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Strict Comparison of Positive Elements in Multiplier Algebras

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Abstract. Main result: If a C^*-algebra $\mathcal{A}$ is simple, σ-unital, has finitely many extremal traces, and has strict comparison of positive elements by traces, then its multiplier algebra $\mathcal{M}(\mathcal{A})$ also has strict comparison of positive elements by traces. The same results holds if *finitely many extremal traces* is replaced by *quasicontinuous scale*. A key ingredient in the proof is that every positive element in the multiplier algebra of an arbitrary σ-unital C^*-algebra can be approximated by a bi-diagonal series. As an application of strict comparison, if $\mathcal{A}$ is a simple separable stable C^*-algebra with real rank zero, stable rank one, and strict comparison of positive elements by traces, then whether a positive element is a positive linear combination of projections is determined by the trace values of its range projection.

1 Introduction

In this paper we study strict comparison of positive elements in multiplier algebras. Comparison theory has a long history. It is a basic fact in von Neumann algebra theory that every finite factor has strict comparison of projections: if p, q are projections in the factor and $\tau(p) < \tau(q)$, then $p \precsim q$ (Murray–von Neumann subequivalence). In the theory of C^*-algebras it was soon realized that both strict comparison of projections and strict comparison of positive operators (mostly formulated in terms of quasitraces) are important properties. Perhaps one can view strict comparison of positive elements as a regularity property in the study of the Cuntz semigroup *e.g.*, [1], [2, III], [46, 4.7]. In recent years, there have been spectacular advances in understanding strict comparison for simple nuclear C^*-algebras and exploring its connections with other properties, *e.g.*, $\mathcal{Z}$-stability (see [37, 46, 50]) or the almost unperforation of the Cuntz semigroup (see [45, 46]).

Comparison theory for multiplier algebras has not been studied systematically, but was often used implicitly in the investigation of the ideal structure and extension theory, *e.g.*, [10–12, 27, 31, 44, 51]

In a previous paper [26, Theorem 3.2] we proved that if $\mathcal{A}$ is a unital, separable, nonelementary simple C^*-algebra with real rank zero with finitely many extremal traces and strict comparison of projections by traces, then $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ has strict comparison of projections by traces provided that the definition is appropriately adapted to the presence of ideals in $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$. The main goal of this paper is to extend

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the above result. In Theorem 5.3 we prove that if $\mathcal{A}$ is a simple σ-unital C^*-algebra that has finitely many extremal traces and strict comparison of positive elements by traces, then $\mathcal{M}(\mathcal{A})$ has strict comparison of positive elements by traces. This theorem has already found an application in the recent proof that the corona algebra $\mathcal{M}(\mathcal{Z} \otimes \mathcal{K})/(\mathcal{Z} \otimes \mathcal{K})$ has real rank zero [30, 32, 53].

The key tool in the proof of Theorem 5.3, but possibly also of independent interest, is Theorem 4.2, where we prove that positive elements in the multiplier algebra of an arbitrary σ-unital C^*-algebra $\mathcal{A}$ can be written as the sum of a bi-diagonal series and a selfadjoint remainder in $\mathcal{A}$ of arbitrarily small norm (see Definition 4.1). The idea of tri-diagonal decomposition of arbitrary elements in $\mathcal{M}(\mathcal{A})$ first occurred in an article by Elliot on AF algebras in 1974 [10, proof of Theorem 3.1]. In 1990, Zhang [51, proof of Theorem 2.2] established a tri-diagonal decomposition of positive elements in $\mathcal{M}(\mathcal{A})$ when the underlying C^*-algebra is of real rank zero. The condition in Theorem 5.3 that the extremal boundary is finite is replaced in Theorem 6.6 by the weaker condition that the algebra has quasicontinuous scale, a notion introduced by Kucerovsky and Perera in [28] (see Section 2.7). However, in a future paper, we will show that in general this condition cannot be further weakened.

Another application of strict comparison of positive elements by traces is the characterization of *positive combination of projections* in the multiplier algebra of simple separable C^*-algebras with real rank zero, stable rank one, strict comparison of projections, and finite extremal boundary (Theorem 7.9).

Positive combination of projections (PCP for short) in a C^*-algebra are sums of the form $\sum_1^n \lambda_j p_j$ where p_j are projections in the algebra, λ_j are positive scalars, and n is a finite integer. This notion has been investigated since 1967 as part of the more general study of linear combination of projections and of sums of commutators, *e.g.*, [14–16, 33, 35, 36, 40]). More recently, interest in that topic was rekindled by its connection with frame theory, *e.g.*, [9]. In [21] and [19] we investigated the notion of PCP in the setting of purely infinite C^*-algebras and W*-algebras, respectively (see also [20, 22–24]). Then we proved [25, Theorem 6.1] that if $\mathcal{A}$ is a simple separable stable σ-unital C^*-algebra with real rank zero, stable rank one, strict comparison of projections by traces and has finitely many extremal traces, then $a \in \mathcal{A}_+$ is a PCP if and only if $\tau(R_a) < \infty$ for all $\tau \in \mathcal{T}(\mathcal{A})$, where R_a denotes the range projection, $\mathcal{T}(\mathcal{A})$ denotes the tracial simplex (see Section 2.2), and τ is extended to the enveloping von Neumann algebra. A key ingredient in the proof was Brown's interpolation theorem [4]. If we further assume that $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ has real rank zero, a similar result holds for $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$: a necessary and sufficient condition for $A \in (\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))_+$ to be a PCP is that either $\tau(R_A) < \infty$ for all those $\tau \in \mathcal{T}(\mathcal{A})$ for which A belongs to the trace ideal I_τ or A is a full element [26, Theorem 6.4], where τ is the extension of a trace on $\mathcal{A}$ to $(\mathcal{A} \otimes \mathcal{K})^{**}$.

To remove the restrictive condition that $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ has real rank zero, a completely different approach is used in Theorem 7.9, based on the strict comparison of positive elements in the multiplier algebra provided by Theorem 5.3.

A key step in the proof of Theorem 7.9 is the extension and reformulation of the "2×2" Lemma, 7.2, which played a key role in obtaining PCP decompositions in purely infinite C^*-algebras and in W^*-algebras (see [19, 21]). This lemma also provides bounds on the number of projections needed for a PCP decomposition.

2 Preliminaries

2.1 The Pedersen Ideal and Approximate Identities

For a simple C^*-algebra $\mathcal{A}$ the Pedersen ideal $\mathrm{Ped}(\mathcal{A})$ is the minimal dense ideal of $\mathcal{A}$ (see [41], and also [29]). It contains all the positive elements with a local unit, *i.e.*, the elements $a \in \mathcal{A}_+$ for which there exists $b \in \mathcal{A}_+$ such that $ba = a$. In fact

$$(\mathrm{Ped}(\mathcal{A}))_+ = \Big\{ x \in \mathcal{A}_+ \mid x \le \sum_1^n y_j \text{ for some } n \in \mathbb{N},\ y_j \in \mathcal{A}_+ \text{ with local unit} \Big\}.$$

Let $\mathcal{B}$ be a σ-unital hereditary sub-algebra of $\mathcal{A}$, let h be a strictly positive element of $\mathcal{B}$ with $\|h\| = 1$, and let $e_n := \phi_{\frac{1}{n}}(h)$ where ϕ_ϵ is the continuous function defined by

$$\phi_\epsilon(t) = \begin{cases} 0 & \text{for } t \in \big[0, \frac{\epsilon}{1+\epsilon}\big], \\ \frac{1+\epsilon}{\epsilon^2} t - \frac{1}{\epsilon} & \text{for } t \in \big(\frac{\epsilon}{1+\epsilon}, \epsilon\big), \\ 1 & \text{for } t \in [\epsilon, 1]. \end{cases} \tag{2.1}$$

It is well known, and routine to verify, that $\{e_n\}_1^\infty$ is an approximate identity of $\mathcal{B}$ satisfying

$$e_{n+1} e_n = e_n \quad \forall\, n \tag{2.2}$$

and $e_n \in \mathrm{Ped}(\mathcal{A})$ for all n.

2.2 Traces and Dimension Functions

For a simple C^*-algebra we denote by $\widetilde{\mathcal{T}}(\mathcal{A})$ the collection of the (norm) lower semi-continuous densely defined tracial weights on $\mathcal{A}_+$, henceforth, traces for short. Explicitly, a trace τ

- is an additive and homogeneous map from $\mathcal{A}_+$ into $[0, \infty]$ (a weight);
- satisfies the trace condition $\tau(xx^*) = \tau(x^*x)$ for all $x \in \mathcal{A}$;
- the cone $\{x \in \mathcal{A}_+ \mid \tau(x) < \infty\}$ is norm dense in $\mathcal{A}_+$ (thus τ is also called densely finite, or semifinite);
- satisfies the condition $\tau(x) \le \underline{\lim}\, \tau(x_n)$ for $x, x_n \in \mathcal{A}_+$ and $\|x_n - x\| \to 0$, or equivalently, $\tau(x) = \lim \tau(x_n)$ for $0 \le x_n \uparrow x$ in norm.

Recall that every trace is finite on $\mathrm{Ped}(\mathcal{A})$, and hence $\tau(e_n) < \infty$ for every $\tau \in \widetilde{\mathcal{T}}(\mathcal{A})$ and every approximate identity $\{e_n\}$ of $\mathcal{B}$ satisfying (2.2).

Using the notations in [48], for every $0 \ne f \in \mathrm{Ped}(\mathcal{A})_+$ set

$$\widetilde{\mathcal{T}}(\mathcal{A})_{f \mapsto 1} := \{\tau \in \widetilde{\mathcal{T}}(\mathcal{A}) \mid \tau(f) = 1\}. \tag{2.3}$$

Then $\widetilde{\mathcal{T}}(\mathcal{A})_{f \mapsto 1}$ is a cone base for $\widetilde{\mathcal{T}}(\mathcal{A})$ and can be viewed as a normalization (or scale) of $\widetilde{\mathcal{T}}(\mathcal{A})$. When equipped with the topology of pointwise convergence on $\mathrm{Ped}(\mathcal{A})$, $\widetilde{\mathcal{T}}(\mathcal{A})_{f \mapsto 1}$ is a Choquet simplex [13, 41]; see also [48, Proposition 3.4]. Set

$$\partial_e(\widetilde{\mathcal{T}}(\mathcal{A})_{f \mapsto 1}) \text{ to be the collection of the extreme points of } \widetilde{\mathcal{T}}(\mathcal{A})_{f \mapsto 1}.$$

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We call $\partial_e(\widetilde{\mathcal{T}}(\mathcal{A})_{f\mapsto 1})$ the *extremal boundary* of $\widetilde{\mathcal{T}}(\mathcal{A})_{f\mapsto 1}$ and call its elements *extreme traces*. (For more details, see [13, 41, 48]).

Given two nonzero elements $f, g \in \mathrm{Ped}(\mathcal{A})_+$, the natural one-to-one map

$$\widetilde{\mathcal{T}}(\mathcal{A})_{g\mapsto 1} \ni \tau \mapsto \frac{1}{\tau(f)}\tau \in \widetilde{\mathcal{T}}(\mathcal{A})_{f\mapsto 1}$$

is a homeomorphism that maps faces onto faces and hence, extreme points onto extreme points. In particular, the cardinality of $\partial_e(\widetilde{\mathcal{T}}(\mathcal{A})_{f\mapsto 1})$ does not depend on the element $f \in \mathrm{Ped}(\mathcal{A})_+$ chosen, *cf.* [25]. To simplify notations, we will henceforth denote $\widetilde{\mathcal{T}}(\mathcal{A})_{f\mapsto 1}$ simply by $\mathcal{T}(\mathcal{A})$, dropping the explicit reference to the element f chosen for the normalization.

If $\mathcal{A}$ is unital, then $\mathrm{Ped}(\mathcal{A}) = \mathcal{A}$ and $\mathcal{T}(\mathcal{A})_{1\mapsto 1}$ coincides with the usual tracial state simplex. Thus the definition of $\mathcal{T}(\mathcal{A})$ that we use coincides with the standard one when $\mathcal{A}$ is unital, and hence, by Brown's stabilization theorem [3], also when $\mathcal{A}$ is stable and has a nonzero projection p.

Furthermore, as remarked in [25, 5.3], by [6, Proposition 4.1, Proposition 4.4] and [39, Proposition 5.2], every $\tau \in \mathcal{T}(\mathcal{A})$ has a unique extension, still denoted by τ, to a lower semicontinuous, *i.e.*, normal, tracial weight (trace for short) on the enveloping von Neumann algebra $\mathcal{A}^{**}$. As usual, the dimension function $d_\tau(\cdot)$ is defined on $\mathcal{M}(\mathcal{A})_+$ as

$$d_\tau(A) =: \lim_n \tau(A^{1/n}) \quad \forall\, A \in \mathcal{M}(\mathcal{A})_+,\ \tau \in \mathcal{T}(\mathcal{A}). \tag{2.4}$$

As shown in [39, Remark 5.3],

$$d_\tau(A) = \tau(R_A), \tag{2.5}$$

where R_A is the range projection of A. In particular

$$d_\tau((A-\delta)_+) = \tau(R_{(A-\delta)_+}) = \tau(\chi_{(\delta,\|A\|]}(A)) \quad \forall\, \delta \ge 0.$$

We will also recall that for all $0 \ne A \in \mathcal{M}(\mathcal{A})_+$ both the maps

$$\mathcal{T}(\mathcal{A}) \ni \tau \mapsto d_\tau(A) \in [0,\infty], \text{ and}$$
$$\mathcal{T}(\mathcal{A}) \ni \tau \mapsto \tau(A) = \widehat{A}(\tau) \in [0,\infty],$$

are affine, lower semicontinuous, and strictly positive.

2.3 Cuntz Subequivalence

Let $\mathcal{A}$ be a C^*-algebra. If p, q are projections in $\mathcal{A}$, $p \sim q$ (resp. $p \preceq q$) denotes Murray–von Neumann equivalence, (resp. subequivalence), *i.e.*, $p = vv^*$, $q = v^*v$ for some $v \in \mathcal{A}$ (resp. $p \sim p' \le q$ for some projection $p' \in \mathcal{A}$). If $a, b \in \mathcal{A}_+$, $a \preceq b$ denotes Cuntz sub-equivalence of positive elements, *i.e.*, $\|a - x_n b x_n^*\| \to 0$ for some sequence $x_n \in \mathcal{A}$. For ease of reference we list here the following known facts that we need in this paper and we cite where they can be found, with no attempt to identify where they were first established.

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Lemma 2.1 *Let $\mathcal{A}$ be a C^*-algebra, $a, b \in \mathcal{A}_+$, $\delta > 0$.*

(i) *If $a \leq b$, then $a \precsim b$* [45, Lemma 2.3].
(ii) *If $\|a - b\| < \delta$, then $(a - \delta)_+ \precsim b$* [45, Proposition 2.2].
(iii) *If $a \precsim b$, then there is $\delta' > 0$ and an element $r \in \mathcal{A}$ such that*

$$(a - \delta)_+ = r(b - \delta')_+ r^*.$$

As a consequence, there is an element $s \in \mathcal{A}$ such that $(a - \delta)_+ = sbs^*$ [45, Proposition 2.4].
(iv) *If $a \precsim b$, $a' \precsim b'$, and $a' \perp b'$, then $a + b \precsim a' + b'$* [7, Proposition 1.1].
(v) *If $a \precsim b$, then $d_\tau(a) \leq d_\tau(b)$ for all $\tau \in \mathcal{T}(\mathcal{A})$* [39, 2.3].

Notice that many of the properties in this lemma were proved for the function $f_\delta(a)$ where

$$f_\delta(t) = \begin{cases} 0 & \text{for } t \in [0, \delta], \\ \frac{t-\delta}{\delta} & \text{for } t \in (\delta, 2\delta], \\ 1 & \text{for } t \in (2\delta, \infty). \end{cases}$$

However, it is immediate to see that the same properties hold for the function $(a-\delta)_+$.

We will need an adaptation of [27, Lemma 1.1].

Lemma 2.2 *Let $\mathcal{A}$ be a C^*-algebra, $a, b \in \mathcal{A}_+$, and $\delta > 0$. If $a \leq (b - \delta)_+$, then for every $\epsilon > 0$, $(a - \epsilon)_+ = xbx^*$ for some $x \in \mathcal{A}$ with $\|x\|^2 \leq \frac{\|a\|}{\delta}$. Furthermore, x can be chosen so that $xx^* \leq c_1(a - \epsilon)_+$ and $x^*x \leq c_2(b - \delta)_+$ for some scalars c_1 and c_2.*

Proof By Lemma 2.1 (iii), there is an $s \in \mathcal{A}$ for which $(a - \epsilon)_+ = s(b - \delta)_+ s^*$. Then $\|s(b - \delta)_+^{1/2}\| = \|(a - \epsilon)_+\|^{1/2} \leq \|a\|^{1/2}$. Let

$$h_\epsilon(t) := \begin{cases} \frac{t}{\epsilon} & t \in [0, \epsilon], \\ 1 & t \in [\epsilon, \|a\|], \end{cases} \quad \text{and} \quad g_\delta(t) := \begin{cases} \frac{1}{\delta} & t \in [0, \delta], \\ \frac{1}{t} & t \in [\delta, \|b\|]. \end{cases}.$$

Then both functions are continuous and

$$\|h_\epsilon(a)\| = 1, \quad (a - \epsilon)_+ = h_\epsilon(a)(a - \epsilon)_+,$$
$$\|g_\delta(b)\| = \frac{1}{\delta}, \quad (b - \delta)_+ = g_\delta(b) b (b - \delta)_+.$$

Set $x = h_\epsilon(a) s (b - \delta)_+^{1/2} g_\delta^{1/2}(b)$. Then

$$\begin{aligned} xbx^* &= h_\epsilon(a) s (b - \delta)_+^{1/2} g_\delta^{1/2}(b) b g_\delta^{1/2}(b)(b - \delta)_+^{1/2} s^* h_\epsilon(a) \\ &= h_\epsilon(a) s (b - \delta)_+ s^* h_\epsilon(a) = h_\epsilon(a)(a - \epsilon)_+ h_\epsilon(a) = (a - \epsilon)_+. \end{aligned}$$

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Moreover,

$$\|x\| \le \|h_\epsilon(a)\| \|s(b-\delta)_+^{1/2}\| \, \|g_\delta(b)^{1/2}\| \le \|a\|^{1/2} \frac{1}{\delta^{1/2}},$$

$$\begin{aligned} xx^* &= h_\epsilon(a)s(b-\delta)_+^{1/2} g_\delta(b)(b-\delta)_+^{1/2} s^* h_\epsilon(a) \\ &\le \frac{1}{\delta} h_\epsilon(a)s(b-\delta)_+ s^* h_\epsilon(a) = \frac{1}{\delta} h_\epsilon(a)(a-\epsilon)_+ h_\epsilon(a) \\ &= \frac{1}{\delta}(a-\epsilon)_+, \end{aligned}$$

and

$$\begin{aligned} x^*x &= g_\delta(b)^{1/2}(b-\delta)_+^{1/2} s^* h_\epsilon(a)^2 s(b-\delta)_+^{1/2} g_\delta^{1/2}(b) \\ &\le \|s\|^2 g_\delta(b)^{1/2}(b-\delta)_+ g_\delta^{1/2}(b) \le \frac{\|s\|^2}{\delta}(b-\delta)_+. \end{aligned}$$ ∎

Notice that if $a, b \in \mathcal{A}$ are selfadjoint and $a \le b$, in general it does not follow that $a_+ \le b_+$. However, we often need less.

