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STABILITY OF THE DEFICIENCY INDICES OF SYMMETRIC OPERATORS UNDER SELF-ADJOINT PERTURBATIONS

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Abstract Let S and T be symmetric unbounded operators. Denote by $\overline{S+T}$ the closure of the symmetric operator $S+T$. In general, the deficiency indices of $\overline{S+T}$ are not determined by the deficiency indices of S and T. The paper studies some sufficient conditions for the stability of the deficiency indices of a symmetric operator S under self-adjoint perturbations T. One can associate with S the largest closed *-derivation δ_S implemented by S. We prove that if the unitary operators $\exp(\mathrm{i}tT)$, for $t \in \mathbb{R}$, belong to the domain of δ_S and $\delta_S(\exp(\mathrm{i}tT)) \to 0$ in the strong operator topology as $t \to 0$, then the deficiency indices of S and $\overline{S+T}$ coincide. In particular, this holds if S and $\exp(\mathrm{i}tT)$ commute or satisfy the infinitesimal Weyl relation.

We also study the case when S and T anticommute: $\exp(-\mathrm{i}tT)S \subseteq S\exp(\mathrm{i}tT)$, for $t \in \mathbb{R}$. We show that if the deficiency indices of S are equal, or if the group $\{\exp(\mathrm{i}tT) : t \in \mathbb{R}\}$ of unitary operators has no stationary points in the deficiency space of S, then S has a self-adjoint extension which anticommutes with T, the operator $S+T$ is closed and the deficiency indices of S and $S+T$ coincide.

Keywords: symmetric operator; self-adjoint operator; deficiency indices

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1. Introduction

Let S be a closed symmetric operator on a Hilbert space H with dense domain $D(S)$ and let S^* be its adjoint. The deficiency spaces of S

$$N_\pm(S) = \{x \in D(S^*) : S^*x = \pm\mathrm{i}x\}$$

are closed in H and the numbers $n_\pm(S) = \dim(N_\pm(S))$ are called the deficiency indices of S. The operator S is self-adjoint if and only if $n_+(S) = n_-(S) = 0$. Let T be another symmetric operator with $D(S) \bigcap D(T)$ dense in H and let $\overline{S+T}$ be the closure of the symmetric operator $S+T$.

Much work has been done on the study of the linear perturbations $\overline{S+T}$ of symmetric operators S and of the stability of their deficiency indices:

$$n_+(\overline{S+T}) = n_+(S), \qquad n_-(\overline{S+T}) = n_-(S). \tag{1.1}$$

In the classical example when T is bounded, not only (1.1) holds but also $\overline{S+T} = S+T$ (see [**1**]). The main thrust of the study was directed towards the important case when

S is self-adjoint. Rellich [**12**] and Kato [**5**, **6**] proved (1.1) when T is S-bounded with the S-bound less than or equal to 1. Putnam [**11**] showed that $S+T$ is self-adjoint if S and T are commuting positive operators. Vasilescu [**13**] and Pedersen [**10**] established that $S+T$ is self-adjoint when S and T are self-adjoint and anticommute: $\exp(-\mathrm{i}tT)S \subseteq S\exp(\mathrm{i}tT)$, for $t \in \mathbb{R}$. Numerous applications of these results to differential operators, to the quantum field theory and to the theory of derivations of C^*-algebras were considered in [**2**, **3**, **6**–**9**, **11**].

In our paper we concentrate on the study of the stability of the deficiency indices of symmetric operators under self-adjoint perturbations, that is, when (1.1) holds if S is symmetric and T is self-adjoint. Unlike the case when both S and T are self-adjoint and the Spectral Theorem can be employed to study the stability, in our case the most suitable tool for the purpose is the theory of indefinite metric spaces: Krein spaces. Using it in Proposition 3.1, we link the deficiency indices of S and $\overline{S+T}$ when both S and T are symmetric. This leads to our first main result (Theorem 4.3), which can be stated in terms of the largest derivation δ_S on $B(H)$ associated with S as follows: (1.1) holds if the group $\{\exp(\mathrm{i}tT) : t \in \mathbb{R}\}$ of unitary operators lies in the domain of δ_S and $\delta_S(\exp(\mathrm{i}tT))x \to 0$, as $t \to 0$ for $x \in D(S^*)$. This is a natural generalization of the condition that S and $\exp(\mathrm{i}tT)$ commute and it shows, in particular, that (1.1) holds if $\exp(\mathrm{i}tT)$ and S satisfy the infinitesimal Weyl relation (4.5). It also points to a link between the theory of perturbation of symmetric operators and the theory of derivations of C^*-algebras.

Our second main result—Theorems 5.3 and 5.4—is the extension of the results of Pedersen and Vasilescu about anticommuting operators to the case when S is symmetric. Here, again, using the theory of Krein spaces, we show that if the deficiency indices of S are finite and equal, or if the group $\{\exp(\mathrm{i}tT) : t \in \mathbb{R}\}$ has no stationary points in $N_+(S) + N_-(S)$, then the operator $S+T$ is closed, (1.1) holds and S has a self-adjoint extension which anticommutes with T.

The above results allow us to extend further the conditions under which the index of the *-derivation δ_S (see [**9**]) is stable: making use of Example 36.3 from [**9**], we conclude that if S is a maximal symmetric operator, then the index of δ_S is stable: $\mathrm{ind}(\delta_S) = \mathrm{ind}(\delta_{S+T})$, under any self-adjoint perturbation T such that S and T satisfy conditions of Theorems 4.3 or 5.3 or 5.4.

