AN ANALOGUE OF THE MULTINOMIAL THEOREM

T. V. Narayana

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1. Let x_i, y_i $(i = 1, 2, \ldots, t)$ and n be non-negative integers. A function $(n; x_1, \ldots, x_t)$ may be defined recursively as follows: let $(0;0, \ldots, 0) = 1$ and

$$(n;x_1, \ldots, x_t) = \begin{cases} 0 & \text{if } \sum_{i=1}^{t} x_i > n \\ \sum_{y_1=0}^{x_1} \cdots \sum_{y_t=0}^{x_t} (n-1; y_1, \ldots, y_t) & \text{otherwise.} \end{cases} \tag{1}$$

If Δ_n denotes the t x t determinant given by

$$\Delta_n(x_1, \ldots, x_t) = \Delta_n = |a_{rs}|_{r, s=1, \ldots, t}; \quad a_{rs} = \begin{cases} -x_r & \text{if } r \neq s \\ n-x_r & \text{if } r=s \end{cases} \tag{2}$$

$$= n^t[1 - \frac{1}{n} \sum_{i=1}^{t} x_i] , \tag{3}$$

then it is easy to verify that, when $\sum_{i=1}^{t} x_i \leqq n$,

$$(n; x_1, \ldots, x_t) = \Delta_{n+1} \prod_{i=1}^{t} (n + 1 + x_i)^{-1} \binom{n+1+x_i}{x_i} \tag{4}$$

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where the last symbol denotes a binomial coefficient. Interest in the relation (4) arose in a study of simple sampling plans of size n and it includes, as special cases, some known ballot theorems in probability [1, 66, 5, 6]. In this note* we confine ourselves to the case t = 3 (apart from a verification of (4) in § 2); our purpose being to exhibit its connection with a certain set T(n) of vectors forming a distributive lattice, partially ordered by a relation of domination [cf. 2, 3]. If $n \geq 1$, and $a_1, \ldots, a_n$ are non-negative integers, T(n) consists of the vectors $A_n = (a_1, \ldots, a_n)$ which satisfy

(i) $0 \leqq a_1 \leqq a_2 \leqq \cdots \leq a_n$,

(ii) $a_i \leqq 3i \ (i = 1, 2, \ldots, n)$.

A_n is said to dominate B_n if, and only if, $a_i \geqq b_i$ $(i = 1, 2, \ldots, n)$ and both A_n and B_n are in T(n). We introduce certain subsets T(n, r) and $S_r(n; x_1, x_2, x_3)$ of T(n), where $r = 0, 1, \ldots, n$, $x_1 + x_2 + x_3 = r$. Let

$$T(n, 0) = \{(0, \ldots, 0)\}, \quad T(n, n) = \{A_n \mid a_1 > 0\}$$
$$T(n, r) = \{A_n \mid a_1 = \ldots = a_{n-r} = 0, \ a_{n-r+1} > 0\}, \quad r = 1, \ldots, n-1. \tag{5}$$

Thus T(n, r), $0 \leq r \leq n$ form a partition of T(n). Also, if $A_n \in T(n, r)$, then it has exactly r <u>positive</u> components $a_{n-r+1}, \ldots, a_n$, by (1). Of these r components, let x_i of them be $\equiv i-1 \pmod 3$ so that

$$x_1 + x_2 + x_3 = r .$$

For fixed $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$ with $x_1 + x_2 + x_3 = r$, let $S_r(n; x_1, x_2, x_3)$ denote the set of all A_n in T(n, r) with this

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*The case t = 2 was treated in [4].

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property. Since the number of solutions of $x_1 + x_2 + x_3 = r$ in non-negative integers is $\binom{r+2}{2}$, we see that $T(n, r)$ is partitioned into $\binom{r+2}{2}$ sets $S_r(n; x_1, x_2, x_3)$.

Let $[n, r; x_1, x_2, x_3]$ denote the number of vectors in $S_r(n; x_1, x_2, x_3)$. Then we shall prove, in § 3, that it is independent of r and, moreover,

$$[n, r; x_1, x_2, x_3] = (n; x_1, x_2, x_3) \, ,$$

where the function on the right is given in (4), with $t = 3$.

We remark that the set $T(n)$ and its relation of domination have a simple geometrical interpretation in the plane, which indicates a connection with ballot theorems [cf. 1, 5]. Consider the triangle Δ ,

$$0 \leqq 3y \leqq x \leqq 3n+4$$

and let $P(n)$ be the path from $O(0, 0)$ to $P(3n + 4, n)$ inside Δ of the form

$$\{ OP_o P_o{}' P_1 P_1{}' \ldots P_{n-1} P'_{n-1} P\} \, ,$$

where $P_{n-j} = (3n + 4 - a_j, \; n - j)$, $P'_{n-j} = P_{n-j} + (0, 1)$. Clearly any vector of $T(n)$ gives a path $P(n)$ of this type and conversely. Also, the relation of domination for $T(n)$ means that if A_n dominates B_n then the path corresponding to B_n is never above that of A_n.

