SUPERSOLVABLE M*-GROUPS

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1. Introduction. A compact bordered Klein surface of genus $g \geq 2$ has *maximal symmetry* [**4**] if its automorphism group is of order $12(g-1)$, the largest possible. An *M*-group* [**8**] acts on a bordered surface with maximal symmetry. The first important result about these groups was that they must have a certain partial presentation [**8**, p. 5]. However, research has tended to focus more on the surfaces with maximal symmetry than on the M*-groups, and results about these groups typically deal with existence.

There is certainly no shortage of M*-groups. One of the earliest results is that there are infinitely many M*-groups [**8**, p. 6]. Many well-known groups are M*-groups. For example, $PSL_2(q)$ is an M*-group if q is a power of 2 [**2**, p. 537] or if q is a prime and $q \equiv 1 \bmod 4$ [**4**, p. 277]. In [**11**] there is the construction of a new infinite family of M*-groups that contains some familiar groups that act on a torus.

There can easily be more than one M*-group of a particular order. In fact, for any positive integer n, there is a positive integer k such that there are at least n non-isomorphic M*-groups of order k [**11**, Th. 9]. Part of the reason for this abundance of M*-groups is that these groups may be solvable as well as non-solvable. Indeed, the classification of M*-groups may well be an enormous problem.

Some small M*-groups, e.g., $C_2 \times S_3$ and $S_3 \times S_3$, are in fact supersolvable. Here we study the especially tractable supersolvable M*-groups. Our main result is that among the M*-groups the supersolvable ones are completely determined by their order. In particular, an M*-group G is supersolvable if and only if $o(G) = 4 \cdot 3^r$ for some positive integer r. We give numerous examples of supersolvable M*-groups from the family of groups in [**11**]. There is a supersolvable M*-group of order $4 \cdot 3^r$ for each positive integer r. We also establish some properties of supersolvable M*-groups and the bordered surfaces on which they act. Finally we classify the topological types of bordered Klein surfaces with maximal symmetry that have a supersolvable automorphism group.

2. M*-groups. We assume that all surfaces are compact and of genus $g \geq 2$. For any Klein surface X, let $A(X)$ denote the group of automorphisms of X.

An M*-group [**8,9**] is a finite group G generated by three distinct non-trivial elements x, u, and z which satisfy the relations

$$u^2 = x^2 = (ux)^2 = (uz)^2 = (xz)^3 = 1. \tag{2.1}$$

A group G of order $12(g-1)$ is an M*-group if and only if G acts on a bordered Klein surface X of genus g [**8**, p. 5]. The order of z is called an *index* of G[**4**], and there is a nice connection between the index and the action of G on X [**4**, p. 282]. If X has k boundary components and $q = o(z)$, then

$$o(G) = 2qk. \tag{2.2}$$

Each distinct topological type of bordered surface of genus g is called a *species* of the genus. An M*-group may well act on more than one species. Note that the index alone does not determine the orientability of the surface on which the M*-group acts. In this regard, also see [**2**, p. 534].

M*-groups may be solvable as well as non-solvable. Infinitely many examples of each type are known. Many of these examples are presented as an extension of an abelian group by an M*-group (see especially [**4**, p. 276] and [**11**]). In this case, then, the larger M*-group has an M*-quotient group.

In general, if G is an M*-group and N is a normal subgroup of G with $[G:N]>6$, then G/N is also an M*-group [**4**, p. 271]. An M*-group is said to be *M*-simple* [**4**] if it has no proper M*-quotient groups. The solvable M*-simple groups have already been classified; the only ones are $C_2 \times S_3$ and S_4 [**4**, p. 278]. Thus a solvable M*-group must have one of these two groups as a quotient. For more on M*-simple groups and the associated surfaces with maximal symmetry, see [**4**, pp. 272, 277, 278] and [**9**, p. 28].