Lemma 2.3 *Let $\mathcal{A}$ be a C^*-algebra and $a, b \in \mathcal{A}$ be selfadjoint. If $a \le b$, then $a_+ \le b_+$. In particular, $d_\tau((a-\delta)_+) \le d_\tau((b-\delta)_+)$ for all $\delta \ge 0$ and $\tau \in \mathcal{T}(\mathcal{A})$.*

Proof Since $a \le b \le b_+$ and since $\delta(t-\delta)_+ \le t(t-\delta)_+$ for all t and $\delta > 0$, then

$$(a-\delta)_+ \le \frac{(a-\delta)_+^{1/2}}{\sqrt{\delta}} a \frac{(a-\delta)_+^{1/2}}{\sqrt{\delta}} \le \frac{(a-\delta)_+^{1/2}}{\sqrt{\delta}} b_+ \frac{(a-\delta)_+^{1/2}}{\sqrt{\delta}} \le b_+.$$

As a consequence, $(a-\delta)_+ \le b_+$ for all δ and hence $a_+ \le b_+$. ∎

Lemma 2.4 *Let $\mathcal{A}$ be a C^*-algebra, $a, b \in \mathcal{A}_+$, $\delta_i \ge 0$ with $\delta_1 \ge \delta_2 + \delta_3$.*

(i) $(a+b-\delta_1)_+ \le (a-\delta_2)_+ + (b-\delta_3)_+$.

(ii) *$d_\tau(a+b) \le d_\tau(a) + d_\tau(b)$ for all $\tau \in \mathcal{T}(\mathcal{A})$. Equality holds if $a \perp b$.*

(iii) *$d_\tau((a+b-\delta_1)_+) \le d_\tau((a-\delta_2)_+) + d_\tau((b-\delta_3)_+)$ for all $\tau \in \mathcal{T}(\mathcal{A})$.*

Proof (i) Without loss of generality, $\delta_1 = \delta_2 + \delta_3$. Then

$$a + b - \delta_1 = (a-\delta_2) + (b-\delta_3) \le (a-\delta_2)_+ + (b-\delta_3)_+,$$

hence the conclusion follows from Lemma 2.3. (ii) is well known, but can also be obtained directly from (2.5) and the fact that $\tau(p \vee q) \le \tau(p) + \tau(q)$ for any pair of projections p, q in a von Neumann algebra and any trace τ, hence

$$d_\tau(a+b) = \tau(R_{a+b}) = \tau(R_a \vee R_b) \le \tau(R_a) + \tau(R_b) = d_\tau(a) + d_\tau(b).$$

If $a \perp b$, then $R_{a+b} = R_a + R_b$, hence equality holds. (iii) follows from (i), the monotonicity of d_τ with respect to $\preceq$, and (ii). ∎

The following simple fact will be used in Section §7.

Lemma 2.5 *Let $\mathcal{A}$ be a C^*-algebra, $a \in \mathcal{A}_+$, $q \in \mathcal{A}$ be a projection, and $\delta > 0$ a real number. If $q \preceq (a-\delta)_+$, then there is a projection $p \sim q$ such that $a \ge \delta p$. If $a \ge \delta p$ for some projection p, then $p \preceq (a-\delta')_+$ for all $0 \le \delta' < \delta$.*

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Proof Assume that $q \precsim (a-\delta)_+$. Since by Lemma 2.1 (iii),

$$\frac{1}{2}q = \Big(q - \frac{1}{2}\Big)_+ = x(a-\delta)_+ x^* \quad \text{for some } x \in \mathcal{A},$$

it follows that $q = (\sqrt{2}x)(a-\delta)_+(\sqrt{2}x)^*$. Thus

$$q \sim p := (a-\delta)_+^{\frac{1}{2}} 2x^* x (a-\delta)_+^{\frac{1}{2}} \le 2\|x\|^2 (a-\delta)_+.$$

Then $p \le R_{(a-\delta)_+} = \chi_{(\delta,\|a\|]}(a) \le \frac{1}{\delta}a$. Assume now that $a \ge \delta p$ and $0 \le \delta' < \delta$; then $a - \delta' \ge (\delta - \delta')p - \delta' p^\perp$. Hence, by Lemma 2.3,

$$(\delta-\delta')p = \big((\delta-\delta')p - \delta' p^\perp\big)_+ \precsim (a-\delta')_+$$

and thus $p \precsim (a-\delta')_+$. ∎

2.4 Strict Comparison

Definition 2.6 Let $\mathcal{A}$ be a simple unital C^*-algebra with nonempty tracial simplex $\mathcal{T}(\mathcal{A})$. We say that $\mathcal{A}$ has strict comparison of positive elements by traces if $a, b \in \mathcal{A}_+$ and $d_\tau(a) < d_\tau(b)$ for all $\tau \in \mathcal{T}(\mathcal{A})$ implies that $a \precsim b$.

When $\mathcal{A}$ is not unital, dimension functions are not necessarily finite valued, so we will use the same definition with the convention that "$\infty < \infty$", or, equivalently, ask that $d_\tau(a) < d_\tau(b)$ for all $\tau \in \mathcal{T}(\mathcal{A})$ for which $d_\tau(b) < \infty$.

Notice that strict comparison of positive elements by traces often denotes the stronger property requiring that the above conditions hold for all $a, b \in M_\infty(\mathcal{A})_+$, or the still stronger property requiring that they hold for all $a, b \in (\mathcal{A} \otimes \mathcal{K})_+$. Also, replacing *traces* by *lower semicontinuous densely defined* 2-*quasitraces* gives the definition of strict comparison of positive elements by quasitraces. Note in passing that if the strongest of the three-mentioned forms of strict comparison of positive elements by traces holds, namely if $a, b \in (\mathcal{A} \otimes \mathcal{K})_+$ and $d_\tau(a) < d_\tau(b)$ for all $\tau \in \mathcal{T}(\mathcal{A})$ implies that $a \precsim b$, then by [38, Theorem 3.6] all lower semicontinuous densely defined 2-quasitraces are traces (see also [25, Theorem 2.9]).

For multiplier algebras we will use the following definition, where we consider only traces on $\mathcal{M}(\mathcal{A})$ that are extensions of (lower semicontinuous densely defined) traces on $\mathcal{A}$ and we take into account that $\mathcal{M}(\mathcal{A})$ is not simple.

Definition 2.7 Let $\mathcal{A}$ be a simple C^*-algebra with nonempty tracial simplex $\mathcal{T}(\mathcal{A})$. We say that $\mathcal{M}(\mathcal{A})$ has strict comparison of positive elements by traces if $A \precsim B$ for all $A, B \in \mathcal{M}(\mathcal{A})_+$ such that

(i) $d_\tau(A) < d_\tau(B)$ for all those $\tau \in \mathcal{T}(\mathcal{A})$ for which $d_\tau(B) < \infty$,
(ii) $A \in I(B)$.

Condition (ii) (*cf.* [46, Corollary 4.7]) is clearly necessary for having $A \precsim B$ and in general it is not implied by condition (i). Indeed if there is any element $b \in \mathcal{A}_+$ such that $d_\tau(b) = \infty$ for all τ (which is always the case when $\mathcal{A}$ is stable), then every element $A \in \mathcal{M}(\mathcal{A})_+ \in \mathcal{A}$ would satisfy condition (i), but not (ii). However, under additional hypotheses, condition (i) implies condition (ii) (Corollary 2.9).

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Some versions of strict comparison of positive elements are related to the almost unperforation of $W(\mathcal{A})$, *e.g.*, [46]. Our version of strict comparison for positive elements for $\mathcal{M}(\mathcal{A})$ implies almost unperforation of $W(\mathcal{M}(\mathcal{A}))$, but the converse is not true. A counterexample will be presented in a future paper.

2.5 Ideals in $\mathcal{M}(\mathcal{A})$

For every $\tau \in \mathcal{T}(\mathcal{A})$, $K_\tau := \{B \in \mathcal{M}(\mathcal{A})_+ \mid \tau(B) < \infty\}$ is a hereditary cone of $\mathcal{M}(\mathcal{A})_+$ which, by the trace property, satisfies the condition that if $X^*X \in K_\tau$, then $XX^* \in K_\tau$. Let $L(K_\tau) := \{X \in \mathcal{M}(\mathcal{A}) \mid X^*X \in K_\tau\}$ be the associated two-sided ideal of $\mathcal{M}(\mathcal{A})$ and let $I_\tau := \overline{L(K_\tau)}$. By [47, Proposition 3.21] (see also [42, Theorem 1.5.2]), it is easy to see that

$$I_\tau := \overline{\{X \in \mathcal{M}(\mathcal{A}) \mid \tau(X^*X) < \infty\}} = \overline{\operatorname{span}\{K_\tau\}},$$

where the closures are in norm.

The following is also well known (for a proof, see [26, Lemma 2.6])

$$B \in (I_\tau)_+ \text{ if and only if } d_\tau\big((B-\delta)_+\big) < \infty \text{ for every } \delta > 0,\ \tau \in \mathcal{T}(\mathcal{A})\,. \tag{2.6}$$

In particular

$$d_\tau\big((a-\delta)_+\big) < \infty \quad \forall\ a \in \mathcal{A},\ \delta > 0,\ \tau \in \mathcal{T}(\mathcal{A}) \tag{2.7}$$

and if $P \in \mathcal{M}(\mathcal{A})_+$ is a projection and $\tau \in \mathcal{T}(\mathcal{A})$, then

$$P \in I_\tau \iff \tau(P) < \infty. \tag{2.8}$$

While the structure of two-sided norm closed ideals of $\mathcal{M}(\mathcal{A})$ is difficult to analyze in general, a case where this structure is well understood is the following.

Theorem 2.8 ([44, Theorem 4.4]) *Let $\mathcal{A}$ be a simple unital infinite dimensional C^*-algebra with strict comparison of positive elements of $\mathcal{A} \otimes \mathcal{K}$ by traces and finite extremal boundary $\partial_e(\mathcal{T}(\mathcal{A}))$ and let $n = |\partial_e(\mathcal{T}(\mathcal{A}))|$.*

(i) *A proper ideal $\mathcal{J}$ of $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ is maximal if and only if $\mathcal{J} = I_\tau$ for some τ in $\partial_e(\mathcal{T}(\mathcal{A}))$.*
(ii) *If $\mathcal{J}$ is a proper ideal of $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$, then either $\mathcal{J} = \mathcal{A} \otimes \mathcal{K}$ or $J = I_\tau$ for some τ in $\mathcal{T}(\mathcal{A})$.*
(iii) *There are exactly $2^n - 1$ proper ideals of $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ properly containing $\mathcal{A} \otimes \mathcal{K}$.*

2.6 Faces of $\mathcal{T}(\mathcal{A})$

We start by recalling that if $\tau = t\tau_1 + (1-t)\tau_2$ for some $0 < t < 1$ and $\tau_i \in \mathcal{T}(\mathcal{A})$, then $I_\tau = I_{\tau_1} \cap I_{\tau_2}$. From this and from (2.6) it is easy to see that for every $B \in \mathcal{M}(\mathcal{A})_+$ the set $\{\tau \in \mathcal{T}(\mathcal{A}) \mid B \in I_\tau\}$ is a face of $\mathcal{T}(\mathcal{A})$ and $\{\tau \in \mathcal{T}(\mathcal{A}) \mid B \notin I_\tau\}$ is convex, but in general is not a face. We will use extensively the following notation: for every $B \in \mathcal{M}(\mathcal{A})_+$, let

$$F(B) := \operatorname{co}\{\tau \in \partial_e(\mathcal{T}(\mathcal{A})) \mid B \notin I_\tau\} \tag{2.9}$$

denote the convex combination of the extremal traces for which $B \notin I_\tau$. Then $F(B)$ is a face by [18, Proposition 10.10] and clearly, $F(B) \subset \{\tau \in \mathcal{T}(\mathcal{A}) \mid B \notin I_\tau\}$.

Let $F(B)'$ be the complementary face of $F(B)$, *i.e.*, the largest face disjoint from $F(B)$ (this is the union of all the faces disjoint from $F(B)$). Either $F(B)$ or $F(B)'$ can be empty. For this and for other basic results on convexity theory and Choquet simplexes, we refer the reader to [18]. Since the face $\{\tau \in \mathcal{T}(\mathcal{A}) \mid B \in I_\tau\}$ is disjoint from $F(B)$ we have

$$\operatorname{co}\{\tau \in \partial_e(\mathcal{T}(\mathcal{A})) \mid B \in I_\tau\} \subset \{\tau \in \mathcal{T}(\mathcal{A}) \mid B \in I_\tau\} \subset F(B)' \tag{2.10}$$

and hence

$$F(B) \cap \partial_e(\mathcal{T}(\mathcal{A})) = \{\tau \in \partial_e(\mathcal{T}(\mathcal{A})) \mid B \notin I_\tau\}, \tag{2.11}$$

$$F(B)' \cap \partial_e(\mathcal{T}(\mathcal{A})) = \{\tau \in \partial_e(\mathcal{T}(\mathcal{A})) \mid B \in I_\tau\}. \tag{2.12}$$

The inclusions in (2.10) are, in general, proper. Clearly, they are equalities in the case when $|\partial_e(\mathcal{T}(\mathcal{A}))| < \infty$. Moreover, we will see in Section 6 that the second inclusion in (2.10) is also an equality in the case of special interest when $\mathcal{A}$ has quasi-continuous scale.

Recall that when $F(B)$ is closed, then by [18, Theorem 11.28],

$$\mathcal{T}(\mathcal{A}) = F(B) \dot{+} F(B)',$$

is the direct convex sum of $F(B)$ and $F(B)'$, that is, $F(B) \cap F(B)' = \varnothing$ and every $\tau \in \mathcal{T}(\mathcal{A}) \setminus F(B) \cup F(B)'$ has a unique decomposition $\tau = t\mu + (1-t)\mu'$ for some $0 < t < 1$, $\mu \in F(B)$, and $\mu' \in F(B)'$.

As a consequence under the hypotheses of Theorem 2.8, which include the condition that $|\partial_e(\mathcal{T}(\mathcal{A}))| < \infty$ when $B \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})_+ \setminus \mathcal{A} \otimes \mathcal{K}$, then

$$I(B) = \bigcap\{I_\tau \mid \tau \in F(B)'\} = \bigcap\{I_\tau \mid \tau \in F(B)' \cap \partial_e(\mathcal{T}(\mathcal{A}))\}. \tag{2.13}$$

Corollary 2.9 *If $\mathcal{A}$ satisfies the conditions of Theorem 2.8, $A \in (\mathcal{M}(\mathcal{A} \otimes \mathcal{K}))_+$, $P \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K}) \setminus \mathcal{A} \otimes \mathcal{K}$ a projection, and $d_\tau(A) \le d_\tau(P)$ for all $\tau \in \mathcal{T}(\mathcal{A})$ for which $d_\tau(P) < \infty$,* i.e., *for all $\tau \in F(P)'$, it follows that $A \in I(P)$.*

In a future paper, we will show that this result fails to hold when $|\partial_e(\mathcal{T}(\mathcal{A}))| = \infty$.

2.7 Quasicontinuous Scale

Kucerovsky and Perera [28] introduced the notion of quasicontinuous scale for simple C^*-algebras of real rank zero in terms of quasitraces. In this paper we will study the same notion for a larger class of algebras, but in terms of traces.

Definition 2.10 Let $\mathcal{A}$ be a C^*-algebra with nonempty tracial simplex $\mathcal{T}(\mathcal{A})$. The function $S := \widehat{1_{\mathcal{M}(\mathcal{A})}}$ is called the scale of $\mathcal{A}$. The scale S is said to be *quasicontinuous* if the following hold:

(i) the set $F_\infty := \{\tau \in \partial_e(\mathcal{T}(\mathcal{A})) \mid S(\tau) = \infty\}$ is finite (possibly empty) and hence $\operatorname{co}(F_\infty)$ is closed;
(ii) the complementary face F'_∞ of $\operatorname{co}(F_\infty)$ is closed (possibly empty);
(iii) the restriction $S|_{F'_\infty} : F'_\infty \to (0, \infty]$ of the scale S to F'_∞ is continuous and hence finite-valued.

