the extended shifts. Therefore, by [1, Proposition 4.4], there exists an integer m such that $a(j) = m^2 b(j)$ for all $j \in \mathbb{Z}$, where $\{a(j) : j \in \mathbb{Z}\}$ and $\{b(j) : j \in \mathbb{Z}\}$ are the sequences associated with the two shifts.

By the theorem each sequence with finite support yields a shift on UHF(p^∞), from which the result follows.

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