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2. Proof of (4). It is clearly true for $n = 0, 1$. Suppose then that (4) holds for $n = k - 1$ $(k \geqq 2)$ and $\Sigma x_i \leqq k - 1$, we will prove that (4) holds for $n = k$ and $\Sigma x_i \leqq k$. Now, by (1),

$$(k; x_1, \ldots, x_t) = \sum_{y_1=0}^{x_1} \cdots \sum_{y_t=0}^{x_t} (k-1; y_1, \ldots, y_t).$$

Since $\Sigma x_i \leqq k$, we have $\Sigma y_i < k$ unless $y_1 = x_1, \ldots, y_t = x_t$ and $x_1 + \ldots + x_t = k$. This is a single term and, by (1), has value 0. Hence we can suppose $\Sigma x_i \leqq k-1$, in which case we can apply (4) to obtain

$$(k;x_1, \ldots, x_t) = \Sigma \ldots \Sigma \prod_{i=1}^{t} (k+y_i)^{-1} \binom{k+y_i}{y_i} \Delta_k(y_1, \ldots, y_i).$$

Taking the factor $(k+y_i)^{-1} \binom{k+y_i}{y_i}$ into the i^{th} column of Δ, we can effect the sum with respect to y_i by adding the t determinants, which differ only in the i^{th} column, in the usual way. Repeating for $i = 1, 2, \ldots, t$, we see that the sum on the right is itself a determinant, say $|d_{rs}|$, where

$$d_{rs} = \begin{cases} -\displaystyle\sum_{y_r=0}^{x_r} \frac{y_r}{k+y_r} \binom{k+y_r}{y_r} & \text{if } r \neq s \\[2ex] \displaystyle\sum_{y_r=0}^{x_r} \frac{k-y_r}{k+y_r} \binom{k+y_r}{y_r} & \text{if } r = s. \end{cases}$$

Since

$$\binom{k+j}{j-1} = \sum_{x=1}^{j} \binom{k+j-x}{j-x},$$

it follows that

$$|d_{rs}| = \prod_{i=1}^{t} (k+1+x_i)^{-1} \binom{k+1+x_i}{x_i} \Delta_{k+1}(x_1, \ldots, x_t).$$

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3. Theorem: $[n, r; x_1, x_2, x_3] = (n; x_1, x_2, x_3)$.

Proof. Since the theorem holds for $n = 0, 1$, it is sufficient to prove that

$$[n, r; x_1, x_2, x_3] = \sum_{y_1=0}^{x_1} \sum_{y_2=0}^{x_2} \sum_{y_3=0}^{x_3} (n-1; y_1, y_2, y_3)$$

for $n \geqq 2$.

Let r be any fixed integer satisfying $0 \leqq r \leqq n$. Then this may be effected by setting up a (1, 1) correspondence between the sets $S_r(n; x_1, x_2, x_3)$ and

$$S^*(n-1; x_1, x_2, x_3) = \bigcup_{y_1, y_2, y_3} S_t(n-1; y_1, y_2, y_3),$$

where $0 \leqq y_i \leqq x_i$ $(i = 1, 2, 3)$ and t denotes $y_1 + y_2 + y_3$. Let

$$\begin{aligned}
S_r^o(n; x_1, x_2, x_3) &= \{A_n \mid A_n \in T(n, r),\ a_i > 3,\ (i=1, 2, \ldots, n)\} \\
S_r^n(n; x_1, x_2, x_3) &= \{A_n \mid A_n \in T(n, r),\ a_i \leqq 3,\ (i=1, 2, \ldots, n)\} \qquad (6)\\
S_r^k(n; x_1, x_2, x_3) &= \{A_n \mid A_n \in T(n, r),\ a_k \leqq 3,\ a_{k+1} > 3\},
\end{aligned}$$

$$k = 1, 2, \ldots, n-1.$$

Then $S_r^k(n; x_1, x_2, x_3)$, $k = 0, 1, \ldots, n$ form a partition of $S_r(n; x_1, x_2, x_3)$. Consider the mapping P_i of $S_r^1(n; x_1, x_2, x_3)$ into the set $S = \{P_i(A_n) \mid A_n \in S_r^i(n; x_1, x_2, x_3)\}$, where $P_i(A_n)$ is obtained from A_n by the following three operations

(a) suppress the first element of A_n,

(b) replace the next $i - 1$ elements of A_n by 0,

(c) subtract 3 from the last $n - i$ elements of A_n.