3. Supersolvable groups. Now we show that among the (solvable) M*-groups, the supersolvable ones are completely determined by their order. We also establish some properties of supersolvable M*-groups.

By the way, none of the supersolvable M*-groups can be nilpotent, because in a nilpotent group each Sylow subgroup is normal. But an M*-group is generated by elements of order two (see [**8**, p. 5] and (2.1)) so that the Sylow 2-subgroup is not normal.

As usual, we denote the Frattini and Fitting subgroups of a group G by $\Phi(G)$ and $F(G)$ respectively.

THEOREM 1. *Let G be an M^*-group. Then G is supersolvable if and only if* $o(G)=4\cdot 3^r$ *for some positive integer r.*

Proof. Since G is an M*-group, we know that $[G:G']$ divides 4 and $[G':G'']$ divides 9 [**4**, p. 278]. It is basic that 12 divides $o(G)$.

First assume that G is supersolvable. Write $o(G')=3^r\cdot b$, where b and 3 are relatively prime. We know $r\geq 1$. Suppose that $b\neq 1$. Let p be a prime dividing b, and let S be the Sylow p-subgroup of G'. Since G is supersolvable, G' is nilpotent [**5**, p. 159]. Thus G' is the direct product of its Sylow subgroups [**5**, p. 155]. Then G' would have S and hence a non-trivial abelian p-group as quotients. But then p would divide $[G':G'']$, a contradiction. Therefore $b=1$ and $o(G')=3^r$. Finally we have $o(G)=4\cdot 3^r$, since 12 divides $o(G)$.

Next assume that G is an M*-group of order $4\cdot 3^r$. The only M*-group of order 12 is $C_2\times S_3$, and it is supersolvable. Suppose then that $r\geq 2$. The group G is solvable by a theorem of Burnside [**5**, p. 143]. Then G has a solvable M*-simple group as a quotient, and it must be $C_2\times S_3$ since $24=o(S_4)$ does not divide $o(G)$. Since $C_2\times S_3$ has $C_2\times C_2$ as a quotient, so does G. Hence $[G:G']=4$.

Now $o(G')=3^r$ so that G' is a nilpotent normal subgroup of G. Set $F=F(G)$ and $\Phi=\Phi(G)$. We have $G'\subset F$, and $F\neq G$ since G is not nilpotent. Suppose $G'\neq F$. Then we must have $o(F)=2\cdot 3^r$, and the nilpotent group F would have a characteristic

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subgroup H of order 2. But H would be normal in G with $o(G/H) = 2 \cdot 3^r > 6$, a contradiction, since G/H could not be an M*-group. Therefore $G' = F$ and $o(F) = 3^r$.

We know $F' \subset \Phi \subset F$ and $F/\Phi = F(G/\Phi)$ [**3**, p. 219]. Then since G/Φ is solvable, $F(G/\Phi) \neq 1$ so that $\Phi \neq F$. Also, $[F:F']$ divides 9, because $F = G'$. Thus $[G:\Phi]$ is equal to 12 or 36. Suppose $[G:\Phi] = 12$. Then $G/\Phi \cong C_2 \times S_3$, the only M*-group of order 12. But $F(C_2 \times S_3) \cong C_2 \times C_3$, so that this is not possible. Therefore $[G:\Phi] = 36$ and $G/\Phi \cong S_3 \times S_3$, the only M*-group of order 36. Now, since G/Φ is supersolvable, so is G, by a theorem of Huppert [**7**].

We collect some facts about supersolvable M*-groups from the proof of the theorem.

Corollary 1. *Let G be a supersolvable M^*-group with order $4 \cdot 3^r$, $r \geq 2$. Then*
(a) $o(G') = 3^r$
(b) $G' = F(G)$
(c) $G'' = \Phi(G)$
(d) $G'/G'' \cong C_3 \times C_3$

We can also say something about the indices of a supersolvable M*-group and the surfaces on which it acts.