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Notice first that while the scale function S depends on the normalization chosen for $\mathcal{T}(\mathcal{A})$, the quasicontinuity of S does not. Indeed, let g, f be two positive nonzero elements in $\mathrm{Ped}(\mathcal{A})$ and S_g and S_f be the scales relative to $\widetilde{\mathcal{T}}(\mathcal{A})_{g\mapsto 1}$ and $\widetilde{\mathcal{T}}(\mathcal{A})_{f\mapsto 1}$, respectively, (see (2.3)). Let ψ be the homeomorphism $\psi: \widetilde{\mathcal{T}}(\mathcal{A})_{g\to 1} \mapsto \widetilde{\mathcal{T}}(\mathcal{A})_{f\mapsto 1}$given by $\psi(\tau) := \frac{1}{\tau(f)}\tau$. Then $S_f(\psi(\tau)) = \frac{S_g(\tau)}{\widehat{f}(\tau)} \quad \forall\, \tau \in \widetilde{\mathcal{T}}(\mathcal{A})_{g\mapsto 1}$. Since $f \in \mathrm{Ped}(\mathcal{A})$, by the definition of the topology on $\widetilde{\mathcal{T}}(\mathcal{A})_{g\to 1}$, $\widehat{f}$ is a continuous function on $\widetilde{\mathcal{T}}(\mathcal{A})_{g\to 1}$ which by the simplicity of $\mathcal{A}$ never vanishes, thus $\frac{1}{\widehat{f}(\tau)}$ is continuous. Furthermore, as stated in §2.2, ψ maps faces onto faces, thus if S_g satisfies conditions (i)–(iii), so does S_f. Because of this, we can drop the reference to the specific normalization used and just refer to the scale S.

Notice also that when $|\partial_e(\mathcal{T}(\mathcal{A}))| < \infty$, the scale is necessarily quasicontinuous. Indeed then all faces are convex hulls of subsets of $\partial_e(\mathcal{T}(\mathcal{A}))$ and hence are closed, and all functions on a face are continuous.

In the notations introduced in §2.5,

$$F_\infty = F(1_{\mathcal{M}(\mathcal{A})}) \cap \partial_e(\mathcal{T}(\mathcal{A})) \quad \text{and} \quad F'_\infty = F(1_{\mathcal{M}(\mathcal{A})})'.$$

Then by (2.12) and (2.8) $F'_\infty \cap \partial_e(\mathcal{T}(\mathcal{A})) = \{\tau \in \partial_e(\mathcal{T}(\mathcal{A})) \mid S(\tau) < \infty\}$. By (ii) we have from (2.10) and the Krein–Millman theorem that

$$\{\tau \in \mathcal{T}(\mathcal{A}) \mid S(\tau) < \infty\} \subset \overline{\mathrm{co}}\{\tau \in \partial_e(\mathcal{T}(\mathcal{A})) \mid S(\tau) < \infty\} = F'_\infty.$$

On the other hand, by (iii) we have that S is finite on F'_∞ and hence

$$F'_\infty = \{\tau \in \mathcal{T}(\mathcal{A}) \mid S(\tau) < \infty\}.$$

In Section 6 we will use the following lemma.

Lemma 2.11 *Let $\mathcal{A}$ be a C^*-algebra with quasicontinuous scale S. Let $B \in \mathcal{M}(\mathcal{A})_+$. Then $F(B) = \mathrm{co}\{\tau \in F_\infty \mid B \notin I_\tau\}$ and $F(B)' = \mathrm{co}\{\tau \in F_\infty \mid B \in I_\tau\} \dot{+} F'_\infty$. Hence both $F(B)$ and $F(B)'$ are closed.*

Proof If $B \notin I_\tau$ for some $\tau \in \partial_e(\mathcal{T}(\mathcal{A}))$, then necessarily $\tau \in F_\infty$. Thus by (2.9), $F(B) = \mathrm{co}\{\tau \in F_\infty \mid B \notin I_\tau\}$. Since $|F_\infty| < \infty$,

$$\mathrm{co}(F_\infty) = F(B) \dot{+} \mathrm{co}\{\tau \in F_\infty \mid B \in I_\tau\}$$

and both $F(B)$ and $\mathrm{co}\{\tau \in F_\infty \mid B \in I_\tau\}$ are closed faces. But then

$$\mathcal{T}(\mathcal{A}) = \mathrm{co}(F_\infty) \dot{+} F'_\infty = F(B) \dot{+} (\mathrm{co}\{\tau \in F_\infty \mid B \in I_\tau\} \dot{+} F'_\infty). \tag{2.14}$$

Since $\mathrm{co}\{\tau \in F_\infty \mid B \in I_\tau\} \dot{+} F'_\infty$ is the direct convex hull of two closed faces, it is a closed face [18, Proposition 5.2]. It is immediate to verify that the direct complement of a face is unique, *i.e.*, if $F \dot{+} G = F \dot{+} H$ where F, G, H are faces, then $G = H$. Thus from (2.14) we conclude that $F(B)' = \mathrm{co}\{\tau \in F_\infty \mid B \in I_\tau\} \dot{+} F'_\infty$ and again by Definition 2.10, that $F(B)'$ is closed. ∎

3 Dimension Functions of Cut-offs of Monotone Sequences

Lemma 3.1 *Let $\mathcal{A}$ be a C^*-algebra, T_n, T be normal elements of $M(\mathcal{A})$, and $K \subseteq \mathbb{C}$ be a compact set for which the spectrum $\sigma(T_n)$ is contained in K for all n and $\sigma(T) \subseteq K$, and assume that $T_n \to T$ strictly. Then $f(T_n) \to f(T)$ strictly for every continuous function $f: K \to \mathbb{C}$.*

Proof This is immediate when f is a polynomial in one complex variable. Then apply the Stone–Weierstrass theorem. ∎

Lemma 3.2 *Let $\mathcal{A}$ be a σ-unital C^*-algebra, $\tau \in \mathcal{T}(\mathcal{A})$, $T_n, T \in \mathcal{M}(\mathcal{A})_+$, and assume that $T_n \to T$ in the strict topology.*

(i) *If $T_n \leq T_{n+1}$ for all n, then $d_\tau((T_n - \delta)_+) \uparrow d_\tau((T - \delta)_+)$ for all $\delta \geq 0$.*
(ii) *If $T = 0$, $T_n \geq T_{n+1}$ for all n, and $T_1 \in I_\tau$, then $d_\tau((T_n - \delta)_+) \downarrow 0$ for all $\delta > 0$.*
(iii) *If $T_n \geq T_{n+1}$ for all n and $T_1 \in I_\tau$, then for all $0 < \epsilon < \delta$*

$$d_\tau((T - \delta)_+) \leq \lim_n d_\tau((T_n - \delta)_+) \leq d_\tau((T - \delta + \epsilon)_+).$$

Proof Assume without loss of generality that $\|T\| \leq 1$. Since strict convergence implies strong convergence in the enveloping W^*-algebra, it is easy to verify that in case (i) $T_n \leq T$, and in case (ii) $T_n \geq T$.

(i) Since $T_n - \delta \leq T_{n+1} - \delta \leq T - \delta$ for every n and hence, by Lemma 2.3, $(T_n - \delta)_+ \preceq (T_{n+1} - \delta)_+ \preceq (T - \delta)_+$, it follows by Lemma 2.1 (v) that

$$d_\tau((T_n - \delta)_+) \leq d_\tau((T_{n+1} - \delta)_+) \leq d_\tau((T - \delta)_+)$$

and hence

$$\lim_n d_\tau((T_n - \delta)_+) \leq d_\tau((T - \delta)_+). \tag{3.1}$$

Now we prove the opposite inequality.

Since $\mathcal{A}$ is σ-unital, there is an approximate identity of $\mathcal{A}$ consisting of an increasing sequence e_n such that $e_{n+1}e_n = e_n$ for all n. As $T_n \to T$ strictly and since $\sigma(T), \sigma(T_n) \subset [0,1]$ for all n, by Lemma 3.1, it follows that for every $N \in \mathbb{N}$,

$$\begin{aligned} (T_n - \delta)_+^{1/N} &\to (T - \delta)_+^{1/N} \quad \text{strictly, and} \\ \lim_n e_k^{1/2}(T_n - \delta)_+^{1/N} e_k^{1/2} &= e_k^{1/2}(T - \delta)_+^{1/N} e_k^{1/2} \quad \text{in norm.} \end{aligned}$$

Now τ is norm continuous on $e_k^{1/2}\mathcal{M}(\mathcal{A})e_k^{1/2} = e_k^{1/2}\mathcal{A}e_k^{1/2}$ because $e_k \in \operatorname{Ped}(\mathcal{A})$, which implies that $\tau(e_k) < \infty$. As a consequence,

$$\lim_n \tau\big(e_k^{1/2}(T_n - \delta)_+^{1/N} e_k^{1/2}\big) = \tau\big(e_k^{1/2}(T - \delta)_+^{1/N} e_k^{1/2}\big) \tag{3.2}$$

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and thus for all $N \in \mathbb{N}$,

$$\begin{aligned}
\tau\big(((T-\delta)_+)^{1/N}\big) &= \lim_k \tau\big((T-\delta)_+)^{1/2N} e_k ((T-\delta)_+)^{1/2N}\big) && \text{(normality of } \tau)\\
&= \lim_k \tau\big(e_k^{1/2}((T-\delta)_+)^{1/N} e_k^{1/2}\big) && \text{(trace property)}\\
&= \lim_k \lim_n \tau\big(e_k^{1/2}((T_n-\delta)_+)^{1/N} e_k^{1/2}\big) && \text{(by (3.2))}\\
&= \lim_k \lim_n \tau\big(((T_n-\delta)_+)^{1/2N} e_k ((T_n-\delta)_+)^{1/2N}\big) && \text{(trace property)}\\
&\le \varliminf_n \tau\big(((T_n-\delta)_+)^{1/N}\big) && \text{(monotonicity of } \tau)\\
&\le \varliminf_n d_\tau((T_n-\delta)_+) && \text{(as } \|(T_n-\delta)_+\| \le 1)\\
&= \lim_n d_\tau((T_n-\delta)_+) && \text{(as } d_\tau((T_n-\delta)_+)\uparrow).
\end{aligned}$$

Hence $\lim_n d_\tau((T_n-\delta)_+) \ge \lim_N \tau\big(((T-\delta)_+)^{1/N}\big) = d_\tau((T-\delta)_+)$, where the last equality is a consequence of the definition (2.4) of d_τ. This, together with (3.1) completes the proof of part (i).

(ii) Let $\epsilon > 0$ and let $Q_n := \chi_{(\delta,\infty)}(T_n)$, $P_\epsilon := \chi_{(\epsilon,\infty)}(T_1^{1/2})$. These spectral projections belong to the von Neumann algebra $\mathcal{A}^{**}$ and commute with T_n and T_1, respectively. Recall that we identify every $\tau \in \mathcal{T}(\mathcal{A})$ with its extension to $\mathcal{A}^{**}$ (see [39, Proposition 5.2] and also §2.2) and that the trace of the range projection of a positive operator is just the dimension function of that operator. In particular,

$$\tau(Q_n) = d_\tau((T_n-\delta)_+) \le d_\tau((T_1-\delta)_+) = \tau(Q_1) < \infty. \tag{3.3}$$

Since $Q_n \le \frac{1}{\delta} T_n Q_n$, it follows that

$$d_\tau((T_n-\delta)_+) \le \frac{1}{\delta}\tau(T_nQ_n) = \frac{1}{\delta}\big(\tau(P_\epsilon(T_nQ_n)P_\epsilon) + \tau(P_\epsilon^\perp(T_nQ_n)P_\epsilon^\perp)\big). \tag{3.4}$$

Also, $\tau(P_\epsilon) = d_\tau((T_1-\epsilon^2)_+) < \infty$ by (2.6) and the hypothesis that $T_1 \in I_\tau$, and hence τ is σ-weakly continuous on $P_\epsilon \mathcal{A}^{**} P_\epsilon$. Therefore

$$\tau(P_\epsilon(T_nQ_n)P_\epsilon) \le \tau(P_\epsilon T_n P_\epsilon) \to 0. \tag{3.5}$$

Since $T_n \le T_1$, there are elements $G_n \in \mathcal{A}^{**}$ such that $T_n^{1/2} = G_n T_1^{1/2} = T_1^{1/2} G_n^*$ and $\|G_n\| \le 1$ [8, Lemme I.1.2]. Then $\|P_\epsilon^\perp T_n^{1/2}\| = \|\chi_{[0,\epsilon)}(T_1^{1/2}) T_1^{1/2} G_n^*\| \le \epsilon$. From here and (3.3) we have

$$\tau(P_\epsilon^\perp T_n Q_n P_\epsilon^\perp) = \tau(Q_n T_n^{1/2} P_\epsilon^\perp T_n^{1/2} Q_n) \le \epsilon^2 \tau(Q_n) \le \epsilon^2 \tau(Q_1).$$

Thus by (3.5) and (3.4), it follows that $d_\tau((T_n-\delta)_+) \to 0$. (iii) By the same argument as in part (i), $d_\tau((T_n-\delta)_+)\downarrow$ and hence $\lim_n d_\tau((T_n-\delta)_+) \ge d_\tau((T-\delta)_+)$. By Lemma 2.4 (iii), for every $0 < \epsilon < \delta$

$$d_\tau((T_n-\delta)_+) \le d_\tau((T-\delta+\epsilon)_+) + d_\tau((T_n-T-\epsilon)_+).$$

By part (ii), $\lim_n d_\tau((T_n-T-\epsilon)_+) = 0$, which concludes the proof. ∎

Remark 3.3 Unlike in (i), for part (ii) we need to assume that $\delta > 0$. Indeed, let P be a projection with $0 < \tau(P) < \infty$. Then $T_n := \frac{1}{n}P \downarrow 0$ in norm, yet $d_\tau(T_n) \equiv \tau(P) \not\to 0$. Similarly, in (iii) we need to assume that $\epsilon > 0$. Indeed, as above, let P be a projection

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with $0 < \tau(P) < \infty$. Then $T_n := (\delta + \frac{1}{n})P \downarrow \delta P = T$ in norm, yet $d_\tau((T_n - \delta)_+) \equiv \tau(P)$ while $d_\tau((T - \delta)_+) = 0$.

For ease of use in the following section, let us single out the following special case.

Corollary 3.4 *Let $\mathcal{A}$ be σ-unital C^*-algebra, $D := \sum_1^\infty d_k \in \mathcal{M}(\mathcal{A})$ be the sum of a series of elements $d_k \in \mathcal{A}_+$ converging in the strict topology, and let $\tau \in \mathcal{T}(\mathcal{A})$. Then*

(i) $\lim_n d_\tau((\sum_{i=m}^n d_i - \delta)_+) = d_\tau((\sum_{i=m}^\infty a_i - \delta)_+)$ *for every $\delta \geq 0$ and $m \in \mathbb{N}$.*
(ii) *If $D \in I_\tau$, then* $\lim_n d_\tau((\sum_{i=n}^\infty d_i - \delta)_+) = 0$ *for every $\delta > 0$.*

4 Bi-diagonal Decomposition

Inspired by Theorem 2.2 of [51], in the following theorem we decompose positive elements in a general σ-unital C^*-algebra into the sum of a *bi-diagonal* series and *small* remainder in $\mathcal{A}$. Notice that the proof in [51, Theorem 2.2] uses the existence of an approximate identity of projections, while we need only approximate identities of positive elements. Also, here we obtain a bi-diagonal decomposition, rather than the tri-diagonal in [51]. By bi-diagonal we mean the following.

Definition 4.1 Let $\mathcal{A}$ be a C^*-algebra. A series $\sum_1^\infty d_k$ that converges in the strict topology of $\mathcal{M}(\mathcal{A})$ is said to be bi-diagonal if $d_n d_m = 0$ for $|n - m| \geq 2$.

Notice that every bi-diagonal series $\sum_1^\infty d_k$ can be decomposed into the sum of two diagonal series ($\sum_1^\infty d_{2k}$, and $\sum_1^\infty d_{2k+1}$), but the sum of two diagonal series is not necessarily bi-diagonal.

Theorem 4.2 *Let $\mathcal{A}$ be a σ-unital C^*-algebra and let $T \in \mathcal{M}(\mathcal{A})_+$. Then for every $\epsilon > 0$ there exist a bi-diagonal series $\sum_1^\infty d_k$ with each $d_k \in \mathcal{A}_+$ and a selfadjoint element $t_\epsilon \in \mathcal{A}$ with $\|t_\epsilon\| < \epsilon$ such that $T = \sum_1^\infty d_k + t_\epsilon$. The elements d_k can be chosen in* $\mathrm{Ped}(\mathcal{A})$.

For every approximate identity $\{e_n\}$ of $\mathcal{A}$ with $e_{n+1}e_n = e_n$, we can choose d_k and t_ϵ that satisfy the above conditions and such that for every $n \in \mathbb{N}$ there is an $N \in \mathbb{N}$ for which $e_n \sum_N^\infty d_k = 0$.