2. Preliminaries

Let F be a closed, densely defined operator on H. Its domain $D(F)$ becomes a Hilbert space with respect to the scalar product

$$\langle x, y\rangle_F = (x, y) + (Fx, Fy), \quad \text{for } x, y \in D(F). \tag{2.1}$$

Lemma 2.1.

(i) *A subset Ω in $D(F)$ is dense in $(D(F), \langle\cdot,\cdot\rangle_F)$ if and only if $\overline{(F|_\Omega)} = F$.*

(ii) *Let A be a bounded operator and $AD(F) \subseteq D(F)$. Then $\tilde{A} = A|_{D(F)}$ is a bounded operator on $(D(F), \langle\cdot,\cdot\rangle_F)$.*

Proof. Part (i) is evident. If $x_n \to x$ and $\tilde{A}x_n \to y$ in $\|\cdot\|_F$, they also converge in $\|\cdot\|$, so $y = Ax = \tilde{A}x$ and $\tilde{A}$ is closed. By the Closed Graph Theorem, $\tilde{A}$ is bounded. $\square$

Let $\boldsymbol{T} = \{T(t) : t \geqslant 0\}$ be a strongly continuous one-parameter semigroup of bounded operators on $H : T(0) = \mathbf{1}$, $T(t+s) = T(t)T(s)$, for $0 \leqslant t,\ s < \infty$, and

$$\|T(t)x - x\| \to 0, \quad \text{as } t \to 0, \quad \text{for } x \in H.$$

Its generator T is a closed operator with dense domain $D(T)$ (see [**3**, Chapter VIII, § 1]) and

$$T(t)D(T) \subseteq D(T) \quad \text{and} \quad TT(t)|_{D(T)} = T(t)T|_{D(T)}, \quad \text{for } t \geqslant 0. \tag{2.2}$$

Proposition 2.2. *Let T be the generator of $\boldsymbol{T}$. If F is a closed operator such that*

$$T(t)D(F) \subseteq D(F), \quad \text{for } t > 0;$$

and

$$\|FT(t)x - Fx\| \to 0, \quad \text{as } t \to 0, \quad \text{for } x \in D(F),$$

then $D(F) \bigcap D(T)$ is dense in $(D(F), \langle \cdot, \cdot \rangle_F)$.

Proof. Set $\tilde{T}(t) = T(t)|_{D(F)}$. By Lemma 2.1, $\tilde{\boldsymbol{T}} = \{\tilde{T}(t) : t \geqslant 0\}$ is a one-parameter semigroup of bounded operators on $(D(F), \langle \cdot, \cdot \rangle_F)$. We have

$$\|\tilde{T}(t)x - x\|_F^2 = \|T(t)x - x\|^2 + \|FT(t)x - Fx\|^2 \to 0, \quad \text{as } t \to 0,$$

for $x \in D(F)$, so $\tilde{\boldsymbol{T}}$ is strongly continuous. Hence the domain $D(\tilde{T})$ of its generator is dense in $(D(F), \langle \cdot, \cdot \rangle_F)$. Since $D(F) \bigcap D(T)$ contains $D(\tilde{T})$, it is dense in $(D(F), \langle \cdot, \cdot \rangle_F)$. $\square$

Let S be a closed symmetric operator and let S^* be its adjoint. With respect to $\langle \cdot, \cdot \rangle_{S^*}$ (see [**3**, Chapter XII, § 4]), $D(S^*)$ is the orthogonal sum of the subspaces $D(S)$, $N_+(S)$ and $N_-(S)$:

$$D(S^*) = D(S) \langle + \rangle_{S^*} N_+(S) \langle + \rangle_{S^*} N_-(S).$$

Define the following indefinite form on $D(S^*)$ (see [**9**, § 28]):

$$[x, y]_S = -\mathrm{i}\{(S^*x, y) - (x, S^*y)\}, \quad \text{for } x, y \in D(S^*).$$

It is degenerate on $D(S) : [x, y]_S = 0$ if $x \in D(S)$ and $y \in D(S^*)$, and $[x, y]_S = \overline{[y, x]}_S$ if $x, y \in D(S^*)$. It is easy to check that

$$[x, y]_S = \begin{cases} \langle x, y \rangle_{S^*} = 2(x, y), & \text{if } x, y \in N_+(S), \\ -\langle x, y \rangle_{S^*} = -2(x, y), & \text{if } x, y \in N_-(S), \\ 0, & \text{if } x \in N_-(S) \text{ and } y \in N_+(S). \end{cases} \tag{2.3}$$

Set

$$N(S) = N_+(S) \langle + \rangle_{S^*} N_-(S).$$

With respect to the form $[\cdot,\cdot]_S$, $N(S)$ is a non-degenerate indefinite metric space: for any $z \in N(S)$, there is $u \in N(S)$ such that $[z,u]_S \neq 0$. The subspaces $N_+(S)$ and $N_-(S)$ are, respectively, uniformly positive and uniformly negative. We also have that, for $x, y \in D(S^*)$,

$$|[x,y]_S| \leqslant \|S^*x\|\,\|y\| + \|x\|\,\|S^*y\| \leqslant 2\|x\|_{S^*}\|y\|_{S^*}. \tag{2.4}$$

3. Sums of symmetric operators

Let S and T be closed symmetric operators. Set

$$D = D(S) \bigcap D(T) \quad \text{and} \quad D^* = D(S^*) \bigcap D(T).$$

We assume that D is dense in H and set $R = \overline{S+T}$. Then $D \subseteq D^* \subseteq D(R^*)$, $R|_D = S+T$ and $R^*|_{D^*} = S^* + T$. If $x, y \in D^*$, then

$$[x,y]_R = -\mathrm{i}\{(R^*x,y) - (x,R^*y)\} = -\mathrm{i}\{(S^*x,y) - (x,S^*y)\} = [x,y]_S. \tag{3.1}$$

