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On the Smallest and Largest Zeros of Müntz–Legendre Polynomials

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Abstract. Müntz–Legendre polynomials $L_n(\Lambda; x)$ associated with a sequence $\Lambda = \{\lambda_k\}$ are obtained by orthogonalizing the system $(x^{\lambda_0}, x^{\lambda_1}, x^{\lambda_2}, \dots)$ in $L_2[0, 1]$ with respect to the Legendre weight. If the λ_k's are distinct, it is well known that $L_n(\Lambda; x)$ has exactly n zeros $l_{n,n} < l_{n-1,n} < \cdots < l_{2,n} < l_{1,n}$ on $(0, 1)$.

First we prove the following global bound for the smallest zero,

$$\exp\left(-4\sum_{j=0}^{n}\frac{1}{2\lambda_j+1}\right) < l_{n,n}.$$

An important consequence is that if the associated Müntz space is non-dense in $L_2[0, 1]$, then

$$\inf_n x_{n,n} \ge \exp\left(-4\sum_{j=0}^{\infty}\frac{1}{2\lambda_j+1}\right) > 0,$$

so the elements $L_n(\Lambda; x)$ have no zeros close to 0.

Furthermore, we determine the asymptotic behavior of the largest zeros; for k fixed,

$$\lim_{n\to\infty} |\log l_{k,n}| \sum_{j=0}^{n}(2\lambda_j + 1) = \left(\frac{j_k}{2}\right)^2,$$

where j_k denotes the k-th zero of the Bessel function J_0.

1 Introduction and Main Results

Müntz polynomials associated with a sequence $\Lambda = \{\lambda_k\}_{k=0}^{\infty}$ are functions of the form

$$\sum_{k=0}^{n} c_k x^{\lambda_k},$$

and the corresponding *Müntz space* is defined by

$$M(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, x^{\lambda_2}, \dots\}.$$

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If the λ_k's are real and satisfy

$$\inf_{k\geq 0}\{\lambda_k\} > -1/2 \qquad \text{and} \qquad \lambda_k \neq \lambda_j, \quad j \neq k, \tag{1.1}$$

then the celebrated Müntz Theorem [1, 2, 4] states that $M(\Lambda)$ is dense in $L_2[0,1]$ if and only if

$$\sum_{k=0}^{\infty} \frac{1}{2\lambda_k + 1} = \infty. \tag{1.2}$$

If the constant functions are included (*i.e.*, $\lambda_0 = 0$) and $\inf_{k\geq 1} \lambda_k > 0$, (1.2) is also equivalent to the denseness of $M(\Lambda)$ in $C[0,1]$.

The *n-th Müntz-Legendre polynomial* $L_n(\Lambda; x)$ is determined by the orthogonality conditions

$$\int_0^1 L_n(\Lambda;x)L_m(\Lambda;x)dx = \frac{\delta_{n,m}}{(2\lambda_n+1)}, \qquad n,m = 0,1,2,\dots$$

and is defined by

$$L_n(\Lambda;x) := \frac{1}{2\pi i}\int_\Gamma \prod_{k=0}^{n-1} \frac{t+\lambda_k+1}{t-\lambda_k}\frac{x^t}{t-\lambda_n}dt,$$

where the simple contour Γ surrounds all the zeros of the denominator of the integrand. If (1.1) is satisfied, then $L_n(\Lambda;x)$ is indeed an element of the Müntz space $M(\Lambda)$, and the Residue Theorem shows that

$$L_n(\Lambda;x) = \sum_{k=0}^{n} c_{k,n}x^{\lambda_k}, \qquad c_{k,n} = \frac{\prod_{j=0}^{n-1}(\lambda_k+\lambda_j+1)}{\prod_{j=0, j\neq k}^{n}(\lambda_k-\lambda_j)}.$$

It is well known ([3]) that if the λ_k's are distinct, then $L_n(\Lambda;x)$ has precisely n zeros on $(0,1)$, and we denote them by

$$0 < l_{n,n} < l_{n-1,n} < \dots < l_{2,n} < l_{1,n} < 1.$$

The zeros of L_n and L_{n+1} strictly interlace, *i.e.*,

$$l_{n+1,n+1} < l_{n,n} < l_{n,n+1} < l_{n-1,n} < \dots < l_{1,n} < l_{1,n+1}. \tag{1.3}$$