Corollary 2. *Let G be a supersolvable M^*-group with order $4 \cdot 3^r$, $r \geq 2$. If q is an index of G, then $q = 2 \cdot 3^t$ for some positive integer t.*

Proof. Let G act on the bordered surface X with index q. Then the M*-group G/Φ acts on X/Φ [**4**, p. 271] and the index of G/Φ (with this action) divides q [**10**, p. 376]. But from the proof of the theorem, $G/\Phi \cong S_3 \times S_3$, and the only index of $S_3 \times S_3$ is 6 [**10**, p. 392]. Now q is a multiple of 6, and by (2.2) $2qk = 4 \cdot 3^r$, where k is the number of boundary components of X. Thus $q = 2 \cdot 3^t$ for some $t \geq 1$.

Corollary 3. *Let G be a supersolvable M^*-group with order $4 \cdot 3^r$, $r \geq 2$. If G acts on the bordered surface X that has k boundary components, then X is orientable and $k = 3^l$ for some positive integer l. If $r \geq 3$, then further $l \geq 2$.*

Proof. Immediately from the previous proof $k = 3^l$ for some $l \geq 1$. Also X is a full covering of the surface X/Φ [**4**, p. 271], a surface of genus 4 with maximal symmetry. Topologically X/Φ is a torus with three holes [**10**, p. 392]. Hence the surface X is orientable [**10**, p. 375].

If $r \geq 3$, then $k \neq 3$ by the theorem that classifies the surfaces with maximal symmetry with $k \leq 5$ [**10**, Th.1, p. 379].

The only M*-group of order 12 is $C_2 \times S_3$, and its actions are well-known. $C_2 \times S_3$ has indices 2 and 6 and acts on a sphere with three holes and a torus with one [**4**, p. 270].

Suppose G is a supersolvable M*-group. Then the commutator subgroup G' is the normal Sylow 3-subgroup of G, and further G' is a subgroup of small index in G. We can exploit the fact that G' is a p-group to gain information about G. Especially useful is an analysis of the lower central series of G' combined with the requirement that G have a presentation of the form (2.1). The fundamental work of Blackburn on p-groups [**1**] is quite helpful.

We define the *lower central series* of a group G recursively, as usual, by

$$\gamma_1(G)=G, \qquad \gamma_2(G)=[G,G], \qquad \gamma_i(G)=[\gamma_{i-1}(G),G] \text{ for } i>2.$$

A group G is nilpotent if there is some integer m such that $\gamma_{m+1}(G)=1$. If n is the least such integer, then n is called the *class* of G; for brevity we write $\text{cl}(G)=n$.

The *upper central series* is also defined recursively. Let $Z_0(G)=1$ and $Z_1(G)=Z(G)$. Then for $i>1$, $Z_i(G)$ is the subgroup of G such that

$$Z_i(G)/Z_{i-1}(G)=Z(G/Z_{i-1}(G)).$$

Then G is nilpotent if and only if $Z_m(G)=G$ for some integer m. If $\text{cl}(G)=n$, then n is the least such integer. Also, if G is nilpotent of class n, then for $i=1,2,\ldots,n+1$,

$$\gamma_i \subset Z_{n-i+1} \quad \text{but} \quad \gamma_i \not\subset Z_{n-i}. \tag{3.1}$$

Finite p-groups are, of course, nilpotent. If $o(G)=p^n$, then the class of G is at most $n-1$. For the basic facts on upper and lower central series, see [**12**, pp. 151–153] and [**1**, §1].

If we consider the lower central series of the Sylow 3-subgroup of a supersolvable M*-group, then we immediately have the following.

PROPOSITION 1. *Let P be the Sylow 3-subgroup of a supersolvable M*-group with $o(P)\geqslant 3^3$. Then*

(a) $P/\gamma_2(P)\cong C_3\times C_3$

(b) $\gamma_2(P)/\gamma_3(P)\cong C_3$

(c) *For $i\geqslant 3$, the exponent of the quotient group $\gamma_i(P)/\gamma_{i+1}(P)$ is at most* 3.