Proposition 3.1. *If $\overline{(S^*|_{D^*})} = S^*$, then $n_+(S) \leqslant n_+(R)$ and $n_-(S) \leqslant n_-(R)$.*

Proof. Let $\{e_i\}$ be an orthonormal basis in $N_+(S)$. By (2.3), $\langle e_i, e_j\rangle_{S^*} = [e_i,e_j]_S = \delta_{ij}$. Choose $k \leqslant n_+(S)$. Since D^* is dense in $(D(S^*), \langle\cdot,\cdot\rangle_{S^*})$, for any $\epsilon > 0$, there are $\{h_i\}_{i=1}^k$ in D^* such that $\|e_i - h_i\|_{S^*} \leqslant \epsilon$, for $i = 1,\dots,k$. By (2.4),

$$\begin{aligned} |[h_i,h_j]_S - \delta_{ij}| &= |[h_i,h_j]_S - [e_i,e_j]_S| \\ &= |[h_i - e_i, h_j - e_j]_S + [e_i, h_j - e_j]_S + [h_i - e_i, e_j]_S| \\ &\leqslant 2\|h_i - e_i\|_{S^*}\|h_j - e_j\|_{S^*} + 2\|e_i\|_{S^*}\|h_j - e_j\|_{S^*} + 2\|h_i - e_i\|_{S^*}\|e_j\|_{S^*} \\ &\leqslant 2\epsilon^2 + 4\epsilon \\ &\leqslant 6\epsilon. \end{aligned} \tag{3.2}$$

Let $x = \sum_{i=1}^k \lambda_i h_i$ for $\lambda_i \in \mathbb{C}$. Then $[x,x]_S = \sum_{i,j=1}^k \lambda_i \overline{\lambda_j}[h_i,h_j]_S$. Choosing sufficiently small ϵ and applying the Principal Minor Test to the matrix $([h_i,h_j]_S)$, we obtain from (3.2) that the quadratic form $[x,x]_S$ is positive definite. Hence all h_i are linearly independent and the subspace M spanned by $\{h_i\}_{i=1}^k$ is positive in $D(S^*)$, that is, $[x,x]_S > 0$, for $x \in M$. By (3.1), M is also positive in $D(R^*)$.

Let Q be the projection on the subspace $N(R)$ in $(D(R^*), \langle\cdot,\cdot\rangle_{R^*})$. By (2.3),

$$[Qx, Qy]_R = [x,y]_R, \quad \text{for } x, y \in D(R^*).$$

Hence QM is a positive subspace in $N(R)$ and $\dim(QM) = k$. It follows from the Law of Inertia for indefinite metric spaces (see [**9**, Corollary 1.12]) that the dimensions of all positive subspaces in $N(R)$ are less than or equal to $\dim(N_+(R))$. Thus $k \leqslant n_+(R)$. Since k is arbitrary, we have $n_+(S) \leqslant n_+(R)$. Similarly, $n_-(S) \leqslant n_-(R)$. □

The condition $\overline{(S^*|_{D^*})} = S^*$ in Proposition 3.1 is sufficient but not necessary for $n_+(S) \leqslant n_+(R)$ and $n_-(S) \leqslant n_-(R)$. Indeed, if $T = S \neq S^*$ then $R = 2S$ and $n_\pm(S) = n_\pm(R)$. However, we have $D^* = D(S^*) \bigcap D(S) = D(S)$ and $\overline{(S^*|_{D^*})} = S \neq S^*$.

We omit the standard proof of the following lemma.

Lemma 3.2.

(i) *If* $\overline{(S|_D)} = S$, *then* $D^* = D(R^*) \bigcap D(T)$.

(ii) *If* $\overline{(S^*|_{D^*})} = S^*$, *then* $D = D(R) \bigcap D(T)$.

Proposition 3.1 and Lemma 3.2 yield the following corollary.

Corollary 3.3. *If* $\overline{(R^*|_{D^*})} = R^*$ *and* $\overline{(S^*|_{D^*})} = S^*$, *then* $n_\pm(S) = n_\pm(R)$.

4. Sum of a symmetric operator and the generator of a one-parameter group

We start with the following result.

Lemma 4.1. *Let* Δ *be a linear manifold in* $D(S)$ *such that* $\overline{(S|_\Delta)} = S$. *Let* A *and* B *be bounded operators such that* $A\Delta \subseteq D(S)$, $B^*\Delta \subseteq D(S)$ *and let* $(SA - BS)|_\Delta$ *extend to a bounded operator* K. *Then* A *and* B^* *preserve* $D(S)$ *and* $D(S^*)$ *and*

$$K|_{D(S^*)} = (S^*A - BS^*)|_{D(S^*)}. \tag{4.1}$$

Proof. Let $x \in D(S)$. Since $\overline{(S|_\Delta)} = S$, there are $x_n \in \Delta$ such that $x_n \to x$ and $Sx_n \to Sx$. Then $Ax_n \to Ax$ and $SAx_n = BSx_n + Kx_n \to BSx + Kx$. Since $Ax_n \in D(S)$ and S is closed, $Ax \in D(S)$ and $(SA - BS)|_{D(S)} = K$. Thus A preserves $D(S)$.