In [2, E.8, §3.4] Borwein and Erdélyi give a global estimate for the zeros. If we let $\lambda_{\min}^{(n)} := \min\{\lambda_0,\dots,\lambda_n\}$ and $\lambda_{\max}^{(n)} := \max\{\lambda_0,\dots,\lambda_n\}$, then

$$\exp\Big(-2\frac{2n+1}{2\lambda_{\min}^{(n)}+1}\Big) < l_{n,n} < \dots < l_{1,n} < \exp\Big(\frac{-j_1^2}{2(2n+1)(2\lambda_{\max}^{(n)}+1)}\Big), \tag{1.4}$$

where j_1 is the smallest positive zeros of the Bessel function J_0 of order 0 (see [6,10]).

D. S. Lubinsky and E. B. Saff [5] determined the zero distribution of the Müntz extremal polynomials $T_{n,p}(\Lambda)$ that satisfy

$$\|T_{n,p}(\Lambda)\|_{L_p[0,1]} = \min_{c_0,\dots,c_{n-1}} \Big\| x^{\lambda_n} - \sum_{j=0}^{n-1} c_j x^{\lambda_j} \Big\|_{L_p[0,1]}.$$

Namely, if

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = \alpha,$$

for some $\alpha > 0$, then the normalized zero counting measure of $T_{n,p}(\Lambda)$ converges weakly to

$$\frac{\alpha}{\pi} \frac{t^{\alpha-1}}{\sqrt{t^\alpha(1-t^\alpha)}} dt,$$

and if $\alpha = 0$ or 1, the limiting measure is a Dirac delta at 0 or 1 respectively. Letting $p = 2$ gives the Müntz–Legendre polynomials. The asymptotics of the spacing between two consecutive zeros $l_{k+1,n} < l_{k,n}$ was studied by the author in [9].

In [7] the author determined the asymptotic behavior of $L_n(\Lambda; x)$ as $n \to \infty$ uniformly for $x \in (0,1)$. The main tool was the following formula, which holds for all real sequences Λ. For $x \in (0,1)$,

$$L_n(\Lambda; x) = \frac{1}{\pi\sqrt{x}} \int_0^\infty \frac{\sin(\Theta_n(t) - t\log x)}{\sqrt{\lambda_n^{*2} + t^2}} dt, \tag{1.5}$$

where

$$\Theta_n(t) = 2\sum_{j=0}^{n-1} \arctan\frac{\lambda_j^*}{t} + \arctan\frac{\lambda_n^*}{t}$$

and $\lambda_k^* = \lambda_k + 1/2$ for all k.

In [8] this formula was revisited and used to compute the endpoint limit asymptotics when $x \longrightarrow 1^-$. The main result was the following. Suppose that $\Lambda : -1/2 < \lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots$ satisfies the regularity condition

$$\lim_{n\to\infty} \frac{\Sigma_n}{2\lambda_n + 1} = \infty, \tag{1.6}$$

where

$$\Sigma_n := \sum_{k=0}^{n-1} (2\lambda_k + 1) + \frac{2\lambda_n + 1}{2}. \tag{1.7}$$

Then uniformly for bounded $y \ge 0$,

$$\lim_{n\to\infty} L_n(e^{-y^2/4\Sigma_n}) = \lim_{n\to\infty} L_n\Big(1 - \frac{y^2}{4\Sigma_n}\Big) = J_0(y), \tag{1.8}$$

and the error term is $\mathcal{O}\big(\sqrt{(2\lambda_n+1)/\Sigma_n}\big)$ as $n \to \infty$.