Proof. First (a) is just a restatement of Corollary 1(d). Then (b) and (c) follow from (a) and basic results on the lower central series of a p-group [**1**, Th.1.5, p. 49].

We obtain additional information by using the special presentation that an M*-group must have. Let G be a supersolvable M*-group with index q and generators u, x, and z that satisfy the relations (2.1). Write $a=xz^{-3}$ and $b=zaz^{-1}$ (This notation is suggested by [**11**].). Let P be the Sylow 3-subgroup of G.

PROPOSITION 2. $P=\langle a, z^2\rangle$.

Proof. Since G is supersolvable, we have $P=G'$. Generators for G' have already been established; $G'=\langle xz, zx\rangle$ [**4**, p. 278]. Set $H=\langle a, z^2\rangle$. Then $z^2=zx\cdot xz$ and $a=xz(z^2)^{-2}$, so that $H\subset G'$. Also $xz=az^4$, $zx=z^2(xz)^{-1}$, and $G'\subset H$. Thus $P=G'=\langle a, z^2\rangle$.

We need some calculations involving commutators. Let $[u,v]=uvu^{-1}v^{-1}$ be the commutator of u and v, and let $\gamma_i=\gamma_i(P)$. Clearly $b=zaz^{-1}\in P$. It is an exercise to check that the relations (2.1) imply $b^{-1}ab=z^6a$ (See the proof of Theorem 1 in [**11**].). Thus

$$z^6=[b^{-1},a] \tag{3.2}$$

and $z^6\in P'$. It is also easy to see that $abz^6=[a,z^{-2}]\in P'$. Now $ab\in P'$ as well. Then

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$z^6 = [b^{-1}, a] = [b^{-1}, ab] \in [P, \gamma_2]$, that is,

$$z^6 \in \gamma_3. \tag{3.3}$$

We note a few consequences of these calculations. In many cases, it is not hard to see that the class of P cannot be too small.

PROPOSITION 3. *Suppose* $o(G) = 4\cdot 3^r$, $r \geq 3$, *and the index* $q = 2\cdot 3^t$, $t \geq 2$. *Then* $\mathrm{cl}(P) \geq t+1$.

Proof. For $j \geq 3$, the exponent of γ_j/γ_{j+1} is at most 3, by Proposition 1. Now $z^6 \in \gamma_3$ by (3.3), so that $z^{18} \in \gamma_4$ and in general for $i > 0$, if $n = 3^i$, then $(z^6)^n \in \gamma_{3+i}$. Since $q = o(z) = 2\cdot 3^t$, $z^{q/3} \neq 1$. Hence $\gamma_{t+1} \neq 1$ and $\mathrm{cl}(P) \geq t+1$.

If the index $q = 6$, then G acts on a torus with holes [**11**, §3]. We briefly consider this important special case. A group H is called *metabelian* if its commutator subgroup H' is abelian.

PROPOSITION 4. *If G is a supersolvable M*-group with index* 6, *then* $G' = P$ *is a metabelian* 3-*group*.

Proof. Let $M = \langle a, b\rangle$. Then $M \subset P$, and it is easy to check that M is normal in $P = \langle a, z^2\rangle$. Since $z^6 = 1$, the order of P/M is at most 3. Thus $P' \subset M$. But from (3.2), $[b^{-1}, a] = 1$ so that M is abelian. Now P' is abelian as well, and P is metabelian.

Among the supersolvable M*-groups with index larger than 6, some of the more interesting groups are those in which $xz^6 = z^6x$. Such a group acts on a bordered surface that is a fully wound covering [**9**] of a torus with holes.

PROPOSITION 5. *If* $xz^6 = z^6x$, *then* $z^{18} \in P''$. *Thus P is not metabelian if the index* $q > 18$.