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Thus $P_i(A_n) = A_{n-1}$ say, where $A_{n-1} \in S^j_{n-i}$ for some $j \geq i - 1$. In fact, if x_k' of the elements of $P_i(A_n)$ are $\equiv k - 1 \pmod 3$, we have

$$P_i(A_n) \in S^j_{n-i}(n-1, x_1', x_2', x_3').$$

We first show that P_i is a (1, 1) mapping of $S^i_r(n; x_1, x_2, x_3)$ into S. Suppose then that $P_i(A_n) = P_i(B_n)$. This means that $P_i(A_n)$ and $P_i(B_n)$ are both elements of some $S^j_{n-i}(n-1; x_1', x_2', x_3')$ and, in particular, that the last n-i elements are identical. Let $A_n = (a_1, \ldots, a_i, a_{i+1}, \ldots a_n)$, $B_n = (b_1, \ldots, b_i, a_{i+1}, \ldots, a_n)$, since these must also have identical elements in the last n-i places by (c). Since $r > n - i$, we consider the elements

$$a_{n-r+1}, \ldots a_i,$$
$$b_{n-r+1}, \ldots b_i.$$

By (5) and (6) we see that they are all positive and ≤ 3. Let x_j of these a's be $\equiv j-1 \pmod 3$. Since $a_k = b_k$ $(k = i + 1, \ldots, n)$, we see that y_j of these b's are $\equiv j-1 \pmod 3$. Moreover, they are all ≤ 3. Hence

either $y_1 \neq 0$ and $a_{n-r+1} = \ldots = a_i = b_{n-r+1} = \ldots = b_i = 3$

$$\text{or} \quad y_1 = 0 \text{ and} \begin{cases} a_{n-r+1} = \ldots = a_{n-r+y_2} = 1;\ a_{n-r+y_2+1} = \ldots = a_i = 2 \\ b_{n-r+1} = \ldots = b_{n-r+y_2} = 1;\ b_{n-r+y_2+1} = \ldots = b_i = 2. \end{cases}$$

Hence $A_n = B_n$. Consider now the mapping P of $S_r(n; x_1, x_2, x_3)$ into $S^*(n-1; x_1, x_2, x_3)$ where $P(A_n) = P_i(A_n)$ whenever $A_n \in S^i_r(n; x_1, x_2, x_3)$, $(i=0, 1, \ldots n)$. Clearly $P_i(A_n) \in S^*$, since $P_i(A_n) \in S^j_{n-i}(n-1; y_1, y_2, y_3)$ for some $j \geq i-1$ and $0 \leq y_i \leq x_i$,

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$(i = 1, 2, 3)$ by (a), (b), (c). We note also that P is (1, 1), for supposing $A_n \in S_r^i$, $B_n \in S_r^k$, we have

$$P(A_n) \in S^j_{n-i} \ (j \geqq i-1), \quad P(B_n) \in S^{\ell}_{n-k} \ (\ell \geqq k-1),$$

and then, $P(A_n) = P(B_n)$ implies that $n - i = n - k$ or $i = k$ and $j = \ell$ and so $P = P_i$.

Finally, we show that P is a (1, 1) map onto S^*. Suppose then that $A_{n-1} \in S^k_{n-j}(n-1; y_1, y_2, y_3)$, where $0 \leqq y_i \leqq x_i$. Write $A_{n-1} = (0, \ldots 0, a'_{j+1}, \ldots, a'_n)$ and consider any $X_n \in S^j_r(n; x_1, x_2, x_3)$, where

$$X_n = (a_1, \ldots, a_j, a'_{j+1}, \ldots, a'_n).$$

Observe that $a_1 = \ldots = a_{n-r} = 0$, and we may choose

$$a_{n-r+1} = \ldots = a_{n-r+(x_2-y_2)} = 1,$$

$$a_{(n-r)+(x_2-y_2)+1} = \ldots = a_{(n-r)+(x_2-y_2)+(x_3-y_3)} = 2,$$

$$a_{(n-r)+(x_3-y_3)+1} = \ldots = a_j = 3,$$

since $(n-r) + (x_2-y_2) + (x_3-y_3) + (x_1-y_1) = j$.

Thus, for this particular X_n, there are exactly $n-r$ zero elements, $(x_2-y_2) + y_2$ elements $\equiv 1 \pmod 3$, $(x_3-y_3) + y_3$ elements $\equiv 2 \pmod 3$ and $(x_1-y_1) + y_1$ elements $\equiv 0 \pmod 3$. This concludes the proof.

4. It is evident that the method of § 3 is rather more general, in that we could, for example, consider sets of vectors $(a_1, \ldots, a_n)$ satisfying

(i)' $1 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_n$

(ii)' $a_i \leqq \lambda i \qquad (i = 1, \ldots, n).$

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Even the condition (ii)' could be weakened to read $a_i \leqq k K_i$, where $0 < K_1 < \ldots < K_n$. But such questions do not have such immediate importance in their applications to statistics.

REFERENCES

1. W. Feller, An introduction to probability theory and its applications, Vol. 1 (New York 1957).

2. T.V. Narayana, Sur les treillis formés par les partitions d'un entier; leurs applications à la théorie des probabilitiés. CR., vol. 240, pp. 1188-89 (1955).

3. T.V. Narayana and G.E. Fulton, A note on the compositions of an integer. Can. Math. Bull. Vol. 1, No. 3, pp. 169-173, Sept. 1958.

4. B. Brainerd and T.V. Narayana, A Note on Simple Binomial Sampling Plans. Annals of Math. Statistics, 32, (1961), 906-8.

5, 6 T.V. Narayana and S.G. Mohanty, Some properties of compositions and their applications to probability and statistics I, II (submitted to Biometrische Zeitschrift).

University of Alberta

and National Institute of Arthritis and Metabolic Diseases.