Using the identity $\arctan y = \pi/2 - \arctan(1/y)$, it is easy to see that we can alternatively write (1.5) in the form

$$L_n(\Lambda; x) = \frac{(-1)^n}{\pi\sqrt{x}} \int_0^\infty \frac{\cos\big(\Phi_n(t) + t\log x\big)}{\sqrt{\lambda_n^{*2} + t^2}}\, dt, \tag{1.9}$$

where

$$\Phi_n(t) = 2\sum_{j=0}^{n-1} \arctan\frac{t}{\lambda_j^*} + \arctan\frac{t}{\lambda_n^*}.$$

This representation will be useful when considering x close to 0.

The main results are presented here. First we get a global bound for the smallest zero.

Theorem 1.1 *Let $\Lambda = \{\lambda_k\}_{k=0}^\infty$ be a sequence of real numbers greater than $-1/2$. Then*

$$\exp\Big(-4\sum_{j=0}^{n-1}\frac{1}{2\lambda_j+1} - 2\frac{1}{2\lambda_n+1}\Big) < l_{n,n}.$$

Remark This considerably improves the lower bound in (1.4) as can be seen from the inequality

$$4\sum_{j=0}^{n-1}\frac{1}{2\lambda_j+1} + 2\frac{1}{2\lambda_n+1} \le 2\frac{2n+1}{2\lambda_{\min}^{(n)}+1}.$$

An important corollary is that for non-dense Müntz spaces, $L_n(\Lambda; x)$ has no zeros close to 0 (compare to [2, E.2, §6.2]).

Corollary 1.2 *Let $\Lambda = \{\lambda_k\}_{k=0}^\infty$ be a sequence of real numbers greater than $-1/2$ such that*

$$T := \sum_{k=0}^{\infty}\frac{1}{2\lambda_k+1} < \infty.$$

Then the smallest zero of $L_n(\Lambda; x)$ for all n is greater than $\exp(-4T) > 0$.

Next we obtain the asymptotic behavior of the largest zeros.

Theorem 1.3 *Let $\Lambda : -1/2 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ be a sequence of real numbers that satisfies* (1.6). *Then for fixed $k \ge 1$,*

$$\lim_{n\to\infty} |\log l_{k,n}|\Sigma_n = \Big(\frac{j_k}{2}\Big)^2,$$

where j_k denotes the k-th positive zero of the Bessel function J_0 and Σ_n was defined in (1.7). *The error term is $\mathcal{O}\big(\sqrt{(2\lambda_n+1)/\Sigma_n}\big)$ as $n \to \infty$.*

Remark Theorem 1.3 gives $l_{1,n} \sim \exp(-j_1^2/4\Sigma_n)$ as $n \longrightarrow \infty$, which, in the asymptotic sense, improves the upper bound in (1.4). We trivially have

$$2\Sigma_n \le (2n+1)\big(2\lambda_{\max}^{(n)}+1\big).$$

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2 Proofs

Proof of Theorem 1.1 For each n we let $\lambda_n^* := \lambda_n + 1/2$ and

$$T_n := \sum_{k=0}^{n-1} \frac{1}{\lambda_k^*} + \frac{1}{2\lambda_n^*}.$$

Now choose any $R_n \geq T_n$, and let $x_n = e^{-2R_n}$ so that $x_n \in (0, e^{-2T_n}]$. We need to show that $L_n(\Lambda; x_n) \neq 0$.

According to (1.9), we can write

$$L_n(\Lambda; x_n) = \frac{(-1)^n e^{R_n}}{\pi} \int_0^\infty \frac{\cos p_n(t)}{(\lambda_n^{*2} + t^2)^{1/2}} dt, \tag{2.1}$$

where $p_n(t) = 2R_n t - \Phi_n(t)$. The first two derivatives of p_n are $p_n'(t) = 2R_n - \Phi_n'(t)$ and $p_n''(t) = -\Phi_n''(t)$, where

$$\Phi_n'(t) = 2\sum_{k=0}^{n-1} \frac{\lambda_k^*}{\lambda_k^{*2} + t^2} + \frac{\lambda_n^*}{\lambda_n^{*2} + t^2}$$

and

$$\Phi_n''(t) = -2t\left(2\sum_{k=0}^{n-1} \frac{\lambda_k^*}{\left[\lambda_k^{*2} + t^2\right]^2} + \frac{\lambda_n^*}{\left[\lambda_n^{*2} + t^2\right]^2}\right).$$