Proof. Obviously z^6 commutes with a and b. Since $z^6 = [b^{-1}, ab]$, $z^{18} = [b^{-1}, ab]^3 = [b^{-3}, ab]$ [**3**, p. 19]. But $b \in P$ and P/P' has exponent 3, so that $b^3 \in P'$. We have already seen that $ab \in P'$. Thus $z^{18} \in [P', P'] = P''$.

4. Examples.

Here we give some examples of supersolvable M*-groups. There is in fact a group of each possible order given in Theorem 1.

Some members of the family of groups constructed in [**11**] are supersolvable. The general construction there forms larger groups from groups that admit an action of $D_6 \cong C_2 \times S_3$, the smallest M*-group. In particular the following groups were obtained. For simplicity we present the results only for an odd integer n.

THEOREM A [**11**, §5]. *Let n be an odd positive integer, and let m divide n. Let $K_{n,m}$ be the group with generators u, x, and z and defining relations*

$$u^2 = x^2 = z^{6m} = (ux)^2 = (uz)^2 = (xz)^3 = 1,\ xz^6 = z^6x,\ a^n = 1,\ a = xz^{-3}. \tag{4.1}$$

Then the order of $K_{n,m}$ is $12n^2m$, and $K_{n,m}$ is an M-group with index $6m$.*

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THEOREM B [**11**, §5]. *Let n be an odd positive integer, and let m divide n. Let $L_{n,m}$ be the group with generators u, x, and z and defining relations*

$$u^2 = x^2 = z^{6m} = (ux)^2 = (uz)^2 = (xz)^3 = 1,\ xz^6 = z^6x,$$
$$a^{3n} = (ba)^n = 1,\ a = xz^{-3},\ b = zaz^{-1}.$$

Then the order of $L_{n,m}$ is $36n^2m$, and $L_{n,m}$ is an M^*-group with index 6m.

By Theorem 1 the groups $K_{n,m}$ and $L_{n,m}$ are supersolvable if and only if n is a power of 3. Thus we have a wealth of examples of supersolvable M*-groups.

THEOREM 2. *For each positive integer i there is a supersolvable M^*-group of order $4 \cdot 3^i$.*

Proof. If $i = 2k$ is even, then set $n = 3^{k-1}$, and $L_{n,1}$ has order $36(3^{k-1})^2 = 4 \cdot 3^i$. If $i = 2k+1$, then let $n = 3^k$, and $K_{n,1}$ has order $12(3^k)^2 = 4 \cdot 3^i$.

The Sylow 3-subgroup is a large, important part of each of these groups. Examining this subgroup is a convenient way to distinguish among them.

First let $i \geqslant 0$, $n = 3^i$, and G be the group $K_{n,1}$ or $L_{n,1}$. Then G has index 6 and acts on a torus with holes that has maximal symmetry [**11**, §§3,5]. The Sylow 3-subgroup of G is metabelian by Proposition 4. This can also be determined by examining the construction of G in [**11**]. The subgroup $M = \langle a, b \rangle$ is an abelian normal subgroup of G and $G/M \cong D_6$. If $G = K_{n,1}$, then $M \cong C_n \times C_n$, while if $G = L_{n,1}$, $M \cong C_n \times C_{3n}$ [**11**, §5]. In either case, it follows from Corollary 1 to Theorem 1 that $G'' \subset M$ (with $[M:G''] = 3$ in general), and consequently G' is a metabelian 3-group.

Now let i and j be integers such that $i \geqslant j \geqslant 1$, and set $n = 3^i$, $m = 3^j$. For each value of n the groups $K_{n,m}$ act on bordered surfaces with maximal symmetry that are in a family of fully wound coverings. Each surface is a fully wound covering of a torus with holes and is therefore orientable [**11**, §5]. If $j \geqslant 2$, then Proposition 5 guarantees that the Sylow 3-subgroup of $K_{n,m}$ is not metabelian.