Proof Let $\{e_n\}$ be an increasing approximate identity of $\mathcal{A}$ and as usual we assume that $e_{n+1}e_n = e_n$ and set $e_0 := 0$ (see (2.2)). As a consequence

$$(e_n - e_{n-1})(e_m - e_{m-1}) = 0 \quad \forall\, |n - m| \geq 2.$$

Assume without loss of generality that $\|T\| = 1$ and let $a_k := T^{1/2}(e_k - e_{k-1})T^{1/2}$. Then $a_k \in \mathcal{A}_+$ for all k and $T = \sum_1^\infty a_k$ where the series converges strictly. We will construct inductively two strictly increasing sequences of positive integers $\{m_k\}_0^\infty$ and $\{n_k\}_0^\infty$ as follows. Start by setting $m_0 := 0$, $n_0 := 0$, $m_1 := 1$, and $b_1 := a_1$. Then choose $n_1 \geq 1$ such that $\|a_1 - e_{n_1} a_1 e_{n_1}\| < \frac{\epsilon}{2}$ since $e_n \to 1$ strictly and $a_1 \in \mathcal{A}$. Now choose $m_2 > m_1$

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and $n_2 > n_1$ such that

$$\|e_{n_1} \sum_{j=m_2+1}^{\infty} a_j\| < \Big(\frac{\epsilon}{2^5}\Big)^2 \qquad \Big(\text{since } \sum_{j=m}^{\infty} a_j \to 0 \text{ strictly and } e_{n_1} \in \mathcal{A}\Big)$$

$$\|(1-e_{n_2}) \sum_{j=m_1+1}^{m_2} a_j\| < \frac{\epsilon}{2^4} \qquad \Big(\text{since } e_n \to 1 \text{ strictly and } \sum_{j=m_1+1}^{m_2} a_j \in \mathcal{A}\Big).$$

Set $b_2 := \sum_{j=m_1+1}^{m_2} a_j$ and iterate the construction.

$$\text{Choose } m_k \text{ so that } \quad \|e_{n_{k-1}} \sum_{j=m_k+1}^{\infty} a_j\| < \Big(\frac{\epsilon}{2^{k+3}}\Big)^2. \tag{4.1}$$

$$\text{Set} \quad b_k := \sum_{j=m_{k-1}+1}^{m_k} a_j,$$

so $\|b_k\| \le \|T\| = 1$.

$$\text{Choose } n_k \text{ so that } \quad \|(1-e_{n_k})b_k\| < \frac{\epsilon}{2^{k+2}}. \tag{4.2}$$

Set for all $k \ge 1$

$$\begin{aligned} c_1 &:= e_{n_1} b_1 e_{n_1} \\ c_k &:= (e_{n_k} - e_{n_{k-2}}) b_k (e_{n_k} - e_{n_{k-2}}) \qquad \forall\, k \ge 2. \end{aligned} \tag{4.3}$$

From (4.1) (applied to $k-1$) we see that

$$\begin{aligned} \|e_{n_{k-2}} b_k\| &\le \|e_{n_{k-2}} b_k^{1/2}\| = \|e_{n_{k-2}} b_k e_{n_{k-2}}\|^{1/2} \\ &\le \|e_{n_{k-2}} \sum_{j=m_{k-1}+1}^{\infty} a_j e_{n_{k-2}}\|^{1/2} \\ &\le \|e_{n_{k-2}} \sum_{j=m_{k-1}+1}^{\infty} a_j\|^{1/2} < \frac{\epsilon}{2^{k+2}}. \end{aligned}$$

From the decomposition

$$b_k - c_k = (1-e_{n_k})b_k + e_{n_k} b_k (1-e_{n_k}) + e_{n_k} b_k e_{n_{k-2}} + e_{n_{k-2}} b_k (e_{n_k} - e_{n_{k-2}})$$

and from the above inequality and (4.2), we thus obtain that $\|b_k - c_k\| < \frac{\epsilon}{2^k}$ $\forall\, k$. As a consequence, the series $t_\epsilon := \sum_{k=1}^{\infty}(b_k - c_k)$ converges in norm and hence $t_\epsilon = t_\epsilon^* \in \mathcal{A}$. Since $T = \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} b_k$, the series $\sum_{k=1}^{\infty} b_k$ converges strictly. Then the series $\sum_{k=1}^{\infty} c_k$, being the sum of the strictly converging series and of a norm converging one, is also strictly converging and $D := \sum_{k=1}^{\infty} c_k = T - t_\epsilon$. Now set

$$d_k := c_{2k-1} + c_{2k} \quad \forall\, k \ge 1. \tag{4.4}$$

Then $D = \sum_{k=1}^{\infty} d_k$ and

$$\begin{aligned} d_1 &= c_1 + c_2 = e_{n_1} b_1 e_{n_1} + e_{n_2} b_2 e_{n_2} \in e_{n_2} \mathcal{A} e_{n_2}, \\ d_k &= (e_{n_{2k-1}} - e_{n_{2k-3}}) b_{2k-1} (e_{n_{2k-1}} - e_{n_{2k-3}}) + (e_{n_{2k}} - e_{n_{2k-2}}) b_{2k} (e_{n_{2k}} - e_{n_{2k-2}}). \end{aligned}$$

As a consequence, $d_n d_m = 0$ for all $|n - m| \ge 2$.

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By construction, all the elements d_k have a local unit and hence belong to $\mathrm{Ped}(\mathcal{A})$. Finally, it is immediate to verify that $e_{n_k}\sum_N^\infty d_j = 0$ for every $N \geq \frac{n_k+4}{2}$. Given $n \in \mathbb{N}$, choose k and N so that $n_k \geq n$ and $N \geq \frac{n_k+4}{2}$. Then $e_n \sum_N^\infty d_j = 0$. ∎

The method of the proof of Theorem 4.2 can be applied to give a joint bi-diagonal form to multiple elements in $\mathcal{M}(\mathcal{A})_+$. Indeed if $T_1, T_2, \dots, T_N \in \mathcal{M}(\mathcal{A})_+$, and we decompose as above each $T_i = \sum_{k=1}^\infty a_{i,k}$, then the sequences $\{m_k\}_0^\infty$ and $\{n_k\}_0^\infty$ can be chosen so to satisfy (4.1)–(4.2) simultaneously for all $1 \leq i \leq N$:

$$\|e_{n_{k-1}} \sum_{j=m_k+1}^{\infty} a_{i,j}\| < \Big(\frac{\epsilon}{2^{k+3}}\Big)^2,$$

$$b_{i,k} := \sum_{j=m_{k-1}+1}^{m_k} a_{i,j},$$

$$\|(1-e_{n_k})b_{i,k}\| < \frac{\epsilon}{2^{k+2}}.$$

Then defining $c_{i,k}$ and $d_{i,k}$ for each $1 \leq i \leq N$ as in (4.3) and (4.4), we see that $d_{i,n}d_{j,m} = 0$ for $|n-m| \geq 2$ and all $1 \leq i,j \leq N$. Thus we obtain the following extension of Theorem 4.2.

Corollary 4.3 *Let $\mathcal{A}$ be a σ-unital C^*-algebra and let $T_1, T_2, \dots, T_N \in \mathcal{M}(\mathcal{A})_+$. Then for every $\epsilon > 0$ there exist N bi-diagonal series $\sum_{k=1}^\infty d_{i,k}$ with $d_{i,k} \in \mathcal{A}_+$ and self-adjoint elements $t_{i,\epsilon} \in \mathcal{A}$, with $\|t_{i,\epsilon}\| < \epsilon$ such that $T_i = \sum_1^\infty d_{i,k} + t_{i,\epsilon}$ and $d_{i,n}d_{j,m} = 0$ for $|n-m| \geq 2$ and all $1 \leq i,j \leq N$. In particular, if $T \in \mathcal{M}(\mathcal{A})$, there is a bi-diagonal series $\sum_{k=1}^\infty d_k$ with $d_k \in \mathcal{A}$ and an element $t_\epsilon \in \mathcal{A}$, with $\|t_\epsilon\| < \epsilon$ such that $T = \sum_1^\infty d_k + t_\epsilon$. If $T = T^*$, the elements d_k and t_ϵ can be chosen selfadjoint.*

Thus, up to a small remainder, every element in $\mathcal{M}(\mathcal{A})_+$ is bi-diagonal and hence the sum of two diagonal series. Diagonal series are used extensively in multiplier algebras. We will need the following result relating Cuntz subequivalence of (cut-offs of) summands in two diagonal series to Cuntz subequivalence of (cut-offs of) their sums. Notice that we do not need to require that the summands belong to $\mathcal{A}$.

Proposition 4.4 *Let $\mathcal{A}$ be a C^*-algebra, $A = \sum_1^\infty A_n$, $B = \sum_1^\infty B_n$ where $A_n, B_n \in \mathcal{M}(\mathcal{A})_+$, $A_nA_m = 0$, $B_nB_m = 0$ for $n \neq m$, and the two series converge in the strict topology. If $A_n \precsim (B_n - \delta)_+$ for some $\delta > 0$ and for all n, then $A \precsim (B-\delta')_+$ for all $0 < \delta' < \delta$.*

Proof Let $\epsilon > 0$. By Lemma 2.2 applied to $A_n \precsim (B_n-\delta)_+ = ((B_n-\delta')_+ - (\delta-\delta'))_+$, for every n there is an $X_n \in \mathcal{A}$ such that

$$\begin{aligned}(A_n-\epsilon)_+ &= X_n(B_n-\delta')_+X_n^*,\\ \|X_n\|^2 &\leq \frac{\|A_n\|}{\delta-\delta'} \leq \frac{\sup_n \|A_n\|}{\delta-\delta'},\\ X_nX_n^* &\leq c_{1,n}(A_n-\epsilon)_+,\\ X_n^*X_n &\leq c_{2,n}(B_n-\delta)_+,\end{aligned}$$

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for some scalars $c_{1,n}$ and $c_{2,n}$. Then

$$X_n X_n^* \le \|X_n\|^2 R_{X_n X_n^*} \le \sup \|X_n\|^2 R_{(A_n-\epsilon)_+} \le \frac{\sup \|X_n\|^2}{\epsilon} A_n.$$

Similarly, $X_n^* X_n \le \frac{\sup \|X_n\|^2}{\delta} B_n$. Since the elements A_n are mutually orthogonal, so are the elements $(A_n - \epsilon)_+$ and hence also the elements $X_n X_n^*$. Similarly, the elements $X_n^* X_n$ are mutually orthogonal and hence $X_n X_m^* = X_n^* X_m = 0$ for all $n \ne m$. Thus for every $m < n \in \mathbb{N}$ and $a \in \mathcal{A}$,

$$\begin{aligned}\Big\| a \sum_{k=m}^{n} X_k \Big\|^2 &= \Big\| a \sum_{k,k'=m}^{n} X_k X_{k'}^* a^* \Big\| = \Big\| a \sum_{k=m}^{n} X_k X_k^* a^* \Big\| \\ &\le \frac{\sup_n \|X_n\|^2}{\epsilon} \Big\| a \sum_{k=m}^{n} A_k a^* \Big\| \le \|a\| \frac{\sup_n \|X_n\|^2}{\epsilon} \Big\| a \sum_{k=m}^{n} A_k \Big\|.\end{aligned}$$

Similarly

$$\Big\| \sum_{k=m}^{n} X_k a \Big\|^2 \le \|a\| \frac{\sup_n \|X_n\|^2}{\delta} \Big\| \sum_{k=m}^{n} B_k a \Big\|.$$

Since the series $\sum_1^\infty A_n$ and $\sum_1^\infty B_n$ converge strictly, it follows that $\sum_1^\infty X_n$ converges strictly. Let $X := \sum_1^\infty X_n$. Then $X \in \mathcal{M}(\mathcal{A})$ and since $X_n = X_n R_{B_n}$ and $R_{B_n}(B_n - \delta')_+ = (B_n - \delta')_+$ for every n,

$$\begin{aligned}(A - \epsilon)_+ &= \sum_1^\infty (A_n - \epsilon)_+ = \sum_1^\infty X_n (B_n - \delta')_+ X_n^* \\ &= \Big(\sum_1^\infty X_n\Big) \sum_1^\infty (B_n - \delta')_+ \Big(\sum_1^\infty X_n^*\Big) = X(B - \delta')_+ X^*.\end{aligned}$$

Since ϵ is arbitrary, it follows that $A \preceq (B - \delta')_+$. ∎

Remark 4.5 From the above proof we see that if the series $\sum_1^\infty A_n$ converges in norm, then the series $\sum_1^\infty X_n$ also converges in norm.

5 Strict Comparison: The Finite Boundary Case

For which simple C^*-algebras $\mathcal{A}$ does strict comparison of positive elements by traces hold for $\mathcal{M}(\mathcal{A})$ when it holds for $\mathcal{A}$? In this section we prove that a sufficient condition is that $\partial_e(\mathcal{T}(\mathcal{A}))$ is finite. In the next section we extend this result to the case when the scale is quasicontinuous (see §2.7).

Our main tool is the following technical lemma, which deals with the case of bi-diagonal series. It is convenient for its further use in Section 6 to formulate this lemma without assuming that the extremal boundary is finite.

Lemma 5.1 *Let $\mathcal{A}$ be a σ-unital nonunital simple C^*-algebra with strict comparison of positive elements by traces. Let $a_i, b_i \in \mathcal{A}_+$ be such that $\sum_{i=1}^\infty a_i$ and $\sum_{i=1}^\infty b_i$ are two bi-diagonal series in $\mathcal{M}(\mathcal{A})_+$. Let F be a closed face of $\mathcal{T}(\mathcal{A})$, F' be its complementary face (either F or F' can be empty), and assume that $|F \cap \partial_e(\mathcal{T}(\mathcal{A}))| < \infty$. Assume also that for some $\epsilon, \delta, \alpha > 0$ we have the following:*

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(i) $(\sum_{i=1}^{\infty} b_i - \delta)_+ \notin \mathcal{A}$;
(ii) $d_\tau((\sum_{i=m}^{\infty} b_i - \delta)_+) = \infty \ \forall\, \tau \in F, m \in \mathbb{N}$;
(iii) $d_\tau((\sum_{i=1}^{\infty} a_i - \epsilon)_+) + \alpha \le d_\tau((\sum_{i=1}^{\infty} b_i - \delta)_+) < \infty \ \forall\, \tau \in F'$;
(iv) $d_\tau((\sum_{i=m}^{n} b_i - \delta)_+) \to d_\tau((\sum_{i=m}^{\infty} b_i - \delta)_+)$ *uniformly on* F', $\forall\, m \in \mathbb{N}$;
(v) $d_\tau((\sum_{i=n}^{\infty} a_i - \epsilon)_+) \to 0$ *uniformly on* F'.

Then $(\sum_{i=1}^{\infty} a_i - 2\epsilon)_+ \precsim (\sum_{i=1}^{\infty} b_i - \delta')_+$ *for all* δ' *with* $0 < \delta' < \delta$.

Proof The main step in the proof is to construct two series $\sum_{k=1}^{\infty} c_k$ and $\sum_{k=1}^{\infty} d_k$, converging strictly and with entries in $\mathcal{A}_+$ such that

(5.1) $$\sum_{k=1}^{\infty} d_k \le \sum_{k=1}^{\infty} b_k \text{ and the series } \sum_{k=1}^{\infty} d_k \text{ is diagonal,}$$

(5.2) $$\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} a_k \text{ and the series } \sum_{k=1}^{\infty} c_k \text{ is bi-diagonal,}$$

(5.3) $$d_\tau\big((c_k - \epsilon)_+\big) < d_\tau\big((d_k - \delta)_+\big) \ \forall\, \tau \in \mathcal{T}(\mathcal{A}),\ k \in \mathbb{N}.$$

We first assume that $F \ne \varnothing$ and $F' \ne \varnothing$. Notice that since the extremal boundary $F \cap \partial_e(\mathcal{T}(\mathcal{A}))$ of F is finite, its convex hull is closed. Thus by the Krein–Millman theorem,

(5.4) $$F = \mathrm{co}(F \cap \partial_e(\mathcal{T}(\mathcal{A}))).$$

Next we construct iteratively three strictly increasing sequences of integers m_k, n_k, and n'_k such that

(5.5) $$m_0 := 0,\ n_0 := -1,\ n'_0 := n_1 \qquad n_k + 2 \le n'_k \le n_{k+1} - 2 \qquad \forall k \ge 1.$$

If we set

$$c_k := \sum_{i=m_{k-1}+1}^{m_k} a_i, \quad g_k := \sum_{i=n_{k-1}+2}^{n'_{k-1}} b_i, \quad h_k := \sum_{i=n'_k+2}^{n_{k+1}} b_i,$$

then for all $k \ge 1$

(5.6) $$d_\tau\Big(\Big(\sum_{i=m_{k-1}+1}^{\infty} a_i - \epsilon\Big)_+\Big) < d_\tau((g_k - \delta)_+) \quad \forall \tau \in F',$$

(5.7) $$d_\tau((c_k - \epsilon)_+) < d_\tau((h_k - \delta)_+) \quad \forall \tau \in F.$$

In the first step of the construction, which is different from the subsequent steps, we choose the integers n_1, n'_1, m_1, and n_2 as follows. By (iii) and (iv), we choose $n_1 \ge 1$ so that

$$d_\tau\Big(\Big(\sum_{i=1}^{n_1} b_i - \delta\Big)_+\Big) > d_\tau\Big(\Big(\sum_{i=1}^{\infty} b_i - \delta\Big)_+\Big) - \alpha \quad \forall\, \tau \in F'.$$

Notice that $\sum_{i=1}^{n_1} b_i = g_1$, so by (iii), condition (5.6) is satisfied for $k = 1$.

Next, by (ii) we have $(\sum_{i=n_1+2}^{\infty} b_i - \delta)_+ \ne 0$. Hence by (iv) (or directly by Lemma 3.1), we can choose $n'_1 \ge n_1 + 2$, so that $(g_2 - \delta)_+ \ne 0$. Since the map $\mathcal{T}(\mathcal{A}) \ni \tau \to d_\tau((g_2 - \delta)_+)$ is lower semicontinuous and strictly positive, we have that

$$\inf_{\tau \in \mathcal{T}(\mathcal{A})} d_\tau((g_2 - \delta)_+) > 0.$$

Then by (v) we choose $m_1 > m_0 = 0$ so that

$$d_\tau\Big(\Big(\sum_{i=m_1+1}^{\infty} a_i - \epsilon\Big)_+\Big) < \inf_{\tau\in\mathcal{T}(\mathcal{A})} d_\tau((g_2-\delta)_+) \le d_\tau((g_2-\delta)_+) \quad \forall\,\tau\in F',$$

that is, condition (5.6) is satisfied for $k = 2$.