For $x, y \in \Delta$,

$$(K^*x, y) = (x, Ky) = (x, (SA - BS)y) = ((A^*S - SB^*)x, y).$$

Since Δ is dense in $H, K^*|_\Delta = (A^*S - SB^*)|_\Delta$. Repeating the argument used above, we prove that $B^*D(S) \subseteq D(S)$ and $K^*|_{D(S)} = (A^*S - SB^*)|_{D(S)}$.

Let $y \in D(S^*)$. For any $x \in D(S)$,

$$(Sx, Ay) = (A^*Sx, y) = (K^*x, y) + (SB^*x, y) = (x, Ky) + (x, BS^*y).$$

Hence $Ay \in D(S^*)$ and (4.1) holds. Similarly, B^* preserves $D(S^*)$. □

Let S and T be symmetric operators and let iT be the generator of a strongly continuous one-parameter semigroup $\{T(t) : t \geqslant 0\}$ of bounded operators.

Proposition 4.2. *Let* $T(t)D(S) \subseteq D(S)$ *and let there exist a family of bounded operators* $\{A(t) : t > 0\}$ *on* H *such that*

(1) $A(t)^*D(S) \subseteq D(S)$, *for* $t > 0$;

(2) $A(t)x \to x$, *as* $t \to 0$, *for* $x \in H$;

(3) $(ST(t) - A(t)S)|_{D(S)}$ *extends to a bounded operator* $K(t)$, *for* $t > 0$;

(4) $K(t)x \to 0$, *as* $t \to 0$, *for any* $x \in D(S^*)$.

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Then

(i) *the linear manifolds* $D = D(S) \bigcap D(T)$ *and* $D^* = D(S^*) \bigcap D(T)$ *are dense in* H*;*

(ii) $\overline{(S|_D)} = S$ *and* $\overline{(S^*|_{D^*})} = S^*$*;*

(iii) $D(\overline{S+T}) \bigcap D(T) = D$*;*

(iv) $n_+(S) \leqslant n_+(\overline{S+T})$ *and* $n_-(S) \leqslant n_-(\overline{S+T})$*.*

Proof. For $x \in D(S)$,

$$\|ST(t)x - Sx\| = \|(A(t)Sx + K(t)x) - Sx\| \leqslant \|A(t)Sx - Sx\| + \|K(t)x\| \to 0,$$

as $t \to 0$. By Proposition 2.2, D is dense in $(D(S), \langle\cdot,\cdot\rangle_S)$, so it is dense in H.

It follows from Lemma 4.1 that $T(t)D(S^*) \subseteq D(S^*)$, for $t > 0$, and

$$\begin{aligned}\|S^*T(t)x - S^*x\| &= \|(A(t)S^*x + K(t)x) - S^*x\| \\ &\leqslant \|A(t)S^*x - S^*x\| + \|K(t)x\| \to 0,\end{aligned}$$

as $t \to 0$, for $x \in D(S^*)$. Hence, by Proposition 2.2, the manifold D^* is dense in $(D(S^*), \langle\cdot,\cdot\rangle_{S^*})$, so it is dense in H. Part (i) is proved. Part (ii) follows from Lemma 2.1. Part (iii) follows from (ii) and Lemma 3.2, and part (iv) follows from (ii) and Proposition 3.1. □

By Stone's Theorem, T is self-adjoint if and only if iT is the generator of a strongly continuous group $\{T(t) : t \in \mathbb{R}\}$ of unitary operators: $T(t) = \exp(\mathrm{i}tT)$.

Theorem 4.3. *Let* T *be a self-adjoint operator. Let*

(i) $T(t)D(S) \subseteq D(S)$ *for each* $t \in \mathbb{R}$*;*

(ii) $(ST(t) - T(t)S)|_{D(S)}$ *extends to a bounded operator* $K(t)$ *for each* $t \in \mathbb{R}$*;*

(iii) $K(t)x \to 0$*, as* $t \to 0$*, for each* $x \in D(S^*)$*.*

Then $n_+(S) = n_+(\overline{S+T})$ *and* $n_-(S) = n_-(\overline{S+T})$*.*

Proof. Since $T(t)^* = T(-t)$, we obtain from Proposition 4.2 that

$$D(R) \bigcap D(T) = D, \quad S = \overline{(S|_D)}, \quad n_+(S) \leqslant n_+(R) \quad \text{and} \quad n_-(S) \leqslant n_-(R). \tag{4.2}$$

It follows from (2.2) that $T(t)D \subseteq D$, for $t \in \mathbb{R}$. Hence

$$(RT(t) - T(t)R)|_D = (ST(t) - T(t)S)|_D + (TT(t) - T(t)T)|_D = K(t)|_D.$$

Since $R = \overline{(R|_D)}$, it follows from Lemma 4.1 that

$$T(t)D(R) \subseteq D(R) \quad \text{and} \quad (RT(t) - T(t)R)|_{D(R)} = K(t)|_{D(R)}. \tag{4.3}$$

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The operator $-\mathrm{i}T$ is the generator of the group $\{T(-t) : t \in \mathbb{R}\}$ and, by (4.3), the group and the operator R satisfy the conditions of Proposition 4.2. Let W be the closure of the operator $(R-T)|_D$. We obtain from (4.2) and from Proposition 4.2 that

$$D(W) \bigcap D(T) = D \quad \text{and} \quad n_+(R) \leqslant n_+(W), \quad n_-(R) \leqslant n_-(W). \tag{4.4}$$

Since $W|_D = (R-T)|_D = (S+T-T)|_D = S|_D$, it follows from (4.2) that $W = \overline{(W|_D)} = \overline{(S|_D)} = S$. Comparing (4.2) and (4.4), we have $n_+(S) = n_+(R)$ and $n_-(S) = n_-(R)$. □