Since $\Phi_n'(0) = 2T_n$, we therefore have $p_n'(0) = 2(R_n - T_n) \geq 0$ and $p_n''(t) > 0$ for $t > 0$. It follows that p_n is a strictly increasing function on $[0, \infty)$ that maps $[0, \infty)$ onto $[0, \infty)$ (note that $\Phi_n(t) \leq \pi n + \pi/2$)

We can therefore use the substitution $u = p_n(t)$ in integral of (2.1), and this gives

$$\int_0^\infty \frac{\cos p_n(t)}{\left(\lambda_n^{*2} + t^2\right)^{1/2}} dt = \int_0^\infty \frac{\cos u}{q_n(u)} du, \tag{2.2}$$

where $q_n(u)$ is determined by

$$q_n(u) = (\lambda_n^{*2} + t^2)^{1/2} p_n'(t).$$

Then $q_n(0) = 2\lambda_n^*(R_n - T_n)$ and since $\lim_{t\to\infty} p_n'(t) = 2R_n$, we have

$$\lim_{u\to\infty} q_n(u) = \lim_{t\to\infty} (\lambda_n^{*2} + t^2)^{1/2} p_n'(t) = \infty.$$

We show that $q_n(u)$ is strictly increasing. The chain rule gives

$$p_n'(t) q_n'(u) = \frac{d}{dt}\left((\lambda_n^{*2} + t^2)^{1/2} p_n'(t)\right) = \frac{t p_n'(t) + (\lambda_n^{*2} + t^2) p_n''(t)}{(\lambda_n^{*2} + t^2)^{1/2}},$$

and since $p_n'(t), p_n''(t) > 0$ for $t > 0$, it follows that $q_n'(u) > 0$ for $u > 0$.

By a standard argument we can write (2.2) as an alternating series $\sum_{k=0}^{\infty} (-1)^k a_k$ with $a_k > a_{k+1} > 0$ and $a_k \to 0$, and the alternating series test shows that $\int_0^\infty \frac{\cos u}{q_n(u)} du \neq 0$. The result follows. ∎

Before we prove Theorem 1.3, we need two lemmas. First we define the function

$$f_n(y) := L_n\big(e^{-y^2/4\Sigma_n}\big), \quad y \geq 0.$$

Then according to (1.8), uniformly for bounded $y \geq 0$,

$$f_n(y) - J_0(y) = \mathcal{O}\Big(\sqrt{\frac{2\lambda_n + 1}{\Sigma_n}}\Big) = o(1), \qquad n \longrightarrow \infty. \tag{2.3}$$

For each n and $k = 1, 2, \ldots, n$, we can write the zeros of $L_n(x)$ in the form

$$l_{k,n} = e^{-r_{k,n}^2/4\Sigma_n}$$

for some $0 < r_{1,n} < r_{2,n} < \cdots < r_{n,n}$. These are precisely the zeros of f_n, *i.e.*,

$$f_n(r_{k,n}) = 0, \qquad k = 1, 2 \ldots, n. \tag{2.4}$$

Below, we let $\|\cdot\|_{[0,y]}$ denote the supremum norm over $[0, y]$.

Lemma 2.1 *For each n and $y \geq 0$, $\|f_n'\|_{[0,y]} \leq \frac{y}{2} \sup_k \|f_k\|_{[0,y]} < \infty$.*

Proof We recall the identity from [3, Corollary 2.6],

$$xL_n'(x) = \lambda_n L_n(x) + \sum_{k=0}^{n-1}(2\lambda_k + 1)L_k(x).$$

It follows that

$$\begin{aligned} f_n'(y) &= -\frac{y}{2\Sigma_n} e^{-y^2/4\Sigma_n} L_n'(e^{-y^2/4\Sigma_n}) \\ &= -\frac{y}{2\Sigma_n}\Big[\lambda_n L_n(e^{-y^2/4\Sigma_n}) + \sum_{k=0}^{n-1}(2\lambda_k + 1)L_k(e^{-y^2/4\Sigma_n})\Big] \\ &= -\frac{y}{2\Sigma_n}\Big[\lambda_n f_n(y) + \sum_{k=0}^{n-1}(2\lambda_k + 1) f_k\Big(y\sqrt{\frac{\Sigma_k}{\Sigma_n}}\Big)\Big]. \end{aligned} \tag{2.5}$$