Lastly, by (ii) and Corollary 3.4 (i), $d_\tau\big(\big(\sum_{i=n_1'+2}^{n} b_i - \delta\big)_+\big) \uparrow \infty$ for all $\tau \in F$ and this convergence is uniform on F because F is the convex hull of a finite set by (5.4). Moreover, $d_\tau\big(\big(\sum_{i=1}^{m_1} a_i - \epsilon\big)_+\big) < \infty$ for all $\tau \in \mathcal{T}(\mathcal{A})$ by (2.7). Therefore, we can choose $n_2 \ge n_1' + 2$ so that

$$d_\tau\Big(\Big(\sum_{i=1}^{m_1} a_i - \epsilon\Big)_+\Big) < d_\tau\Big(\Big(\sum_{i=n_1'+2}^{n_2} b_i - \delta\Big)_+\Big) \quad \forall\,\tau\in F,$$

i.e., condition (5.7) is satisfied for $k = 1$.

In the second step of the construction, we choose the integers n_2', m_2, and n_3 as follows. Reasoning as in step 1, we choose $n_2' \ge n_2 + 2$ so that $(g_3 - \delta)_+ \ne 0$ and hence $\inf_{\tau\in\mathcal{T}(\mathcal{A})} d_\tau((g_3-\delta)_+) > 0$. Then by (v), we choose $m_2 > m_1$ so that

$$d_\tau\Big(\Big(\sum_{i=m_2+1}^{\infty} a_i - \epsilon\Big)+\Big) < \inf_{\tau\in\mathcal{T}(\mathcal{A})} d_\tau((g_3-\delta)_+) \quad \forall\,\tau\in F',$$

and hence condition (5.6) is satisfied for $k = 3$. Again by (ii), Corollary 3.4 (i), and (5.4), we have $d_\tau\big(\big(\sum_{i=n_2'+2}^{n} b_i - \delta\big)_+\big) \uparrow \infty$ uniformly on F and $d_\tau((c_2-\epsilon)_+) < \infty$ for all $\tau \in \mathcal{T}(\mathcal{A})$ by (2.7). Thus we can choose $n_3 \ge n_2' + 2$ so that condition (5.7) is satisfied for $k = 2$.

The construction now continues as for the case of $k = 2$, that is, assuming we have $\{n_j\}_0^k$, $\{n_j'\}_0^{k-1}$, $\{m_j\}_0^{k-1}$ that satisfy (5.5), (5.6), and (5.7), we choose $n_k' \ge n_k + 2$ to have $\inf_{\tau\in TA} d_\tau((g_{k+1}-\delta)_+) > 0$, $m_k > m_{k-1}$ to satisfy condition (5.6) for $k+1$, and n_{k+1} to satisfy condition (5.7) for k.

Now we draw two conclusions from this construction. First, since nonconsecutive terms in a bi-diagonal series are orthogonal, it is immediate to see that

$$\begin{cases} g_i g_j = 0 & \forall\, i \ne j,\\ g_i h_j = 0 & \forall\, i,\ j,\\ h_i h_j = 0 & \forall\, i\ne j,\\ c_i c_j = 0 & \forall\, |i-j| \ge 2. \end{cases} \tag{5.8}$$

Set $d_k := g_k + h_k$. Then conditions (5.1) and (5.2) are satisfied. Next, we see from Lemma 2.3 and (5.6) that for all $k \ge 1$

$$d_\tau((c_k-\epsilon)_+) < d_\tau((g_k-\delta)_+) \quad \forall\,\tau\in F'. \tag{5.9}$$

But then if $\tau \in F \cup F'$, we have by (5.9) and (5.7) that

$$d_\tau((c_k-\epsilon)_+) < d_\tau((g_k-\delta)_+) + d_\tau((h_k-\delta)_+).$$

Now

$$(g_k-\delta)_+ \perp (h_k-\delta)_+ \tag{5.10}$$

as $g_k \perp h_k$ by (5.8), and thus by Lemma 2.4 (ii),

$$\begin{aligned} d_\tau((g_k-\delta)_+) + d_\tau((h_k-\delta)_+) &= d_\tau((g_k-\delta)_+ + (h_k-\delta)_+) \\ &= d_\tau((g_k+h_k-\delta)_+) \\ &= d_\tau((d_k-\delta)_+). \end{aligned}$$

The second equality follows also from (5.10). Thus we have

$$d_\tau((c_k-\epsilon)_+) < d_\tau((d_k-\delta)_+) \quad \forall\, \tau \in F \cup F'.$$

This inequality extends immediately to all $\tau \in \mathcal{T}(\mathcal{A})$ because, since F is closed, every $\tau \in \mathcal{T}(\mathcal{A})$ is a convex combination of elements in F and F'. Thus (5.3) also holds. This concludes the construction of the two series $\sum_{k=1}^\infty c_k$ and $\sum_{k=1}^\infty d_k$ satisfying (5.1), (5.2), and (5.3) for the case that F and F' are both nonempty.

The cases when F or F' are empty are considerably simpler. Assume that $F' = \varnothing$ and hence $F = \mathcal{T}(\mathcal{A})$. Then $d_\tau((a_k-\epsilon)_+) < \infty$ for all τ by (2.7) and for every m, $d_\tau((\sum_{j=m}^n b_j - \delta)_+) \uparrow \infty$ uniformly because $|\partial_e(\mathcal{T}(\mathcal{A}))| < \infty$. Thus we can choose a strictly increasing sequence of integers n_k starting with $n_0 = 0$, such that if we set $c_k := a_k$ and $d_k := \sum_{j=n_{k-1}+2}^{n_k} b_j$, then

$$d_\tau((c_k-\epsilon)_+) < d_\tau((d_k-\delta)_+) \quad \forall\, \tau \in \mathcal{T}(\mathcal{A}),\ k \in \mathbb{N}.$$

Then the series $\sum_{k=1}^\infty c_k$ and $\sum_{k=1}^\infty d_k$ satisfy conditions (5.1), (5.2), and (5.3).

Assume next that $F = \varnothing$, *i.e.*, $F' = \mathcal{T}(\mathcal{A})$, and by (iii) and (iv) choose n_1 as in the first part of the proof, so that $d_\tau((\sum_{j=1}^\infty a_j - \epsilon)_+) < d_\tau((\sum_{j=1}^{n_1} b_j - \delta)_+)$. By Lemma 2.4 (i), we have for every $m \in \mathbb{N}$, $(\sum_{i=1}^\infty b_i - \delta)_+ \leq \sum_{i=1}^{m-1} b_i + (\sum_{i=m}^\infty b_i - \delta)_+$. Since $\sum_{i=1}^{m-1} b_i \in \mathcal{A}$, it follows by (i) that $(\sum_{i=m}^\infty b_i - \delta)_+ \neq 0$. Since by Lemma 3.1,

$$\Big(\sum_{i=m}^n b_i - \delta\Big)_+ \to \Big(\sum_{i=m}^\infty b_i - \delta\Big)_+$$

in the strict topology, we can find a strictly increasing sequence n_k such that if we set $d_k := \sum_{j=n_{k-1}+2}^{n_k} b_j$, then $(d_k - \delta)_+ \neq 0$. Notice that $d_i d_j = 0$ for all $i \neq j$. By (v) we can choose a strictly increasing sequence m_k starting with $m_0 = 0$ such that

$$d_\tau\Big(\Big(\sum_{i=m_{k-1}+1}^\infty a_j - \epsilon\Big)_+\Big) < d_\tau((d_k-\delta)_+) \quad \forall \tau \in \mathcal{T}(\mathcal{A}).$$

Then set $c_k := \sum_{j=m_{k-1}+1}^{m_k} a_j$ for all k and *a fortiori*, we have

$$d_\tau((c_k-\epsilon)_+) < d_\tau((d_k-\delta)_+) \quad \forall\, \tau \in \mathcal{T}(\mathcal{A}).$$

In this case, too, the series $\sum_{k=1}^\infty c_k$ and $\sum_{k=1}^\infty d_k$ satisfy conditions (5.1), (5.2), and (5.3).

Now if we have two series $\sum_{k=1}^\infty c_k$ and $\sum_{k=1}^\infty d_k$ satisfying conditions (5.1), (5.2), and (5.3), by the hypothesis that $\mathcal{A}$ has strict comparison of positive elements, we obtain that $(c_k-\epsilon)_+ \preceq (d_k-\delta)_+ \ \forall\, k$. Since $\sum_{k=1}^\infty c_k$ is bi-diagonal, both the series $\sum_k c_{2k}$ and $\sum_k c_{2k+1}$ are diagonal and hence, so are the series $\sum_k (c_{2k}-\epsilon)_+$, $\sum_k (c_{2k+1}-\epsilon)_+$. Since $\sum_{k=1}^\infty d_k$ is diagonal, so are also the series $\sum_k d_{2k}$, and $\sum_k d_{2k+1}$. Then by Proposition

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4.4, for every $0 < \delta' < \delta$ we have

$$(5.11)\qquad \Big(\sum_k c_{2k} - \epsilon\Big)_+ = \sum_k (c_{2k} - \epsilon)_+ \preceq \Big(\sum_k d_{2k} - \delta'\Big)_+,$$
$$\Big(\sum_k c_{2k+1} - \epsilon\Big)_+ = \sum_k (c_{2k+1} - \epsilon)_+ \preceq \Big(\sum_k d_{2k+1} - \delta'\Big)_+.$$

As a consequence,

$$\begin{aligned}\Big(\sum_1^\infty a_k - 2\epsilon\Big)_+ &\preceq \Big(\sum_k c_{2k} - \epsilon\Big)_+ + \Big(\sum_k c_{2k+1} - \epsilon\Big)_+ && \text{(by Lemma 2.4 (i))}\\ &\preceq \Big(\sum_k d_{2k} - \delta'\Big)_+ \oplus \Big(\sum_k d_{2k+1} - \delta'\Big)_+ && \text{(by (5.11), Lemma 2.1 (iv))}\\ &= \Big(\sum_k d_k - \delta'\Big)_+ && \text{(since } \textstyle\sum_k d_{2k} \perp \sum_k d_{2k+1})\\ &\preceq \Big(\sum_k b_k - \delta'\Big)_+ && \text{(since } \textstyle\sum_k d_k \preceq \sum_k b_k),\end{aligned}$$

which concludes the proof. ■

For the rest of this section we focus on the case when the extremal boundary $\partial_e(\mathcal{T}(\mathcal{A}))$ is finite. Recall from (2.9) that $F(B) := \mathrm{co}\{\tau \in \partial_e(\mathcal{T}(\mathcal{A})) \mid B \notin I_\tau\}$ and that by the finiteness of $\partial_e(\mathcal{T}(\mathcal{A}))$,

$$(5.12)\qquad F(B)' = \mathrm{co}\{\tau \in \partial_e(\mathcal{T}(\mathcal{A})) \mid B \in I_\tau\} = \{\tau \in \mathcal{T}(\mathcal{A}) \mid B \in I_\tau\}.$$

Lemma 5.2 *Let $\mathcal{A}$ be a σ-unital simple C^*-algebra with $|\partial_e(\mathcal{T}(\mathcal{A}))| < \infty$. Let $A, B \in \mathcal{M}(\mathcal{A})_+$ such that $A \in I(B)$ and $d_\tau(A) < d_\tau(B)$ for all those $\tau \in \mathcal{T}(\mathcal{A})$ for which $d_\tau(B) < \infty$. Then for every $\epsilon > 0$, there are $\delta > 0$ and $\alpha > 0$ such that*

(i) *$d_\tau((A-\epsilon)_+) + \alpha \le d_\tau((B-\delta)_+) < \infty$ if $\tau \in F(B)'$;*
(ii) *$d_\tau((B-\delta)_+) = \infty$ if $\tau \in F(B)$.*

Proof For every $\tau \in \mathcal{T}(\mathcal{A})$ for which $B \notin I_\tau$, by (2.6) there is a $\delta'_\tau > 0$ such that $d_\tau((B-\delta'_\tau)_+) = \infty$. Let

$$\delta' := \begin{cases}\min\{\delta'_\tau \mid \tau \in \partial_e(\mathcal{T}(\mathcal{A})),\ B \notin I_\tau\} & \text{if } \{\tau \in \partial_e(\mathcal{T}(\mathcal{A})),\ B \notin I_\tau\} \neq \varnothing,\\ 1 & \text{if } \{\tau \in \partial_e(\mathcal{T}(\mathcal{A})),\ B \notin I_\tau\} = \varnothing.\end{cases}$$

Since $\partial_e(\mathcal{T}(\mathcal{A}))$ is finite, it follows that $\delta' > 0$. *A fortiori*, $d_\tau((B-\delta')_+) = \infty$ for every $\tau \in \{\tau \in \partial_e(\mathcal{T}(\mathcal{A})) \mid B \notin I_\tau\}$ and hence, by the definition of $F(B)$, for every $\tau \in F(B)$, *i.e.*, (ii) holds for δ'.

Now assume that $B \in I_\tau$ for some $\tau \in \partial_e(\mathcal{T}(\mathcal{A}))$; hence $A \in I_\tau$. By (2.6) we have that $d_\tau((A-\epsilon)_+) < \infty$ and $d_\tau((B-\delta)_+) < \infty$ for every δ. In the case when $d_\tau(B) = \infty$, we can find $\delta''_\tau > 0$:

$$(5.13)\qquad d_\tau((A-\epsilon)_+) < d_\tau((B-\delta''_\tau)_+).$$

By Lemma 3.2 $d_\tau((B-\delta)_+) \to d_\tau(B)$. In the case when $d_\tau(B) < \infty$, we can use the same fact and the inequalities $d_\tau((A-\epsilon)_+) \le d_\tau(A) < d_\tau(B)$ to find a $\delta''_\tau > 0$ for which (5.13) also holds. Let

$$\delta'' := \begin{cases} \min\{\delta''_\tau \mid \tau \in \partial_e(\mathcal{T}(\mathcal{A})),\, B \in I_\tau\} & \text{if } \{\tau \in \partial_e(\mathcal{T}(\mathcal{A})) \mid B \in I_\tau\} \neq \varnothing, \\ 1 & \text{if } \{\tau \in \partial_e(\mathcal{T}(\mathcal{A})) \mid B \in I_\tau\} = \varnothing. \end{cases}$$

Again $\delta'' > 0$, and *a fortiori*, (5.13) holds if we replace δ_τ by δ''.

In case $F(B)' \neq \varnothing$, let

$$\alpha := \min\{d_\tau((B-\delta'')_+) - d_\tau((A-\epsilon)_+) \mid \tau \in \partial_e(\mathcal{T}(\mathcal{A})),\, B \in I_\tau\}.$$

Then $\alpha > 0$ and by (5.12) it is immediate to see that

$$d_\tau((A-\epsilon)_+) + \alpha \leq d_\tau((B-\delta'')_+) \quad \forall\, \tau \in F(B)'.$$

Now set $\delta := \min\{\delta', \delta''\}$. *A fortiori*, (ii) and the first inequality in (i) hold for δ. The second inequality in (i) holds by (2.6). ∎

We are now in position to state and prove our main theorem.

Theorem 5.3 *Let $\mathcal{A}$ be a σ-unital simple C^*-algebra with strict comparison of positive elements by traces and with $|\partial_e(\mathcal{T}(\mathcal{A}))| < \infty$. Then strict comparison of positive element by traces holds in $\mathcal{M}(\mathcal{A})$.*

Proof Let $A, B \in \mathcal{M}(\mathcal{A})_+$ such that $A \in I(B)$ and $d_\tau(A) < d_\tau(B)$ for all those $\tau \in \mathcal{T}(\mathcal{A})$ for which $d_\tau(B) < \infty$. Since strict comparison holds on $\mathcal{A}$, we can assume without loss of generality that $B \notin \mathcal{A}$. Since $(B-\delta)_+ \to B$ in norm as $\delta \to 0$, there is some $\delta' > 0$ such that $(B-\delta')_+ \notin \mathcal{A}$. Let $\epsilon > 0$. By Lemma 5.2 we can choose $\delta'' > 0$ and $\alpha > 0$ such that

$$\begin{cases} d_\tau((A-\epsilon)_+) + \alpha \leq d_\tau((B-\delta'')_+) < \infty & \text{if } \tau \in F(B)', \\ d_\tau((B-\delta'')_+) = \infty & \text{if } \tau \in F(B). \end{cases} \tag{5.14}$$

For every $\tau \in F(B)'$, $B \in I_\tau$ by (2.10) and hence $d_\tau((B-\nu)_+) < \infty$ for all $\nu > 0$ by (2.6). Thus (5.14) holds also if we replace δ'' with $\min\{\delta', \delta''\}$. Notice also that then

$$(B-\delta)_+ \notin \mathcal{A}. \tag{5.15}$$

By Theorem 4.2 we can find bi-diagonal decompositions $A = \sum_{i=1}^\infty a_i + a_0$ and $B = \sum_{i=1}^\infty b_i + b_0$, where the series converge strictly, $a_i, b_i \in \mathcal{A}_+$ (in fact they are in $\mathrm{Ped}(\mathcal{A})$), $a_i a_j = 0$, $b_i b_j = 0$ for $|i-j| \geq 2$, $a_0, b_0 \in \mathcal{A}_{sa}$, $\|a_0\| < \epsilon$, and $\|b_0\| < \frac{\delta}{4}$. Our next step is to verify that the hypotheses of Lemma 5.1 are satisfied for the two bi-diagonal series $\sum_{i=1}^\infty a_i$, $\sum_{i=1}^\infty b_i$, the face $F = F(B)$, and the scalars 2ϵ, $\frac{\delta}{2}$, and α. First of all, notice that $|F \cap \partial_e(\mathcal{T}(\mathcal{A}))| \leq |\partial_e(\mathcal{T}(\mathcal{A}))| < \infty$ and that the face F is closed as are all the faces of $\mathcal{T}(\mathcal{A})$. Furthermore, pointwise convergence of affine finite-valued functions on F' is necessarily uniform.