A self-adjoint operator T and a symmetric operator S commute if

$$\exp(\mathrm{i}tT)D(S) \subseteq D(S) \quad \text{and} \quad S\exp(\mathrm{i}tT)|_{D(S)} = \exp(\mathrm{i}tT)S|_{D(S)}, \quad \text{for } t \in \mathbb{R}.$$

They satisfy the *infinitesimal Weyl relation* (see [**4**]) if $\exp(\mathrm{i}tT)D(S) \subseteq D(S)$ and

$$(S\exp(\mathrm{i}tT) - \exp(\mathrm{i}tT)S)|_{D(S)} = t\exp(\mathrm{i}tT)|_{D(S)}, \quad \text{for } t \in \mathbb{R}. \tag{4.5}$$

Corollary 4.4. *Let S be a symmetric operator and T be a self-adjoint operator. If T and S commute or satisfy the infinitesimal Weyl relation (4.5), then*

$$n_+(\overline{S+T}) = n_+(S) \quad \textit{and} \quad n_-(\overline{S+T}) = n_-(S).$$

Even if S is self-adjoint and commutes with $\boldsymbol{T}(T)$, the operator $S+T$ is not necessarily closed. If, for example, $T=-S$, then $S+T = \mathbf{0}|_{D(S)}$ is not closed. Putnam [**11**] showed that if S and T are positive and commute, then $S+T$ is self-adjoint.

5. Anticommuting operators

A self-adjoint operator T and a symmetric operator S anticommute (cf. [**10**, **13**]) if $\exp(\mathrm{i}tT)D(S) \subseteq D(S)$ and

$$S\exp(\mathrm{i}tT)|_{D(S)} = \exp(-\mathrm{i}tT)S|_{D(S)}, \quad \text{for } t \in \mathbb{R}. \tag{5.1}$$

For self-adjoint S, Vasilescu [**13**] and Pedersen [**10**] proved that the operator $S+T$ is closed and self-adjoint. We study the case when S is symmetric.

We have from Proposition 4.2 that, for any anticommuting operators S and T, $n_+(S) \leqslant n_+(\overline{S+T})$ and $n_-(S) \leqslant n_-(\overline{S+T})$. Hence if $n_+(S) = n_-(S) = \infty$, then

$$n_+(S) = n_+(\overline{S+T}) \quad \text{and} \quad n_-(S) = n_-(\overline{S+T}).$$

We will extend this to all symmetric S. Set $T(t) = \exp(\mathrm{i}tT)$. From Lemma 4.1 we have

$$T(t)D(S^*) \subseteq D(S^*) \quad \text{and} \quad S^*T(t)|_{D(S^*)} = T(-t)S^*|_{D(S^*)}, \quad \text{for } t \in \mathbb{R}. \tag{5.2}$$

Set $\tilde{T}(t) = T(t)|_{D(S^*)}$. As in Proposition 2.2, $\{\tilde{T}(t) : t \in \mathbb{R}\}$ is a strongly continuous one-parameter group of bounded operators on $(D(S^*), \langle\cdot,\cdot\rangle_{S^*})$.

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Lemma 5.1. *All operators $\tilde{T}(t)$ are unitary and preserve $N(S)$.*

Proof. Since all $T(t)$ are unitary, it follows from (5.2) that

$$\begin{aligned}\langle \tilde{T}(t)x, \tilde{T}(t)y\rangle_{S^*} &= (T(t)x, T(t)y) + (S^*T(t)x, S^*T(t)y)\\ &= (x,y) + (T(-t)S^*x, T(-t)S^*y) = (x,y) + (S^*x, S^*y) = \langle x,y\rangle_{S^*},\end{aligned}$$

for $x, y \in D(S^*)$. Hence all operators $\tilde{T}(t)$ are unitary. Since they preserve the subspace $D(S)$ of $(D(S^*), \langle\cdot,\cdot\rangle_{S^*})$, they also preserve its orthogonal complement $N(S)$. □

Set $U(t) = \tilde{T}(t)|_{N(S)}$ and $J = -\mathrm{i}S^*|_{N(S)}$. Then $\boldsymbol{U} = \{U(t) : t \in \mathbb{R}\}$ is a strongly continuous one-parameter group of unitary operators on the Hilbert space $(N(S), \langle\cdot,\cdot\rangle_{S^*})$,

$$Jx = x \quad \text{if } x \in N_+(S), \qquad Jx = -x \quad \text{if } x \in N_-(S),$$

and

$$JU(t) = U(-t)J, \quad \text{for } t \in \mathbb{R}.$$

Let $\mathrm{i}W$ be the generator of the group $\boldsymbol{U}$. Then W is a self-adjoint operator on $N(S)$ and the operators W and J anticommute (see (5.1)). It follows from Proposition 1.1 in [**10**] that

$$JD(W) \subseteq D(W), \qquad JW|_{D(W)} = -WJ|_{D(W)}. \tag{5.3}$$

Since $J^2 = \mathbf{1}_{N(S)}$, we have $JD(W) = D(W)$. Set $D_\pm = D(W) \bigcap N_\pm(S)$.

Lemma 5.2.