Therefore, since $0 \leq y\sqrt{\Sigma_k/\Sigma_n} \leq y$ for all $k = 0, 1, \ldots, n$,

$$|f_n'(y)| \leq \frac{y}{2\Sigma_n}\Big[\lambda_n + \sum_{k=0}^{n-1}(2\lambda_k + 1)\Big] \max_{0\leq k\leq n} \|f_k\|_{[0,y]} \leq \frac{y}{2} \sup_k \|f_k\|_{[0,y]}.$$

Since f_k is continuous on $[0, y]$ for each k and $f_n(t) \to J_0(t)$ uniformly for t bounded, it follows from the inequality $\|f_k\|_{[0,y]} \leq \|J_0\|_{[0,y]} + \|f_k - J_0\|_{[0,y]} = 1 + \|f_k - J_0\|_{[0,y]}$ that

$$\sup_k \|f_k\|_{[0,y]} < \infty.$$

The result now follows from the trivial inequality $\frac{t}{2} \sup_k \|f_k\|_{[0,t]} \leq \frac{y}{2} \sup_k \|f_k\|_{[0,y]}$ for each $t \leq y$. ■

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Lemma 2.2 *For each n and* $y \geq 0$, *we have*

$$\|f_n''\|_{[0,y]} \leq \frac{1}{2}\Big(1+\frac{y^2}{2}\Big) \sup_k \|f_k\|_{[0,y]} < \infty.$$

In particular, the family $\{f_n''\}$ *is uniformly bounded on bounded sets* $[0, y]$.

Proof Using the identity (2.5) for $f_n'(y)$, we obtain

$$\begin{aligned} f_n''(y) &= -\frac{1}{2\Sigma_n}\Big[\lambda_n f_n(y) + \sum_{k=0}^{n-1}(2\lambda_k+1) f_k\Big(y\sqrt{\frac{\Sigma_k}{\Sigma_n}}\Big)\Big] \\ &\quad - \frac{y}{2\Sigma_n}\Big[\lambda_n f_n'(y) + \sum_{k=0}^{n-1}(2\lambda_k+1)\sqrt{\frac{\Sigma_k}{\Sigma_n}} f_k'\Big(y\sqrt{\frac{\Sigma_k}{\Sigma_n}}\Big)\Big] \\ &= \frac{f_n'(y)}{y} - \frac{y}{2\Sigma_n}\Big[\lambda_n f_n'(y) + \sum_{k=0}^{n-1}(2\lambda_k+1)\sqrt{\frac{\Sigma_k}{\Sigma_n}} f_k'\Big(y\sqrt{\frac{\Sigma_k}{\Sigma_n}}\Big)\Big] \end{aligned}$$

If we let $A := \frac{1}{2}\sup_k \|f_k\|_{[0,y]}$, then since $0 \leq y\sqrt{\Sigma_k/\Sigma_n} \leq y$ for all n and $k = 0, 1, \dots, n$, Lemma 2.1 gives

$$\Big| f_k'\Big(y\sqrt{\frac{\Sigma_k}{\Sigma_n}}\Big)\Big| \leq \frac{y}{2}\sqrt{\frac{\Sigma_k}{\Sigma_n}} \sup_k \|f_k\|_{[0,y\sqrt{\Sigma_k/\Sigma_n}]} \leq Ay.$$

It follows that

$$|f_n''(y)| \leq A + \frac{y}{2}\cdot\frac{\lambda_n + \sum_{j=0}^{n-1}(2\lambda_j+1)}{\Sigma_n} Ay \leq \Big(1+\frac{y^2}{2}\Big)A.$$

The result now follows from the trivial inequality $\sup_k \|f_k\|_{[0,t]} \leq \sup_k \|f_k\|_{[0,y]} = 2A$ for each $t \leq y$. ∎

Proof of Theorem 1.3 Let $0 < j_1 < j_2 < \cdots$ denote the zeros of J_0 on the positive axis. According to the interlacing property (1.3), for fixed k, $\{r_{k,n}\}_n$ is a decreasing sequence bounded below by 0, and thus has a limit. Then from (2.3) it is clear that for each k,

$$\lim_{n\to\infty} r_{k,n} = j_m$$

for some integer $m = m(k) \geq 1$. By the intermediate value theorem, for n large enough, f_n has a zero close to each j_k. Therefore, its smallest zero $r_{1,n}$ necessarily has j_1 as limit.