By Lemma 2.1 (ii) and (i),

$$(B-\delta)_+ \leq \Big(\sum_{i=1}^\infty b_i - \frac{3\delta}{4}\Big)_+ \leq \Big(\sum_{i=1}^\infty b_i - \frac{\delta}{2}\Big)_+, \tag{5.16}$$

$$\Big(\sum_{i=1}^\infty a_i - 2\epsilon\Big)_+ \leq (A-\epsilon)_+. \tag{5.17}$$

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From (5.16) we see that $(\sum_{i=1}^{\infty} b_i - \frac{\delta}{2})_+ \notin \mathcal{A}$, thus satisfying Lemma 5.1 (i).

By Lemma 2.4 (iii) and (5.14), for every $m \in \mathbb{N}$ and $\tau \in F$,

$$\infty = d_\tau\Big(\big(\sum_{i=1}^{\infty} b_i - \frac{3\delta}{4}\big)_+\Big) \leq d_\tau\Big(\big(\sum_{i=1}^{m-1} b_i - \frac{\delta}{4}\big)_+\Big) + d_\tau\Big(\big(\sum_{i=m}^{\infty} b_i - \frac{\delta}{2}\big)_+\Big).$$

Since $d_\tau\Big(\big(\sum_{i=1}^{m-1} b_i - \frac{\delta}{4}\big)_+\Big) < \infty$ for all $\tau \in \mathcal{T}(\mathcal{A})$ by (2.7), it follows that

$$d_\tau\Big(\big(\sum_{i=m}^{\infty} b_i - \frac{\delta}{2}\big)_+\Big) = \infty \quad \forall\, \tau \in F,\ m \in \mathbb{N}, \tag{5.18}$$

which satisfies Lemma 5.1 (ii).

From (5.14), (5.16), and (5.17) we have

$$d_\tau\Big(\big(\sum_{i=1}^{\infty} a_i - 2\epsilon\big)_+\Big) + \alpha \leq d_\tau\Big(\big(\sum_{i=1}^{\infty} b_i - \frac{\delta}{2}\big)_+\Big) \quad \forall\, \tau \in F'. \tag{5.19}$$

Moreover, $B - \sum_{i=1}^{\infty} b_i \in \mathcal{A}$, thus for every $\tau \in F'$, $\sum_{i=1}^{\infty} b_i \in I_\tau$ and hence by (2.6), $d_\tau((\sum_{i=1}^{\infty} b_i - \frac{\delta}{2})_+) < \infty$. Thus Lemma 5.1 (iii) is satisfied.

By Corollary 3.4 (i), for all $m \in \mathbb{N}$

$$d_\tau\Big(\big(\sum_{i=m}^{n} b_i - \frac{\delta}{2}\big)_+\Big) \to d_\tau\Big(\big(\sum_{i=m}^{\infty} b_i - \frac{\delta}{2}\big)_+\Big) \quad \forall\, \tau \in \mathcal{T}(\mathcal{A}).$$

As stated above, the convergence is uniform on F'. Thus Lemma 5.1 (iv) is satisfied.

For all $\tau \in F'$ we have $B \in I_\tau$, hence $A \in I_\tau$. Since $a_0 \in \mathcal{A} \subset I_\tau$, it follows that $\sum_1^{\infty} a_i = A - a_0 \in I_\tau$. By Corollary 3.4 (ii), $d_\tau((\sum_{j=n}^{\infty} a_j - 2\epsilon)_+) \to 0$ for every $\tau \in F'$ and again the convergence is uniform on F'. Thus Lemma 5.1 (v) is satisfied.

Thus all the conditions of Lemma 5.1 being satisfied, it follows that

$$\big(\sum_{i=1}^{\infty} a_i - 4\epsilon\big)_+ \preceq \big(\sum_{i=1}^{\infty} b_i - \frac{\delta}{4}\big)_+.$$

By Lemma 2.1 (ii) we have

$$(A - 5\epsilon)_+ \preceq \big(\sum_{i=1}^{\infty} a_i - 4\epsilon\big)_+ \quad \text{and} \quad \big(\sum_{i=1}^{\infty} b_i - \frac{\delta}{4}\big)_+ \preceq B.$$

Thus $(A - 5\epsilon)_+ \preceq B$ for every $\epsilon > 0$, and hence $A \preceq B$. ∎

6 Strict Comparison: The Quasicontinuous Scale Case

In the previous section we have shown that if a σ-unital simple C^*-algebra with strict comparison of positive elements by traces has finite extremal boundary, then strict comparison of positive elements by traces holds in $\mathcal{M}(\mathcal{A})$ (Theorem 5.3). In this section we prove that the same result holds in the more general case when the algebra has a quasicontinuous scale (see Definition 2.10).

Now we start with the following lemmas.

Lemma 6.1 *Let $\mathcal{A}$ be a simple C^*-algebra, $K \subset \mathcal{T}(\mathcal{A})$ a closed set, and $A \preceq B \in \mathcal{M}(\mathcal{A})_+$. If $\widehat{B}|_K$ is continuous, then $\widehat{A}|_K$ also is continuous.*

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Proof As $\widehat{B}\mid_K = \widehat{A}\mid_K + \widehat{B-A}\mid_K$ and since the first function is continuous and the second two functions are lower semicontinuous, it is immediate to see that both must be continuous. ∎

Lemma 6.2 *Let $\mathcal{A}$ be a σ-unital simple C^*-algebra, $K \subset \mathcal{T}(\mathcal{A})$ a closed set, $A, B \in \mathcal{M}(\mathcal{A})_+$ with $\widehat{A}\mid_K$ continuous, and assume that $d_\tau(A) < d_\tau(B)$ for all $\tau \in K$ for which $d_\tau(B) < \infty$. Then for every $\epsilon > 0$, there exist $\delta > 0$ and $\alpha > 0$ such that*

$$d_\tau((A-\epsilon)_+) + \alpha \le d_\tau((B-\delta)_+) \quad \forall\, \tau \in K.$$

Furthermore, if $\widehat{B}\mid_K$ is continuous, then $d_\tau((B-\delta)_+) < \infty$ for all $\tau \in K$.

Proof Assume without loss of generality that $\|A\| = 1$ and let ϕ_ϵ be the function defined in (2.1). Then $\chi_{(\epsilon,1]}(t) \le \phi_\epsilon(t) \le \min\{\chi_{(0,1]}(t), \frac{t}{\epsilon}\}$, and hence

$$R_{(A-\epsilon)_+} \le \phi_\epsilon(A), \tag{6.1}$$

$$\phi_\epsilon(A) \le \frac{1}{\epsilon}A, \tag{6.2}$$

$$\phi_\epsilon(A) \le R_A. \tag{6.3}$$

From (6.1) we have

$$\widehat{\phi_\epsilon(A)}(\tau) \ge d_\tau((A-\epsilon)_+) \quad \forall \tau \in \mathcal{T}(\mathcal{A}). \tag{6.4}$$

From (6.2) and Lemma 6.1 it follows that $\widehat{\phi_\epsilon(A)}\mid_K$ is continuous. From (6.3) it follows that $\widehat{\phi_\epsilon(A)}(\tau) \le \tau(R_A) = d_\tau(A) \le d_\tau(B)$, with the last inequality being strict when $d_\tau(B) < \infty$. As a consequence, the function $(d_\tau(B) - \widehat{\phi_\epsilon(A)}(\tau))\mid_K$ is strictly positive lower semicontinuous, and hence

$$\alpha := \frac{1}{2}\min\{d_\tau(B) - \widehat{\phi_\epsilon(A)}(\tau))\mid_K\} > 0.$$

Let $B_n = (B - \frac{1}{n})_+$. Then $0 \le B_n \uparrow B$ (in norm) and hence $d_\tau(B_n) \uparrow d_\tau(B)$ for every $\tau \in \mathcal{T}(\mathcal{A})$. Since all the functions $d_\tau(B_n)$ are lower semicontinuous, by the compactness of K there is an n such that $d_\tau(B_n) \ge \widehat{\phi_\epsilon(A)}(\tau) + \alpha\ \forall \tau \in K$. Thus for $\delta := \frac{1}{n}$ we have

$$d_\tau((B-\delta)_+) \ge \widehat{\phi_\epsilon(A)}(\tau) + \alpha \ge d_\tau((A-\epsilon)_+) + \alpha \quad \forall \tau \in K,$$

where the last inequality follows from (6.4).

If in addition $\widehat{B}$ is continuous on K, then by the same reasoning as for A, for every $\delta > 0$ we have $d_\tau((B-\delta)_+) \le \widehat{\phi_\delta(B)}(\tau) \le \frac{1}{\delta}\widehat{B}(\tau)$. Thus $d_\tau((B-\delta)_+) < \infty$ for every $\tau \in K$. ∎

The previous two lemmas permit us to extend Lemma 5.2 to the case when $\mathcal{A}$ has quasicontinuous scale.

Lemma 6.3 *Let $\mathcal{A}$ be a σ-unital simple C^*-algebra with quasicontinuous scale. Let $A, B \in \mathcal{M}(\mathcal{A})_+$ such that $A \in I(B)$ and $d_\tau(A) < d_\tau(B)$ for all those $\tau \in \mathcal{T}(\mathcal{A})$ for which $d_\tau(B) < \infty$. Then for every $\epsilon > 0$ there are $\delta > 0$ and $\alpha > 0$ such that*

(i) $d_\tau((A-\epsilon)_+) + \alpha \le d_\tau((B-\delta)_+) < \infty$ *if* $\tau \in F(B)'$,
(ii) $d_\tau((B-\delta)_+) = \infty$ *if* $\tau \in F(B)$.

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Proof Assume without loss of generality that $\|A\| \leq 1$, $\|B\| \leq 1$. For every $\tau \in \mathcal{T}(\mathcal{A})$ for which $B \notin I_\tau$, by (2.6) there is a $\delta'_\tau > 0$ such that $d_\tau((B - \delta'_\tau)_+) = \infty$. Let

$$\delta' := \begin{cases} \min\{\delta'_\tau \mid \tau \in F_\infty, B \notin I_\tau\} & \text{if } \{\tau \in F_\infty, B \notin I_\tau\} \neq \varnothing, \\ 1 & \text{if } \{\tau \in F_\infty, B \notin I_\tau\} = \varnothing. \end{cases}$$

Since F_∞ is finite, $\delta' > 0$ is finite, and *a fortiori*, for every $\tau \in \{\tau \in F_\infty \mid B \notin I_\tau\}$ and hence for every $\tau \in \operatorname{co}\{\tau \in F_\infty \mid B \notin I_\tau\}$, it follows that $d_\tau((B - \delta')_+) = \infty$. Since by Lemma 2.11 the latter set coincides with $F(B)$, we conclude that (ii) holds for δ' and hence for any $0 < \delta \leq \delta'$.

Now assume that $B \in I_\tau$ for some $\tau \in F_\infty$. Hence $A \in I_\tau$. By (2.6) we have that $d_\tau((A - \epsilon)_+) < \infty$ and $d_\tau((B - \delta)_+) < \infty$ for every δ. Since $d_\tau((B - \delta)_+) \to d_\tau(B)$ we can find $\delta''_\tau > 0$ such that

$$d_\tau((A - \epsilon)_+) < d_\tau((B - \delta''_\tau)_+). \tag{6.5}$$

In the case when $d_\tau(B) = \infty$, this is obvious, and in the case when $d_\tau(B) < \infty$, this follows from the inequality $d_\tau((A - \epsilon)_+) \leq d_\tau(A) < d_\tau(B)$. Let

$$\delta'' := \begin{cases} \min\{\delta''_\tau \mid \tau \in F_\infty, B \in I_\tau\} & \text{if } \{\tau \in F_\infty \mid B \in I_\tau\} \neq \varnothing, \\ 1 & \text{if } \{\tau \in F_\infty \mid B \in I_\tau\} = \varnothing. \end{cases}$$

Again $\delta'' > 0$, and *a fortiori*, (6.5) holds if we replace δ''_τ by δ''.

If $\{\tau \in F_\infty \mid B \in I_\tau\} \neq \varnothing$, let

$$\alpha'' := \min\{d_\tau((B - \delta'')_+) - d_\tau((A - \epsilon)_+) \mid \tau \in F_\infty, B \in I_\tau\}.$$

Then $\alpha'' > 0$ and

$$d_\tau((A - \epsilon)_+) + \alpha'' \leq d_\tau((B - \delta'')_+) \quad \forall\, \tau \in \operatorname{co}\{\tau \in F_\infty \mid B \in I_\tau\}. \tag{6.6}$$

Since $A, B \leq 1_{\mathcal{M}(\mathcal{A})}$ and since $S|_{F'_\infty}$ is continuous by hypothesis, the functions $\widehat{A}|_{F'_\infty}$ and $\widehat{B}|_{F'_\infty}$ are also continuous by Lemma 6.1. Thus by Lemma 6.2 there are $\delta''' > 0$ and $\alpha''' > 0$ such that

$$d_\tau((A - \epsilon)_+) + \alpha''' \leq d_\tau((B - \delta''')_+) < \infty \quad \forall\, \tau \in F'_\infty. \tag{6.7}$$

Now set $\delta := \min\{\delta', \delta'', \delta''\}$ and $\alpha := \min\{\alpha'', \alpha'''\}$. It is obvious that (6.6) holds if we replace α'' and δ'' by α and δ, respectively. Also, by (2.6) we have that $d_\tau((B - \delta'')_+) < \infty$ for all $\tau \in \operatorname{co}\{\tau \in F_\infty \mid B \in I_\tau\}$. Similarly, (6.7) holds if we replace α''' and δ''' by α and δ, respectively. Thus

$$d_\tau((A - \epsilon)_+) + \alpha \leq d_\tau((B - \delta)_+) < \infty \quad \forall\, \tau \in \operatorname{co}\{\tau \in F_\infty \mid B \in I_\tau\} \cup F'_\infty. \tag{6.8}$$

Since by Lemma 2.11 $F(B)' = \operatorname{co}\{\tau \in F_\infty \mid B \in I_\tau\} \dot{+} F'_\infty$, it is immediate to see that (6.8) holds for all $\tau \in F(B)'$, that is, (i) holds. ■

Lemma 6.4 *Let $\mathcal{A}$ be a σ-unital nonunital simple C^*-algebra, $P \in \mathcal{M}(\mathcal{A})$ a projection, $K \subset \mathcal{T}(\mathcal{A})$ a closed set such that $\widehat{P}\,|_K$ is continuous, and let $\sum_{j=1}^\infty A_j$ be the strictly converging sum of elements $A_j \in (P\mathcal{M}(\mathcal{A})P)_+$. Assume furthermore that there exists an increasing approximate identity $\{e_n\}_{n=1}^\infty$ for $(P\mathcal{A}P)_+$ with $e_{n+1}e_n = e_n$ for all $n \in \mathbb{N}$ such that for all $m \geq 1$, there exists $N \in \mathbb{N}$ with $e_m \sum_{j=N}^\infty A_j = 0$. Then for every $\delta \geq 0$,*

(i) $d_\tau((\sum_{j=n}^\infty A_j - \delta)_+) \to 0$ *uniformly on K,*

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(ii) $d_\tau((\sum_{j=1}^n A_j - \delta)_+) \to d_\tau((\sum_{j=1}^\infty A_j - \delta)_+)$ *uniformly on* K.

Proof Assume without loss of generality that $\| \sum_{j=1}^\infty A_j\| \le 1$ and let $\epsilon > 0$ be given. (i) Since $d_\tau((\sum_{j=n}^\infty A_j - \delta)_+) \le d_\tau(\sum_{j=n}^\infty A_j)$ for every n by Lemma 2.3, it is enough to prove the statement for $\delta = 0$.

Since e_n has a local unit, it belongs to the Pedersen ideal and hence by the definition of the topology on $\mathcal{T}(\mathcal{A})$, $\widehat{e_n}$ is continuous. As $\widehat{e_n} \uparrow \widehat{P}$, and $\widehat{P}\,|_K$ is continuous, by Dini's theorem the convergence is uniform on K. Thus choose m such that $0 \le \widehat{P} - \widehat{e_{m-1}} < \epsilon$ on K. Now choose N such that $e_m \sum_{j=N}^\infty A_j = 0$. Then for every $n \ge N$

$$\sum_{j=n}^\infty A_j = (P - e_m)\Big(\sum_n^\infty A_j\Big)(P - e_m) \le (P - e_m)^2 \le P - e_m.$$

Since $R_{\sum_{j=n}^\infty A_j} \le R_{P-e_m} \le P - e_{m-1}$, because $(P - e_{m-1})(P - e_m) = (P - e_m)$, we thus have for every $\tau \in K$ that $d_\tau(\sum_{j=n}^\infty A_j) \le \tau(P - e_{m-1}) < \epsilon$, which proves (i).

(ii) By Lemma 2.3 and Lemma 2.4 (iii) we have, for all $n \ge 1$ and $\tau \in K$, that

$$\begin{aligned} d_\tau\Big(\Big(\sum_{j=1}^n A_j - \delta\Big)_+\Big) &\le d_\tau\Big(\Big(\sum_{j=1}^\infty A_j - \delta\Big)_+\Big) \\ &\le d_\tau\Big(\Big(\sum_{j=1}^n A_j - \delta\Big)_+\Big) + d_\tau\Big(\sum_{j=n+1}^\infty A_j\Big). \end{aligned}$$

Thus (ii) follows from (i). ∎

Remark 6.5 The condition that for every n there exists an $N \in \mathbb{N}$ such that

$$e_n \sum_{j=N}^\infty A_j = 0$$

cannot be removed for $\delta = 0$. Consider for instance an element $b \in \mathcal{A}_+$ such that $\|b\| = 1$ and $R_b = P$ and let $A_n := \frac{1}{2^n}\phi_{1/n}(b)$ (see (2.1)). Then $\sum_1^\infty A_n$ converges in norm, hence strictly, but since $R_{\sum_n^\infty A_n} = R_b$ for all n, it follows that $d_\tau(\sum_n^\infty A_n) \not\to 0$.

