A GENERALIZATION OF HURWITZ'S THEOREM

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Hurwitz's theorem says that the limit of schlicht functions, in the topology of compact convergence, is again a schlicht function or a constant function. We generalize this to mappings between Riemann surfaces and get more precise information on the relation between the distribution of values of analytic functions and the topology of compact convergence on the space of all analytic maps.

Let X and Y be two Riemann surfaces. Let $A_c(X, Y)$ be the set of all analytic maps $f: X \to Y$ with the topology of compact convergence. Let K be a compact part of X and V an open part of Y ; then the subsets $\Omega(K, V)$ of $f \in A_c(X, Y)$ such that $f(K) \subset V$ form a base for the topology on $A_c(X, Y)$. See Bourbaki [**2**, pp. 18–19].

We describe the value distribution of an analytic map with the following notations. Let $x \in X$, $y \in Y$, and $f \in A_c(X, Y)$; then we define $\operatorname{ord}_{x,y}(f) = n$ as follows :

(i) if $f(x) \neq y$, then $n = 0$, and

(ii) if $f(x) = y$, then f can be represented as

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + \ldots \quad (a_n \neq 0),$$

in a coordinate neighbourhood of x. (If $f(x) = y$ for all $x \in X$, then $\operatorname{ord}_{x,y}(f) = \infty$ by convention.) Clearly $\operatorname{ord}_{x,y}(f)$ is the generalization of the notion of the order of a zero of a numerical function. Finally, let

$$\deg_y(f) = \sum_{x \in X} \operatorname{ord}_{x,y}(f).$$

Then $\deg_y(f)$ is the number of times that f takes the value y counted with proper multiplicities. It could happen that $\deg_y(f) = \infty$.

Let $x \in X$ and $y \in Y$. Then, for all natural numbers n, we let $E_n(x, y)$ denote the set of all $f \in A_c(X, Y)$ such that $\operatorname{ord}_{x,y}(f) = n$, and $F_n(y)$ denote the set of all $f \in A_c(X, Y)$ such that $\deg_y(f) = n$. $\bar{E}_n(x, y)$ (resp. $\bar{F}_n(y)$) is $E_n(x, y)$ (resp. $F_n(y)$) together with the constant functions.

THEOREM. *Let X and Y be two Riemann surfaces and let $x \in X$ and $y \in Y$; then $E_n(x, y)$ is open, $F_n(y)$ is open, and $\bigcup_{r \leqslant n} \bar{F}_r(y)$ is closed in the topology on $A_c(X, Y)$.*

Proof. Let $\{f_k\}$ be an arbitrary sequence in $A_c(X, Y)$ converging to $f \in A_c(X, Y)$. Let $f^{-1}(y) = \{x_i\} = A$, which is a discrete set. Let V' be a coordinate neighbourhood of y with coordinate function z' such that $z'(y) = 0$, and where V' is determined by $|z'| < 1$. Let V_i be a family of coordinate neighbourhoods with coordinate functions z_i such that $z_i(x_i) = 0$, where V_i is determined by $|z_i| < 1$, V_i is compact, $\{\bar{V}_i\}$ are disjoint, and $f(\bar{V}_i) \subset V'$.

(i) $E_n(x, y)$ is open. For $n = 0$, $f(x) \neq y$, so that there exists an N, such that $f_k(x) \neq y$ for $k > N$; therefore $E_0(x, y)$ is open. For $n > 0$, let $x = x_q \in A$. For $k > N_1$, we have $f_k(V_q) \subset V'$, and we can represent f_k and f as complex-valued functions of a complex variable. Moreover, the derivatives $f'_k(z_q)$ tend to $f'(z_q)$ in the topology of compact convergence. Let a $(0 < a < 1)$ be such that no $f_k(z_q)$ has a zero on $|z_q| = a$. By the argument principle [**1**],

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we have

$$\operatorname{ord}_{x,y}(f) = \frac{1}{2\pi i}\int_{|z_q|=a} \frac{f_k'(z_q)}{f_k(z_q)}\,dz_q \to \frac{1}{2\pi i}\int_{|z_q|=a} \frac{f'(z_q)}{f(z_q)}\,dz_q = \operatorname{ord}_{x,y}(f).$$

Since $\operatorname{ord}_{x,y}$ is discrete-valued, there exists an $N > N_1$, such that $\operatorname{ord}_{x,y}(f_k) = \operatorname{ord}_{x,y}(f)$ for $k > N$. This proves that $E_n(x, y)$ is open.

(ii) $F_n(y)$ is open. $F_n(y)$ is the union of $E_{n_1}(x_1, y) \cap \dots \cap E_{n_p}(x_p, y)$ consisting of $f \in A_c(X, Y)$ with $f^{-1}(y) = \{x_1, \dots, x_p\}$ and $n = n_1 + \dots + n_p$, where $n_i = \operatorname{ord}_{x_i,y}(f)$ $(1 \leqslant i \leqslant p)$. The sets $E_{n_i}(x_1, y) \cap \dots \cap E_{n_p}(x_p, y)$ are open by (i); therefore their union $F_n(y)$ is open.

(iii) $\bigcup_{r \leqslant n} \bar{F}_r(y)$ is closed. Suppose that $f_k \in \bigcup_{r \leqslant n} \bar{F}_r(y)$ for all $k > 0$ and $f \notin \bigcup_{r \leqslant n} \bar{F}_r(y)$; then we have $x_1, \dots, x_p \in A$, such that

$$\sum_{1 \leqslant j \leqslant p} \operatorname{ord}_{x_j,y}(f) = m > n.$$

For each j with $1 \leqslant j \leqslant p$, there exists N_j such that, if $k > N_j$, then we have $f_k(V_j) \subset V'$ and

$$\frac{1}{2\pi i}\int_{|z_j|=a_j} \frac{f_k'(z_j)}{f_k(z_j)}\,dz_j = \frac{1}{2\pi i}\int_{|z_j|=a_j} \frac{f'(z_j)}{f(z_j)}\,dz_j$$

for an appropriate a_j. Let $N = \max_{1 \leqslant j \leqslant p} N_j$. Then, if $k > N$, we have

$$\deg_y(f_k) \geqslant \sum_{1 \leqslant j \leqslant p} \frac{1}{2\pi i}\int_{|z_j|=a_j} \frac{f_k'(z_j)}{f_k(z_j)}\,dz_j = \sum_{1 \leqslant j \leqslant p} \frac{1}{2\pi i}\int_{|z_j|=a_j} \frac{f'(z_j)}{f(z_j)}\,dz_j = m > n,$$

which is absurd. Hence $f \in \bigcup_{r \leqslant n} \bar{F}_r(y)$, and $\bigcup_{r \leqslant n} \bar{F}_r(y)$ is closed.

Let $f \in A_c(X, Y)$. We recall that f is defined to be schlicht if $x_1 \neq x_2$ implies that $f(x_1) \neq f(x_2)$. The family of schlicht functions and constants in $A_c(X, Y)$ is just the set $\bigcap_{y \in Y} \bar{F}_1(y)$, which is closed by the theorem. We obtain Hurwitz's theorem as a corollary.

COROLLARY (Hurwitz). *If f_k is a convergent sequence of schlicht functions in $A_c(X, Y)$, then the limit function f is schlicht or a constant.*

REFERENCES

1. L. V. Ahlfors, *Complex analysis* (New York, 1953).
2. N. Bourbaki, *Topologie générale* (Paris, 1949), Chap. X, §2.

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