We need to show that $r_{2,n}$ does not approach j_1 as well. Suppose to the contrary that

$$\lim_{n\to\infty} r_{2,n} = j_1.$$

Then by the mean value theorem, there exists some $c_n \in (r_{1,n}, r_{2,n})$ such that

$$f_n'(c_n) = 0 \tag{2.6}$$

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and of course by hypothesis $c_n \to j_1$ as $n \to \infty$.

Define a point $a_n = j_1 + \delta_n$, where the error δ_n is chosen so that

$$\sqrt{(2\lambda_n + 1)/\Sigma_n} = o(\delta_n) = o(1)$$

(say $\delta_n = \log\big((2\lambda_n + 1)/\Sigma_n\big)$). Then, since $f_n(y) \to J_0(y)$ uniformly for bounded y with error $\mathcal{O}((2\lambda_n + 1)/\Sigma_n)$, and $J_0(j_1) = 0$, we have for some ξ_n between j_1 and a_n,

$$\begin{aligned}(2.7)\qquad f_n(a_n) &= J_0(a_n) + f_n(a_n) - J_0(a_n) = J_0'(\xi_n)(a_n - j_1) + \mathcal{O}\left(\sqrt{\frac{2\lambda_n + 1}{\Sigma_n}}\right)\\ &= J_0'(j_1)\delta_n[1 + o(1)]\end{aligned}$$

as $n \to \infty$ (it is well known, see Olver [6, §7.6], that the zeros the Bessel functions are simple, so $J_0'(\xi_n) \longrightarrow J_0'(j_1) \neq 0$). On the other hand, using (2.3) again with $J_0(j_1) = 0$ yields

$$(2.8)\qquad f_n(a_n) = f_n(j_1) + f_n'(\nu_n)(a_n - j_1) = \mathcal{O}\left(\sqrt{\frac{2\lambda_n + 1}{\Sigma_n}}\right) + f_n'(\nu_n)\delta_n.$$

for some ν_n between j_1 and a_n. Expanding f' about the point c_n from (2.6) gives

$$f_n'(\nu_n) = f_n''(\eta_n)(\nu_n - c_n)$$

for some η_n between ν_n and c_n, and according to Lemma 2.2, since $c_n, \nu_n \to j_1$ as $n \to \infty$, we have $f_n'(\nu_n) = o(1)$ as $n \to \infty$. Therefore, (2.8) gives $f_n(a_n) = o(\delta_n)$, which contradicts (2.7). Hence $\lim_{n\to\infty} r_{2,n} \neq j_1$.

Since f_n has a zero close to j_2 for n large enough, it follows that $r_{2,n} \to j_2$. Now we can repeat the proof for $r_{3,n}$ and so on, and we have established that $\lim_{n\to\infty} r_{k,n} = j_k$ for each fixed k. The result now follows from $-4\Sigma_n \log l_{k,n} = r_{k,n}^2$.

As for the error, a linear approximation yields

$$J_0(r_{k,n}) = J_0(r_{k,n}) - J_0(j_k) = J_0'(\xi_n)(r_{k,n} - j_k)$$

for some $\xi_{k,n}$ between $r_{k,n}$ and j_k, and thus since the zeros of J_0 are simple, (2.3) and (2.4) yield

$$r_{k,n} - j_k = \mathcal{O}\left(J_0(r_{k,n})\right) = \mathcal{O}\left(\sqrt{\frac{2\lambda_n + 1}{\Sigma_n}}\right), \quad n \longrightarrow \infty. \qquad \blacksquare$$

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