Set $A := \sum_{j=1}^\infty A_j$. Then by Lemma 6.1, $\widehat{A}$ is continuous on K. Notice that if we substitute the continuity of $\widehat{P}\,|_K$ with the weaker condition of the continuity of $\widehat{A}\,|_K$, we still obtain uniform convergence on K for every $\delta > 0$. Indeed, by Dini's theorem, $\tau(\sum_n^\infty A_j) \to 0$ uniformly on K and hence for every $\delta > 0$ so does

$$d_\tau\Big(\Big(\sum_n^\infty A_j - \delta\Big)_+\Big) \le \frac{1}{\delta}\tau\Big(\sum_n^\infty A_j\Big).$$

However, this convergence does not hold for $\delta = 0$ as we see by considering the case of $\mathcal{A} := \mathcal{B}\otimes\mathcal{K}$ with $\mathcal{B}$ unital and simple, $K = \mathcal{T}(\mathcal{A})$, $P = 1_{\mathcal{M}(\mathcal{A})}$, $A_k := \frac{1}{2^k}1_{\mathcal{B}} \otimes e_{k,k}$, and $A := \sum_1^\infty A_k$. Then $\widehat{A}(\tau) = 1$ for all $\tau \in \mathcal{T}(\mathcal{A})$, hence it is continuous, and for all m we have $1_{\mathcal{B}} \otimes e_{m,m} \sum_{m+1}^\infty A_k = 0$, but $d_\tau(\sum_n^\infty A_k) = \infty$ for all $\tau \in \mathcal{T}(\mathcal{A})$ and $n \in \mathbb{N}$.

Theorem 6.6 *Let $\mathcal{A}$ be a σ-unital simple C^*-algebra with strict comparison of positive elements by traces and with quasicontinuous scale. Then strict comparison of positive element by traces holds in $\mathcal{M}(\mathcal{A})$.*

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Proof Let $A, B \in \mathcal{M}(\mathcal{A})_+$ such that $A \in I(B)$ and $d_\tau(A) < d_\tau(B)$ for all those $\tau \in \mathcal{T}(\mathcal{A})$ for which $d_\tau(B) < \infty$. Let $\epsilon > 0$. Assume without loss of generality that $\|A\| \leq 1$, $\|B\| \leq 1$, and that $B \notin \mathcal{A}$. Then

$$(B - \delta')_+ \notin \mathcal{A} \tag{6.9}$$

for some $\delta' > 0$. By Lemma 6.3, there is an $\alpha > 0$ and $\delta'' > 0$ such that

$$\begin{aligned} d_\tau((A-\epsilon)_+) + \alpha \leq d_\tau((B-\delta'')_+) < \infty \qquad & \forall \tau \in F(B)' \\ d_\tau((B-\delta'')_+) = \infty \qquad & \forall \tau \in F(B). \end{aligned} \tag{6.10}$$

Clearly, (6.9) and (6.10) hold if we replace δ' and δ'' with $\delta := \min\{\delta', \delta''\}$.

Since we have obtained the same conditions as in (5.14) and (5.15), now we can proceed as in the remainder of the proof of Theorem 5.3. Thus we decompose A and B into bidiagonal series:

$$A = \sum_{i=1}^{\infty} a_i + a_0 \quad \text{and} \quad B = \sum_{i=1}^{\infty} b_i + b_0, \quad \|a_0\| < \epsilon, \|b_0\| < \frac{\delta}{4}.$$

Here we will also use the fact that both bidiagonal series can be chosen so that for every $n \in \mathbb{N}$ there is an $N \in \mathbb{N}$ for which $e_n \sum_N^\infty a_k = e_n \sum_N^\infty b_k = 0$ for some approximate identity $\{e_n\}$ satisfying the condition $e_{n+1}e_n = e_n$ for all n (see Theorem 4.2). Then we proceed to verify that the hypotheses of Lemma 5.1 are satisfied for the two bidiagonal series $\sum_{i=1}^\infty a_i$, $\sum_{i=1}^\infty b_i$, the face $F = F(B)$, and the scalars 2ϵ, $\frac{\delta}{2}$, and α.

By (2.11), $F(B) \cap \partial_e(\mathcal{T}(\mathcal{A})) \subset F_\infty$ and hence it is finite. By Lemma 2.11, $F(B)'$ is closed. By the same reasoning as in the proof of Theorem 5.3, we obtain (5.16), (5.17), (5.18), and (5.19), which show that conditions (i), (ii), and (iii) of Lemma 5.1 are satisfied.

Finally, conditions (iv) and (v) are also satisfied, because by Lemma 6.4 applied to $P = 1_{\mathcal{M}(\mathcal{A})}$ and $K = F'_\infty$, the convergence of both limits is uniform on F'_∞. Furthermore, by Corollary 3.4, the convergence is pointwise on $\{\tau \in F_\infty \mid B \in I_\tau\}$ and by the finiteness of F_∞, it is uniform on $\operatorname{co}\{\tau \in F_\infty \mid B \in I_\tau\}$. Thus all the conditions of Lemma 5.1 are satisfied and the rest of the proof of Theorem 5.3 applies without change. ∎

In a future paper, we will study the case when the extremal boundary is infinite, and we show that for a large class of C^*-algebras $\mathcal{A}$, strict comparison of positive elements by traces holds in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ if and only if $|\partial_e(\mathcal{T}(\mathcal{A}))| < \infty$, which for stable algebras is equivalent to the quasicontinuity of the scale. That class includes simple separable C^*-algebras of real rank zero and stable rank one with strict comparison of positive elements by traces and simple separable finite $\mathcal{Z}$-stable C^*-algebras.

7 Positive Linear Combinations of Projections

It is well known that every element of $B(\mathcal{H})$ is a linear combination of projections. The same property holds for all von Neumann algebras without a finite type I direct summand with infinite dimensional center [17]. However this property may fail even for C^*-algebras of real rank zero [25, Proposition 5.1].

In the process of investigating linear combination of projections in C^*-algebras, we found it convenient to consider the following stronger condition.

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Definition 7.1 A C^*-algebra $\mathcal{A}$ has a LCP constant V if every selfadjoint element a in $\mathcal{A}$ is a linear combination of N projections $p_j \in \mathcal{A}$ with $a = \sum_1^N \lambda_j p_j$ for some $N \in \mathbb{N}$, and $\lambda_j \in \mathbb{R}$, satisfying the condition $\sum_1^N |\lambda_j| \leq V\|a\|$. Furthermore if N can be chosen independently of the element a, we say that $\mathcal{A}$ has an LCP pair of constants V and N.

The LCP constant V was first introduced in $B(H)$ by Fong [16].

C^*-algebras that have LCP pairs (V, N) of constants include the following.

- von Neumann algebras without a finite type I summand with infinite dimensional center. More precisely, the following estimates for (V, N) are implicit in [17].
  - If $\mathcal{A}$ is a properly infinite von Neumann algebra, then $(8, 6)$ is an LCP pair.
  - If $\mathcal{A}$ is a type II_1 von Neumann algebra, then $(14, 12)$ is an LCP pair.
  - If $\mathcal{A}$ is the direct sum of m matrix algebras, then $(m+4, m+4)$ is an LCP pair.
- unital properly infinite C^*-algebras [21, Propositions 2.6, 2.7];
- unital simple separable C^*-algebras with real rank zero, stable rank one, strict comparison of projections, and finite extremal tracial boundary [25, Theorem 4.4];
- corners $P\mathcal{M}(\mathcal{A} \otimes \mathcal{K})P$ with P a projection in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ for a unital simple separable C^*-algebra $\mathcal{A}$ with real rank zero, stable rank one, strict comparison of projections, and finite extremal tracial boundary [26, Theorem 4.4].

Estimates of LCP pairs (V, N) for C^*-algebras are mostly missing or very far from sharp ("horrendous" according to Marcoux [35]). Suffice it to quote the case when $\mathcal{A}$ is a unital simple C^*-algebra of real rank zero with a unique tracial state and with strict comparison of projections by traces. Then by [34, Remark 5.3], every element of $a \in \mathcal{A}$ can be written as a linear combination of 113 projections and the coefficients are bounded by $9,537,600\|a\|$. Thus $(9,537,600, 113)$ is an LCP pair!

A linear combination $A = \sum_1^n \alpha_j p_j$ with projections $p_j \in \mathcal{A}$ and scalars $\alpha_j > 0$ will be called a *positive linear combination of projections* or PCP for short. This notion was studied in [21, 23, 25, 26], and, in particular, we proved that if a C^*-algebra $\mathcal{A}$ has such LCP constants and if furthermore $\mathcal{A}_+$ is the closure of PCPs in $\mathcal{A}$, then every positive invertible element of $\mathcal{A}$ is a PCP [21, Proposition 2.7].

Thus if both conditions hold for all corners $p\mathcal{A}p$ of $\mathcal{A}$, then all positive locally invertible elements are PCP. A key tool for the further investigation of PCP elements is the fact that a direct sum of projection and of a "small" positive perturbation is also PCP [21, Lemma 2.2].

We can obtain the following result under less restrictive conditions.

Lemma 7.2 *Let $\mathcal{A}$ be a C^*-algebra, $p \in \mathcal{A}$ be a projection such that the corner algebra $p\mathcal{A}p$ has LCP constant V.*

(i) *$p + b$ is a PCP for every $b = b^* \in p\mathcal{A}p$ with $\|b\| \leq \frac{1}{V}$. If the corner algebra $p\mathcal{A}p$ has an LCP pair of constants V and N, the number of projections needed in the PCP is $N + 1$.*

(ii) *$p + b$ is a PCP for every $b \in \mathcal{A}_+$ with $b = qb = bq$ for some projection $q \in \mathcal{A}$ such that $q \perp p$ with $q \prec p$ and $\|b\| \leq \frac{1}{1+V}$. Furthermore, if $p\mathcal{A}p$ has an LCP pair of constants V and N, $p + b$ can be decomposed as a PCP of $N + 4$ projections.*

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(iii) *$p+b$ is a PCP for every $b \in \mathcal{A}_+$ with $b = qb = bq$ for some projection $q \in \mathcal{A}$ such that $q \perp p$ with $m[q] \le [p]$ for some $m \in \mathbb{N}$ with $m \ge \|b\|(1+V)$.*

Proof (i) By hypothesis we can find N real numbers λ_j and projections $q_j \in \mathcal{A}$ with $q_j \le p$ such that $b = \sum_{j=1}^N \lambda_j q_j$ and $\sum_{j=1}^N |\lambda_j| \le V\|b\| \le 1$. Thus

$$p + b = \sum_{\lambda_j \ge 0} \lambda_j q_j + \sum_{\lambda_j < 0} (-\lambda_j)(p - q_j) + \Big(1 + \sum_{\lambda_j<0} \lambda_j\Big) p$$

is a PCP of $N+1$ projections.

(ii) Assume without loss of generality that $b \ne 0$. Notice that $V\|b\| < 1$ and let $\beta := \frac{1}{1 - V\|b\|}$. Then $1 < \beta \le \frac{1}{\|b\|}$. Following the proof of [21, Lemma 2.9], let $v \in \mathcal{A}$ be a partial isometry such that $v^*v = q$ and $vv^* = p' \le p$. Define

$$r_1 := \beta b + v\sqrt{\beta b - (\beta b)^2} + \sqrt{\beta b - (\beta b)^2}\, v^* + p' - \beta v b v^*,$$
$$r_2 := \beta b - v\sqrt{\beta b - (\beta b)^2} - \sqrt{\beta b - (\beta b)^2}\, v^* + p' - \beta v b v^*.$$

Then r_1 and r_2 are projections in $\mathcal{A}$ and $\beta b = \frac{1}{2}(r_1 + r_2) - p' + \beta v b v^*$. Hence

$$p + b = \frac{1}{2\beta}(r_1 + r_2) + \frac{1}{\beta}(p - p') + \Big(1 - \frac{1}{\beta}\Big)\Big(p + \frac{vbv^*}{1 - \frac{1}{\beta}}\Big).$$

Now $0 \le \frac{vbv^*}{1-1/\beta} \in p'\mathcal{A}p' \subset p\mathcal{A}p$ and $\|\frac{vbv^*}{1-1/\beta}\| = \frac{\|b\|}{1-1/\beta} = \frac{1}{V}$. Then by part (i) $p + \frac{vbv^*}{1-1/\beta}$ is a PCP, and hence so is $p+b$. Furthermore, if $p\mathcal{A}p$ has an LCP pair of constants V and N, by part (i) $p + \frac{vbv^*}{1-1/\beta}$ can be decomposed as a PCP of $N+1$ projections and hence $p+b$ can be decomposed as a PCP of $N+4$ projections.

(iii) Decompose $p = \bigoplus_{i=1}^m p_i$ into projections $p_i \in \mathcal{A}$ with $q \prec p_i$ for each i. Then $p + b = \sum_{i=1}^m \big(p_i + \frac{1}{m} b\big)$. For each i it follows from part (ii) that $p_i + \frac{1}{m} b$ is a PCP and hence so is $p+b$. ∎

Our next lemma permits us to embed isomorphically certain σ-unital hereditary sub-algebras of $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ into unital corners of $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ *with control on the "size" of the corner*. When $B \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})_+$ we use the following notations.

- $\mathrm{her}(B) := \overline{B(\mathcal{A}\otimes\mathcal{K})B}$ hereditary subalgebra of $\mathcal{A}\otimes\mathcal{K}$,
- $\mathrm{Her}(B) = \overline{B\,\mathcal{M}(\mathcal{A}\otimes\mathcal{K})\,B}$ hereditary subalgebra of $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$.

Lemma 7.3 *Let $\mathcal{A}$ be a C^*-algebra and $B \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})_+$ be such that the hereditary algebra* $\mathrm{her}(B)$ *of $\mathcal{A}\otimes\mathcal{K}$, has an approximate unit $\{f_j\}$ consisting of an increasing sequence of projections. Then there is a partial isometry $W \in (\mathcal{A}\otimes\mathcal{K})^{**}$ such that*

(i) $W^*W = R_B$,
(ii) $WW^* \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$,
(iii) $WB \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$,
(iv) $W\,\mathrm{Her}(B)W^* \subseteq R\,\mathcal{M}(\mathcal{A}\otimes\mathcal{K})\,R$ *where* $R := WW^*$.
(v) *The onto map* $\mathrm{Her}(B) \ni X \to \Phi(X) := WXW^* \in \mathrm{Her}(\Phi(B))$ *is a $*$-isomorphism of hereditary algebras.*

Proof Let $e_j := f_j - f_{j-1}$ (with $f_0 := 0$) and let $I_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} = \sum_1^\infty E_j$ be a decomposition of the identity into a strictly converging series of mutually orthogonal projections

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$E_j \sim I_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}$. As $e_j \le E_j$, there are partial isometries $v_j \in \mathcal{A}\otimes\mathcal{K}$ such that $v_j^* v_j = e_j$ and $v_j v_j^* \le E_j$. Let $W := \sum_1^\infty v_j$. The series converges in the strong topology of $(\mathcal{A}\otimes\mathcal{K})^{**}$ because both the range projections of the partial isometries v_j are mutually orthogonal and so are the range projections of the partial isometries v_j^*. Then $W^*W = \sum_1^\infty e_j = \lim_j f_j = R_B$ and the convergence is again in the strong topology of $(\mathcal{A}\otimes\mathcal{K})^{**}$. On the other hand, $WW^* = \sum_1^\infty v_j v_j^*$ in the strict topology because $v_j v_j^* \le E_j$ and $\sum_1^\infty E_j$ converges strictly. Thus the projection $R := WW^*$ belongs to $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$.

Next we show that $WB \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$. Let $a \in \mathcal{A}\otimes\mathcal{K}$. Then $Baa^*B \in \operatorname{her}(B)$. Hence $f_k Ba \to Ba$ in norm, or equivalently $\sum_1^n e_j Ba$ converges in norm to Ba. Since $We_j = v_j$ for all j, we have $W\sum_1^n e_j Ba = \sum_1^n v_j Ba \to WBa \in \mathcal{A}\otimes\mathcal{K}$ since the convergence is in norm. On the other hand, since $\sum_1^\infty v_j v_j^*$ converges strictly, $\|a\sum_n^\infty v_j v_j^*\| \to 0$ for $n\to\infty$, and hence $\|a\sum_n^\infty v_j v_j^* W\| = \|a\sum_n^\infty v_j\| \to 0$. Thus $aW \in \mathcal{A}\otimes\mathcal{K}$, and hence $aWB \in \mathcal{A}\otimes\mathcal{K}$. This concludes the proof of (i)–(iii).

Next $BW^* \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$. Hence $WB\,\mathcal{M}(\mathcal{A}\otimes\mathcal{K})\,BW^* \subset \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ and hence $W\operatorname{Her}(B)W^* \subseteq R\,\mathcal{M}(\mathcal{A}\otimes\mathcal{K})\,R$, *i.e.*, (iv) holds. Finally, proving (v) is routine. ■

Remark 7.4 The above result can be seen as the construction of a projection $P \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ that is equivalent to the open projection R_B in the sense of Peligrad and Zsido [43] (see also [39]).

$\mathcal{A}$ has real rank zero if and only if every hereditary subalgebra of $\mathcal{A}$ has an approximate identity of projections [5].