(i) *The linear manifolds D_+ and D_- are dense in $N_+(S)$ and $N_-(S)$, respectively,*

$$WD_+ \subseteq N_-(S), \quad WD_- \subseteq N_+(S) \quad \textit{and} \quad D(W) = D_+\langle+\rangle_{S^*}D_-. \tag{5.4}$$

(ii) *If one of the deficiency indices of S is finite, then $D_+ = N_+(S)$, $D_- = N_-(S)$, W is a bounded operator and the group $\boldsymbol{U}$ is uniformly continuous.*

Proof. If $x \in D_+$, then $J(Wx) = -WJx = -Wx$, so $Wx \in N_-(S)$. Thus $WD_+ \subseteq N_-(S)$. Similarly, $WD_- \subseteq N_+(S)$. If $x \in D(W)$, then $Jx \in D(W)$ and $J(x + Jx) = x + Jx$, so that $x + Jx \in D(W) \bigcap N_+(S) = D_+$. Similarly, $x - Jx \in D_-$. Thus $D(W) = D_+\langle+\rangle_{S^*}D_-$. Since $D(W)$ is dense in $N(S)$, D_- is dense in $N_-(S)$ and D_+ is dense in $N_+(S)$.

Assume now that $n_+(S) < \infty$. Since D_+ is dense in $N_+(S)$, we have $D_+ = N_+(S)$. Set

$$K_\pm = \{x \in D_\pm : Wx = 0\}. \tag{5.5}$$

Since $WD_- \subseteq D_+$, we have $\dim(WD_-) \leqslant n_+(S) < \infty$, so the quotient space D_-/K_- is finite dimensional. Since W is closed and D_- is dense in $N_-(S)$, we have $D_- = N_-(S)$. Thus $D(W) = N(S)$ and W is bounded. By Corollary VIII.1.9 in [**3**], the group $\boldsymbol{U}$ is uniformly continuous. □

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Let L be a subspace of $N(S)$. Its $[\cdot,\cdot]_S$-orthogonal 'complement' is defined by

$$L^{[\perp]} = \{y \in N(S) : [x,y]_S = 0, \text{ for } x \in L\}.$$

L is called *neutral* if $L \subseteq L^{[\perp]}$, that is, $[x,y]_S = 0$, for $x,y \in L$. It is maximal neutral if it is not contained in any larger neutral subspace.

Similarly, a subspace $\mathcal{L}$ of $D(S^*)$ is *neutral* if $[x,y]_S = 0$ for $x,y \in \mathcal{L}$. If L is a neutral space in $N(S)$ then, by (2.3), $\mathcal{L} = D(S)\langle + \rangle_{S^*} L$ is a neutral space in $D(S^*)$. The operator $\tilde{S} = S^*|_{\mathcal{L}}$ is a symmetric extension of S (see [**3**, § XII.4]) and

$$n_+(\tilde{S}) = n_+(S) - \dim(L) \quad \text{and} \quad n_-(\tilde{S}) = n_-(S) - \dim(L). \tag{5.6}$$

The operator $\tilde{S}$ is self-adjoint, that is, $n_+(\tilde{S}) = n_-(\tilde{S}) = 0$, if and only if

$$L = L^{[\perp]}. \tag{5.7}$$

It follows from (2.2) and from the properties of the operator J that

$$[x,y]_S = \langle Jx, y\rangle_{S^*}, \quad \text{for } x,y \in N(S). \tag{5.8}$$

Theorem 5.3. *Let a symmetric operator S and a self-adjoint operator T anticommute. Set $G = \{x \in H : \exp(itT)x = x, \text{ for } t \in \mathbb{R}\}$. If*

$$G \bigcap N(S) = \{0\}, \tag{5.9}$$

then

(i) $n_-(S) = n_+(S)$;

(ii) *S has a self-adjoint extension which anticommutes with T;*

(iii) *if the deficiency indices of S are finite, the operator $S+T$ is closed;*

(iv) $n_+(S) = n_+(\overline{S+T})$ *and* $n_-(S) = n_-(\overline{S+T})$.

Proof. Since W is closed, $\mathrm{Ker}(W)$ is closed in $N(S)$. By (2.2), $\mathrm{Ker}(W)$ is invariant for all $U(t)$. Therefore, if $x \in \mathrm{Ker}(W)$, then $\exp(itT)x = U(t)x = x$, for $t \in \mathbb{R}$. Since $G \bigcap N(S) = \{0\}$, we have $\mathrm{Ker}(W) = \{0\}$, so that $K_- = K_+ = \{0\}$. If $n_+(S) < \infty$, then, by Lemma 5.2, $D_- = N_-(S)$ and $WD_- \subseteq N_+(S)$. Since $K_- = \{0\}$, we have

$$n_-(S) = \dim(D_-) = \dim(WD_-) \leqslant n_+(S).$$

Similarly, $n_+(S) = \dim(D_+) = \dim(WD_+) \leqslant n_-(S)$. Part (i) is proved.

Let $E(t)$ be the spectral function of W. Since $\mathrm{Ker}(W) = \{0\}$, we have $E(0) = \lim_{t\to 0+} E(t)$. Set $M = E(0)N(S)$ and $L = (\mathbf{1}_{N(S)} - E(0))N(S)$. Then

$$N(S) = M\langle + \rangle_{S^*} L. \tag{5.10}$$

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Since J and W anticommute and since J is bounded, it follows from Proposition 1.4 in [**10**] that $E(t)J = J(\mathbf{1}_{N(S)} - E(-t))$, for $t \in \mathbb{R}$. Thus

$$JL = J(\mathbf{1}_{N(S)} - E(0))N(S) = E(0)JN(S) = E(0)N(S) = M. \tag{5.11}$$

Since $J^2 = \mathbf{1}_{N(S)}$, we have $JM = L$.

If $x \in L$, then $Jx \in M$. Making use of (5.8) and (5.10), we obtain that $[x, y]_S = \langle Jx, y\rangle_{S^*} = 0$, for $y \in L$, so that L is a neutral subspace. Similarly, M is neutral and, moreover, L and M are maximal neutral subspaces in $N(S)$ and they coincide with their $[\cdot,\cdot]_S$-orthogonal 'complements' in $N(S)$: $L^{[\perp]} = L$ and $M^{[\perp]} = M$.

Set $\mathcal{L} = D(S)\langle+\rangle_{S^*}L$ and $\tilde{S} = S^*|_{\mathcal{L}}$. We obtain from (5.6) and (5.7) that $\tilde{S}$ is a self-adjoint extension of S and that

$$0 = n_+(\tilde{S}) = n_+(S) - \dim(L) \quad \text{and} \quad 0 = n_-(\tilde{S}) = n_-(S) - \dim(L). \tag{5.12}$$

Since the projection $E(0)$ commutes with all operators $U(t)$, the subspaces L and M are invariant for $U(t)$ with $t \in \mathbb{R}$. Hence

$$\begin{aligned}\exp(\mathrm{i}tT)D(\tilde{S}) &= \exp(\mathrm{i}tT)\mathcal{L} = \exp(\mathrm{i}tT)(D(S)\langle+\rangle_{S^*}L)\\ &\subseteq D(S)\langle+\rangle_{S^*}U(t)L = D(\tilde{S})\end{aligned}$$

and, by (5.2),

$$\begin{aligned}\tilde{S}\exp(\mathrm{i}tT)|_{D(\tilde{S})} &= S^*\exp(\mathrm{i}tT)|_{D(\tilde{S})}\\ &= \exp(-\mathrm{i}tT)S^*|_{D(\tilde{S})} = \exp(-\mathrm{i}tT)\tilde{S}|_{D(\tilde{S})}.\end{aligned}$$

Thus the operators $\tilde{S}$ and T anticommute. Part (ii) is proved.

Assume now that the deficiency indices of S are finite. It follows from Lemma 5.2 that $N(S) \subset D(T)$. Set $D = D(S) \bigcap D(T)$. We obtain from Proposition 4.2 that $D(\overline{S+T}) \bigcap D(T) = D$. Since $L \subset N(S) \subset D(T)$ and $L \bigcap D = \{0\}$, we have $D(\overline{S+T}) \bigcap L = \{0\}$.

On the other hand, the operator $\tilde{S} + T$ is defined on

$$\begin{aligned}\mathcal{M} &= D(\tilde{S})\bigcap D(T) = (D(S)\langle+\rangle_{S^*}L)\bigcap D(T)\\ &= \Big(D(S)\bigcap D(T)\Big)\langle+\rangle_{S^*}L = D\langle+\rangle_{S^*}L.\end{aligned}$$

Since $\tilde{S}$ and T anticommute, it follows from Theorem 2.1 in [**10**] that $\tilde{S} + T$ is self-adjoint and hence closed: $D(\tilde{S} + T) = \mathcal{M}$. Clearly, $\tilde{S} + T$ is a self-adjoint extension of $S + T$. Thus $S + T \subseteq \overline{S+T} \subseteq \tilde{S} + T$. If $S + T \neq \overline{S+T}$, then $D \subset D(\overline{S+T}) \subseteq D\langle+\rangle_{S^*}L$, so $D(\overline{S+T}) \bigcap L \neq \{0\}$. This contradiction shows that $S + T$ is closed. Part (iii) is proved.

If $n_\pm(S) = \infty$, then (iv) follows from Proposition 4.2. Let $n_\pm(S)$ be finite. By (iii), $R = S + T$ is closed, $D(R) = D$ and $R \subseteq \tilde{S} + T$. Since $\tilde{S} + T$ is self-adjoint, $\tilde{S} + T \subseteq R^*$, so that $\tilde{S} + T = R^*|_{\mathcal{M}}$. Since L is neutral, we have from (2.3) that $\mathcal{M}$ is a neutral subspace in $D(S^*)$. Hence, by (3.1), $\mathcal{M}$ is a neutral subspace in $D(R^*)$. Since

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$D(R^*) = D(R)\langle+\rangle_{R^*} N(R) = D\langle+\rangle_{R^*} N(R)$, we obtain that $\mathcal{M} = D\langle+\rangle_{R^*} L'$, where $\dim(L) = \dim(L')$ and L' is a neutral subspace in $N(R)$. By (5.6),

$$n_+(R^*|_{\mathcal{M}}) = n_+(R) - \dim(L') \quad \text{and} \quad n_-(R^*|_{\mathcal{M}}) = n_-(R) - \dim(L').$$

Since $\tilde{S} + T = R^*|_{\mathcal{M}}$ is self-adjoint, we have $n_+(R^*|_{\mathcal{M}}) = n_-(R^*|_{\mathcal{M}}) = 0$. Therefore,

$$n_+(R) = n_-(R) = \dim(L') = \dim(L).$$

Comparing this with (5.12), we complete the proof. □

In the next theorem we consider the case when $n_-(S) = n_+(S) < \infty$ and prove the results of Theorem 5.3 without the restriction given in (5.9).