Proposition 7.5 *Let $\mathcal{A}$ be a simple separable C^*-algebra with real rank zero, stable rank one, strict comparison of projections, and finite extremal boundary. Let $P \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})\setminus\mathcal{A}\otimes\mathcal{K}$ be a projection. Then for every $B \in (P^\perp\mathcal{M}(\mathcal{A}\otimes\mathcal{K})P^\perp)_+$ such that $\tau(R_B) < \infty$ for all those $\tau \in \mathcal{T}(\mathcal{A})$ for which $\tau(P) < \infty$, it follows that $P + B$ is a PCP.*

Proof Let $\partial_e(\mathcal{T}(\mathcal{A})) = \{\tau_j\}_1^n$ and notice that $F(B)' = \{\tau \in \mathcal{T}(\mathcal{A}) \mid \tau(P) < \infty\}$. Since $\mathcal{A}$ has real rank zero and R_B is an open projection, R_B has a decomposition $R_B = \bigoplus_1^\infty r_j$ into the sum of mutually orthogonal projections $r_j \in \mathcal{A}\otimes\mathcal{K}$ converging strongly in $(\mathcal{A}\otimes\mathcal{K})^{**}$. By [26, Theorem 5.1], $P\,\mathcal{M}(\mathcal{A}\otimes\mathcal{K})\,P$ has an LCP constant V. Let $m > \|B\|(1+V)$ be an integer. Since $\tau(\bigoplus_1^\infty r_j) < \infty$ for all those $\tau \in \partial_e(\mathcal{T}(\mathcal{A}))$ for which $\tau(P) < \infty$ and since there are only finitely many extremal traces, there exists a k such that $\tau(\bigoplus_k^\infty r_j) < \frac{1}{m}\tau(P)$ for all $\tau \in F(B)'$. Let

$$B' := B^{1/2}\Big(\bigoplus_1^{k-1} r_j\Big)B^{1/2} \quad\text{and}\quad B'' := B - B' = B^{1/2}\Big(\bigoplus_k^{\infty} r_j\Big)B^{1/2}.$$

Then $B' \in \mathcal{A}\otimes\mathcal{K}_+$ and $B'' \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})_+$. Moreover,

$$R_{B'} \preceq \bigoplus_1^{k-1} r_j \quad\text{and}\quad R_{B''} \preceq \bigoplus_k^{\infty} r_j$$

where the Murray–von Neumann subequivalences $\preceq$ are in $(\mathcal{A}\otimes\mathcal{K})^{**}$. Thus

$$\tau(R_{B'}) < \infty \ \forall\tau \in \partial_e(\mathcal{T}(\mathcal{A})) \quad\text{and}\quad \tau(R_{B''}) < \frac{1}{m}\tau(P)\ \forall\tau \in F(B)'.$$

By [25, Theorem 6.1], B' is a PCP. Thus it remains to prove that $P + B''$ is also a PCP.

By Lemma 7.3 there is a partial isometry $W \in (\mathcal{A} \otimes \mathcal{K})^{**}$ with $W^*W = R_{B''}$, $R := WW^* \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ and that induces an isomorphism

$$\mathrm{Her}(B'') \ni X \to \Phi(X) := WXW^* \in \mathrm{Her}(\Phi(B'')) \subset R\,\mathcal{M}(\mathcal{A} \otimes \mathcal{K})\,R.$$

Notice that $R \sim R_{B''}$ in $(\mathcal{A} \otimes \mathcal{K})^{**}$ and hence

$$\tau(R) = \tau(R_{B''}) \le \tau(R_B) = \tau(\bigoplus_1^\infty r_j) \le \tau(P^\perp) \;\forall\, \tau \in \mathcal{T}(\mathcal{A})\,.$$

Thus if $\tau(P^\perp) < \infty$, then $\tau(\bigoplus_1^\infty r_j) < \infty$, and hence $\tau(R_{B''}) \le \tau(\bigoplus_k^\infty r_j) < \tau(P^\perp)$. By strict comparison of projections in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ (see [26, Theorem 3.2]), or as a consequence of Corollary 2.9 and Theorem 5.3, it follows that $R \precsim P^\perp$. Without loss of generality we can assume that $R \le P^\perp$. Now let $W' := P \oplus W$. Then the map

$$\mathrm{Her}(P \oplus B'') \ni X \to \Phi'(X) = W'XW'^* \in \mathrm{Her}(\Phi'(P \oplus B''))$$

is a $*$-isomorphism. Now $\Phi'(P \oplus B'') = P \oplus \Phi(B'') = P \oplus R\Phi(B'')R$. Furthermore, $\tau(R) < \frac{1}{m}\tau(P)$ for all $\tau \in F(B)'$ and since $P \notin \mathcal{A} \otimes \mathcal{K}$, by the strict comparison of projections in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$, $m[R] \le [P]$. Then $\Phi'(P \oplus B'')$ is a PCP by Lemma 7.2 (iii). Since Φ' is an isomorphism of hereditary algebras, $P + B''$ is also a PCP and hence so is $P + B$. ∎

Next we need some results on principal ideals.

Lemma 7.6 *Let $\mathcal{A}$ be a real rank zero C^*-algebra such that $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ has only finitely many ideals. Then every ideal of $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ is generated by a projection.*

Proof Let $\{J_k\}_1^n$ be the collection of all the ideals of $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$, including the zero ideal. Let J be a nonzero ideal, and let $S := \{k \mid J \not\subset J_k\}$. Notice that S is nonempty. Since by [51, Theorem 2.2] every ideal in the multiplier algebra of a real rank zero algebra is the closed linear span of its projections, for every $k \in S$ there must be a projection $P_k \in J \setminus J_k$. Let $P := \bigoplus_{k \in S} P_k$. Then $P \in J$ and hence $I(P) \subset J$. Assume by contradiction that $I(P) \neq J$, hence $J \not\subset I(P)$, and hence $I(P) = J_k$ for some $k \in S$, a contradiction, since $P \notin J_k$ because $P_k \notin J_k$. ∎

By [44, Theorem 4.4] (see Theorem 2.8) the conditions of the above lemma are satisfied if $\mathcal{A}$ is simple, unital, with real rank zero, strict comparison of positive elements by traces, and finite extremal boundary. But in that case we can say more.

Proposition 7.7 *Let $\mathcal{A}$ be a simple unital C^*-algebra with real rank zero, strict comparison of positive elements by traces, and with finite extremal boundary, and let $T \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})_+ \setminus \mathcal{A} \otimes \mathcal{K}$. Then there is a $\delta > 0$ such that*

(i) $I(T) = I((T - \delta)_+)$,

(ii) *there is a projection P such that $I(P) = I(T)$ and $T \ge \delta P$.*

Proof The case when $\mathcal{A} \otimes \mathcal{K} = \mathcal{K}$, and hence $\mathcal{M}(\mathcal{A} \otimes \mathcal{K}) = B(\mathcal{H})$, follows from standard operator theory, so assume without loss of generality that $\mathcal{A}$ is nonelementary.

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(i) By definition, for every $\tau \in F(T) \cap \partial_e(\mathcal{T}(\mathcal{A}))$, $T \notin I_\tau$; hence by (2.6), there is a $\delta_\tau > 0$ such that $(T - \delta_\tau)_+ \notin I_\tau$. Let $\delta := \inf\{\delta_\tau \mid \tau \in F(T) \cap \partial_e(\mathcal{T}(\mathcal{A}))\}$. By the assumption that $\partial_e(\mathcal{T}(\mathcal{A}))$ is finite, $\delta > 0$. Thus $(T - \delta)_+ \notin I_\tau$ for all $\tau \in F(T) \cap \partial_e(\mathcal{T}(\mathcal{A}))$. It follows that we have the inclusion $F(T) \subset F((T - \delta)_+)$. On the other hand, $(T - \delta)_+ \leq T$. Hence if $\tau \in F((T - \delta)_+) \cap \partial_e(\mathcal{T}(\mathcal{A}))$, *i.e.*, $(T - \delta)_+ \notin I_\tau$, then $T \notin I_\tau$, *i.e.*, $\tau \in F(T)$. Thus $F((T - \delta)_+) \subset F(T)$, whence $F((T - \delta)_+) = F(T)$, and hence $F((T - \delta)_+)' = F(T)'$. By (2.13), $I((T - \delta)_+) = I(T)$.

(ii) By (i), $I((T - \delta)_+) = I(T)$, and hence

$$d_\tau((T - \delta)_+) < \infty \text{ for } \tau \in F(T)',$$
$$d_\tau((T - \delta)_+) = \infty \text{ for } \tau \in F(T).$$

By Lemma 7.6, there is a projection $Q \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ such that $I(Q) = I(T)$. By [54, Theorem 1.1], for every $n \in \mathbb{N}$ we can find a projection Q' such that $\bigoplus_1^{2^n} Q' = Q$. Notice that $I(Q') = I(Q)$. By choosing n large enough, and using the fact that $|\partial_e(\mathcal{T}(\mathcal{A}))| < \infty$, we obtain $\tau(Q') < d_\tau((T - \delta)_+)$ for all $\tau \in F((T - \delta)_+)$. Then by strict comparison of positive elements in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ (Theorem 5.3) it follows that $Q' \precsim (T - \delta)_+$. Thus by Lemma 2.5, there is a projection $P \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ such that $T \geq \delta P$ and $P \sim Q$, and hence $I(P) = I(Q) = I(T)$. ∎

We list here a property we will need in the proof of our the next theorem

Lemma 7.8 *Let $\mathcal{B}$ be a C^*-algebra. For every $g \in C([0,1])$ the function $\mathcal{B} \ni b \mapsto g(b)$ is uniformly continuous on the positive part of the unit ball of $\mathcal{B}$.*

Proof Let a, b be in the unit ball of $\mathcal{B}$ and let $\epsilon > 0$. Find a polynomial p_n such that $\|g - p_n\|_\infty < \frac{\epsilon}{3}$. Then $\|g(a) - p_n(a)\| < \frac{\epsilon}{3}$ and $\|g(b) - p_n(b)\| < \frac{\epsilon}{3}$. Moreover, $\|p_n(a) - p_n(b)\| \leq c\|a - b\|$ where $p_n(t) = \sum_0^n \alpha_j t^j$ and $c = \sum_1^n j|\alpha_j|$. Indeed, since

$$\|a^n - b^n\| \leq \|a^{n-1}\|\|a - b\| + \|a^{n-2}\|\|b\|\|a - b\| + \cdots + \|b^{n-1}\|\|a - b\| \leq n\|a - b\|$$

and hence

$$\|p_n(a) - p_n(b)\| = \|\sum_1^n \alpha_j(a^j - b^j)\| \leq \sum_1^n |\alpha_j|\,\|a^j - b^j\| \leq \sum_1^n j|\alpha_j|\,\|a - b\|.$$

Set $\delta = \frac{\epsilon}{3c}$. For every $\|a - b\| < \delta$ it follows that $\|p_n(a) - p_n(b)\| < \frac{\epsilon}{3}$. Thus

$$\|g(a) - g(b)\| < \epsilon.$$ ∎

Theorem 7.9 *Let $\mathcal{A}$ be a simple separable C^*-algebra with real rank zero, stable rank one, strict comparison of projections, and finite extremal boundary, and let $T \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})_+$. Then T is a PCP if and only if $\tau(R_T) < \infty$ for all $\tau \in F(T)'$, that is, for all τ for which $T \in I_\tau$.*

Proof We first prove the necessity. Assume that $T = \sum_{j=1}^n \lambda_j P_j$ for some $\lambda_j > 0$ and projections $P_j \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ and assume that $T \in I_\tau$ for some $\tau \in \mathcal{T}(\mathcal{A})$. Let $R = \bigvee_1^n P_j \in (\mathcal{A} \otimes \mathcal{K})^{**}$. Since $P_j \leq \frac{1}{\lambda_j} T$, it follows that $P_j \in I_\tau$ and thus $\tau(P_j) < \infty$. Also, $P_j \leq R_T$ for all j, hence $R \leq R_T$. On the other hand $RT = T$, and hence $R_T = R$. Then $\tau(R_T) = \tau(R) \leq \sum_1^n \tau(P_j) < \infty$.

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Now we prove the sufficiency. If $T \in \mathcal{A} \otimes \mathcal{K}$, the result was proved in [25, Theorem 6.1]. Thus assume that $T \notin \mathcal{A} \otimes \mathcal{K}$ and further that $\|T\| \leq 1$. By Proposition 7.7, there is a $0 < \delta < 1$ and a projection $P \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ for which $I(P) = I(T)$ and $T \geq \delta P$. Assume further that $\delta < \frac{6}{7}$. Since $P \notin \mathcal{A} \otimes \mathcal{K}$, by [54, Theorem 1.1 (ii)] P can be decomposed into the sum $P = P_1 + P_2$ of two projections $P_1 \sim P_2$. Then for $i = 1, 2$

$$\tau(P_i) < \infty \text{ for } \tau \in F(T)',$$
$$\tau(P_i) = \infty \text{ for } \tau \in F(T),$$

and hence $I(P_1) = I(P_2) = I(T)$. Now set $T' = T - \frac{\delta}{2}P_1 = T - \delta P + \frac{\delta}{2}P_1 + \delta P_2$. Since $T = T' + \frac{\delta}{2}P_1$, it is enough to prove that T' is a PCP. Notice that $R_{T'} \leq R_T$. Let f_1 and f_2 be the continuous functions defined by

$$f_1(t) = \begin{cases} t & t \in [0, \frac{2}{3}\delta], \\ 0 & t \in [\frac{5}{6}\delta, 1], \\ \text{linear} & t \in [\frac{2}{3}\delta, \frac{5}{6}\delta], \end{cases} \quad \text{and} \quad f_2(t) = \begin{cases} 0 & t \in [0, \frac{2}{3}\delta], \\ t & t \in [\frac{5}{6}\delta, 1], \\ \text{linear} & t \in [\frac{2}{3}\delta, \frac{5}{6}\delta]. \end{cases}$$

Now consider the continuous functions g_1 and g_2 defined by

$$g_1(t) = \begin{cases} 0 & t \in [0, \frac{\delta}{3}] \cup [\frac{2\delta}{3}, 1], \\ \frac{\delta}{2} & t = \frac{\delta}{2}, \\ \text{linear} & \text{elsewhere}, \end{cases} \quad \text{and} \quad g_2(t) = \begin{cases} 0 & t \in [0, \frac{5\delta}{6}] \cup [\frac{7\delta}{6}, 1], \\ \delta & t = \delta, \\ \text{linear} & \text{elsewhere}. \end{cases}$$

Then for all $t \in [0, 1]$,

$$f_1(t) + f_2(t) = t, \tag{7.1}$$
$$g_1(t) \leq f_1(t) \quad \text{and} \quad g_2(t) \leq f_2(t), \tag{7.2}$$
$$g_1(t) f_2(t) = 0 \quad \text{and} \quad g_2(t) f_1(t) = 0, \tag{7.3}$$
$$f_1(t) \geq \frac{\delta}{3} \quad \text{where } g_1(t) \neq 0, \tag{7.4}$$
$$f_2(t) \geq \frac{5\delta}{6} \quad \text{where } g_2(t) \neq 0. \tag{7.5}$$

Since the functions g_1 and g_2 are both continuous on $[0, 1]$, by Lemma 7.8 they are uniformly continuous on the set of positive contractions. Thus there is an integer n such that $\|g_i(A) - g_i(B)\| \leq \frac{\delta}{4}$ whenever $0 \leq A \leq 1$, $0 \leq B \leq 1$, and $\|A - B\| \leq \frac{1}{n}$.

Reasoning as in the first part of the proof, we can subdivide the projections P_1 and P_2 into an orthogonal sum of n projections $P_1 = \sum_1^n P_{1,j}$ and $P_2 = \sum_1^n P_{2,j}$ such that $I(P_{i,j}) = I(T)$ for all $i = 1, 2$ and $1 \leq j \leq n$. Then

$$T' = \sum_{j=1}^{n} \Big(\frac{1}{n}(T - \delta P) + \frac{\delta}{2} P_{1,j} + \delta P_{2,j} \Big).$$

Thus it is enough to prove that for every pair of projections $Q_1 \perp Q_2$, with $Q_i \leq R_T$, and $I(Q_i) = I(T)$ for $i = 1, 2$, we have that the positive element

$$T'' := \frac{1}{n}(T - \delta P) + \frac{\delta}{2} Q_1 + \delta Q_2$$

is a PCP. Notice that $R_{T''} \le R_T$. Now

$$g_1\Big(\frac{\delta}{2}Q_1 + \delta Q_2\Big) = \frac{\delta}{2}Q_1 \quad \text{and} \quad g_2\Big(\frac{\delta}{2}Q_1 + \delta Q_2\Big) = \delta Q_2.$$

Since $\|\frac{1}{n}(T - \delta P)\| \le \frac{1}{n}$, it follows that

$$\Big\|g_1(T'') - \frac{\delta}{2}Q_1\Big\| = \Big\|g_1(T'') - g_1\Big(\frac{\delta}{2}Q_1 + \delta Q_2\Big)\Big\| \le \frac{\delta}{4}.$$

Then $\|\frac{2}{\delta}g_1(T'') - Q_1\| \le \frac{1}{2}$ and by Lemma 2.1 (ii), $\frac{1}{2}Q_1 = (Q_1 - \frac{1}{2})_+ \le \frac{2}{\delta}g_1(T'')$. Hence $Q_1 \le g_1(T'')$. As a consequence and by (7.4), there is a projection $Q_1' \le R_{g_1(T'')} \le \frac{1}{\delta/3}f_1(T'')$ with $Q_1' \sim Q_1$ and hence $I(Q_1') = I(T)$. Similarly, there is a projection $Q_2' \le R_{g_2(T'')} \le \frac{1}{5\delta/6}f_2(T'')$ with $Q_2' \sim Q_2$ and hence $I(Q_2') = I(T)$.

Notice that $T'' = f_1(T'') + f_2(T'')$ by (7.1). Then

$$T'' = \Big(\Big(f_1(T'') - \frac{\delta}{3}Q_1'\Big) + \frac{5\delta}{6}Q_2'\Big) + \Big(\Big(f_2(T'') - \frac{5\delta}{6}Q_2'\Big) + \frac{\delta}{3}Q_1'\Big)$$

is a decomposition of T'' into the sum of two positive elements. From (7.3), it follows that $g_1(T'')f_2(T'') = 0$ and hence $Q_2' \perp R_{f_1(T'')}$. Moreover, $\tau(R_{f_1(T'')}) \le \tau(R_T) < \infty$ for all $\tau \in F(T)'$ and hence for all τ for which $\tau(Q_2') < \infty$. Similarly $Q_1' \perp R_{f_2(T'')}$ and $\tau(R_{f_2(T'')}) < \infty$ for all τ for which $\tau(Q_1') < \infty$. Thus both summands of T'' satisfy the conditions of Proposition 7.5 and hence are a PCP, which concludes the proof. ∎

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