Theorem 5.4. *Let the operators S and T be the same as in Theorem 5.3. If $n_-(S) = n_+(S) < \infty$, then*

(i) *S has a self-adjoint extension which anticommutes with T;*

(ii) *the operator $S + T$ is closed;*

(iii) *$n_+(S) = n_+(S + T)$ and $n_-(S) = n_-(S + T)$.*

Proof. Since $n_\pm(S) < \infty$, it follows from Lemma 5.2 that $D_\pm = N_\pm(S)$, so $K_\pm = \{x \in N_\pm(S) : Wx = 0\}$ (see (5.5)). Let P_+ be the orthogonal complement of K_+ in $N_+(S)$ and let P_- be the orthogonal complement of K_- in $N_-(S)$. By Lemma 5.2, $WP_+ \subseteq N_-(S)$. Since W is self-adjoint, if $y \in P_+$ and $x \in K_-$, then $0 = \langle Wx, y\rangle_{S^*} = \langle x, Wy\rangle_{S^*}$. Hence $WP_+ \subseteq P_-$. Similarly, $WP_- \subseteq P_+$. Thus

$$\dim(P_+) = \dim(WP_+) \leqslant \dim(P_-) \quad \text{and} \quad \dim(P_-) = \dim(WP_-) \leqslant \dim(P_+).$$

Therefore, $\dim(P_+) = \dim(P_-)$, so that $\dim(K_+) = \dim(K_-)$.

The subspace $P = P_+\langle+\rangle_{S^*} P_-$ is invariant for W. If e is an eigenvector of W in P with eigenvalue λ, it follows from (5.3) that $W(Je) = -JWe = -\lambda Je$, so Je is an eigenvector of W with eigenvalue $-\lambda$. Since $\operatorname{Ker}(W \mid P) = \{0\}$, there is an orthonormal basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ in $(P, \langle\cdot,\cdot\rangle_{S^*})$ such that

$$f_i = Je_i, \quad We_i = \lambda_i e_i, \quad Wf_i = -\lambda_i f_i \quad \text{and} \quad \lambda_i > 0.$$

We obtain from (5.8) that

$$[e_i, e_j]_S = \langle Je_i, e_j\rangle_{S^*} = \langle f_i, e_j\rangle_{S^*} = 0$$

for all i, j. Thus the subspace $\mathcal{M}$ spanned by all $\{e_i\}_{i=1}^n$ is neutral, invariant for W and its $[\cdot,\cdot]_S$-orthogonal complement in P coincides with $\mathcal{M}$.

Let $K = K_+\langle+\rangle_{S^*} K_-$, let $m = \dim(K_+) = \dim(K_-)$ and let $\{h_i^-\}_{i=1}^m$ and $\{h_i^+\}_{i=1}^m$ be orthonormal bases in K_- and K_+, respectively. By (5.8),

$$\begin{aligned}[h_i^- + h_i^+, h_j^- + h_j^+]_S &= \langle J(h_i^- + h_i^+), h_j^- + h_j^+\rangle_{S^*} = \langle -h_i^- + h_i^+, h_j^- + h_j^+\rangle_{S^*} \\ &= -\langle h_i^-, h_j^-\rangle_{S^*} + \langle h_i^+, h_j^+\rangle_{S^*} = 0,\end{aligned}$$

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for all i, j. Hence the subspace $\mathcal{N}$ spanned by $\{h_i^- + h_i^+\}_{i=1}^m$ is neutral and its $[\cdot,\cdot]_S$-orthogonal complement in K coincides with $\mathcal{N}$. Since $\mathcal{N} \subset \mathrm{Ker}(W)$, we have that $\mathcal{N}$ is invariant for W.

The subspace $L = \mathcal{M}\langle + \rangle_{S^*}\mathcal{N}$ is invariant for W. Since $J\mathcal{M} \subset P$ and $J\mathcal{N} \subset K$, the subspaces $\mathcal{M}$ and $\mathcal{N}$ are orthogonal with respect to $[\cdot,\cdot]_S$. Hence L is neutral and $L^{[\perp]} = L$. Setting $\mathcal{L} = D(S)\langle + \rangle_{S^*}L$ and $\tilde{S} = S^*|_{\mathcal{L}}$ and repeating the argument used in the proof of Theorem 5.3, we complete the proof of the theorem. □

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References

1. N. I. Ahiezer and I. M. Glazman, *The theory of linear operators in Hilbert spaces* (Ungar, New York, 1961).
2. J. M. Cook, Asymptotic properties of a boson field with a given source, *J. Math. Phys.* **2** (1961), 33–45.
3. N. Dunford and J. T. Schwartz, *Linear operators*, vols I and II (Wiley Interscience, New York, 1963).
4. P. E. T. Jorgensen and P. S. Muhly, Self-adjoint extensions satisfying the Weyl operator commutation relations, *J. Analyse Math.* **37** (1980), 46–99.
5. T. Kato, On the convergence of the perturbation method, *J. Fac. Sci. Univ. Tokyo (1)* **6** (1951), 145–226.
6. T. Kato, *Perturbation theory for linear operators* (Springer, 1966).
7. Y. Kato, Some converging examples of the perturbation series in the quantum field theory, *Prog. Theor. Phys.* **26** (1961), 99–122.
8. Y. Kato and N. Mugibayashi, Regular perturbation and asymptotic limits of operators in quantum field theory, *Prog. Theor. Phys.* **30** (1963), 103–133.
9. E. Kissin and V. Shulman, *Representations on Krein spaces and derivations of C^*-algebras* (Addison-Wesley, 1997).
10. S. Pedersen, Anticommuting self-adjoint operators, *J. Funct. Analysis* **89** (1990), 428–443.
11. C. R. Putnam, *Commutation properties of Hilbert space operators* (Springer, 1967).
12. F. Rellich, Storungstheorie der Spektralzerlegung, III, *Math. Ann.* **116** (1939), 555–570.
13. F.-H. Vasilescu, Anticommuting self-adjoint operators, *Rev. Roumaine Math. Pures Appl.* **28** (1983), 77–91.

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