The case $j = 1$ is special. In fact the Sylow 3-subgroup P of $K_{n,3}$ is metabelian with $P' \cong C_3 \times C_{n/3} \times C_n$. This can be seen by analyzing the construction in [**11**, §5].

Similar comments hold for the groups $L_{n,m}$ and the surfaces on which they act.

All these groups are quite intimately related, of course. Some of the most important relationships are in [**11**, §5]. It is quite natural to consider these groups as a single family.

There are, however, an abundance of supersolvable M*-groups outside this family. The constructions of [**4**, §4] may be used to establish the existence of other supersolvable M*-groups with metabelian Sylow 3-subgroups. One of these constructions is the following.

Let X be an orientable Klein surface with topological genus p and k boundary components, with $k > 1$. Let r and s be positive integers such that r divides s. Then there is a full cover $Y_s^r(X)$ of X with covering group $(C_s)^{2p} \times (C_r)^{k-1}$ and boundary degree r. Further the surface $Y_s^r(X)$ is orientable [**4**, Th.9, p. 274]. It is easy to determine the topological type of $Y_s^r(X)$ [**4**, p. 273]. In addition, if X has maximal symmetry, then so does the full cover $Y_s^r(X)$ [**4**, p. 271].

Now let X be a torus with three holes that has maximal symmetry, so that $A(X) \cong S_3 \times S_3$. Let i and j be integers with $i \geq j \geq 1$, and set $s = 3^i$, $r = 3^j$. Then the surface $Z = Y_s^r(X)$ has maximal symmetry, and $G = A(Z)$ is an M*-group of order $36s^2r^2$. Further G has a normal subgroup $N \cong (C_s)^2 \times (C_r)^2$ such that $G/N \cong S_3 \times S_3$. Now G is supersolvable, of course, and it follows from Corollary 1 to Theorem 1 that $G'' = N$. Thus G' is a metabelian 3-group. This fact, combined with the structure of G'', shows that G is not isomorphic to one of the groups $K_{n,m}$, $L_{n,m}$.

Hence this example alone shows that there are infinitely many supersolvable M*-groups outside our family of groups. The constructions of [**4**, §4] could be used to produce other examples as well. The problem of classifying the supersolvable M*-groups [**10**, Problem 2, p. 392] may well be intractable.

5. Species with maximal symmetry. Nevertheless, we are able to classify the species of bordered Klein surfaces with maximal symmetry that have a supersolvable automorphism group [**10**, Problem 3, p. 393]. Perhaps the surprising thing here is that this problem is not too difficult. We use the groups $K_{n,m}$ and $L_{n,m}$ to produce all the species and then apply results about supersolvable groups and full coverings to show that there are no others.

First, we use a result from [**9**] to find a second index for each of these groups.

PROPOSITION 6. *Let i and j be integers such that* $i \geq j \geq 0$, $n = 3^i$, *and* $m = 3^j$. *Then the M*-group* $K_{n,m}$ *has indices* $6m$ *and* $2n$. *The M*-group* $L_{n,m}$ *has indices* $6m$ *and* $6n$.

Proof. Let $G = K_{n,m}$ have the presentation (4.1). G has index $6m$ by Theorem A. Let $w = xuz$. Then the order of w is an index of G [**9**, p. 24]. Let P be the Sylow 3-subgroup of G. We know $P = \langle a, z^2 \rangle$ and P is normal in G. Now $w = xuz = (az^3)uz = az^2(zuz) = az^2u$. If w were in P, then $u \in P$ as well. But $o(u) = 2$. Hence $w \notin P$ and $o(w)$ is not a power of 3.

Write $b = zaz^{-1}$, as usual. Then $o(b) = o(a) = n$. (See (4.1) and the construction of G in [**11**, §5].) Now $w^2 = (xuz)(xuz) = x(uzu)(uxu)z = (xz^{-1}x)z = (zxz)z = z(xz^{-3})z^{-1}z^6 = bz^6$, using (4.1). Since b and z^6 commute and $o(z^6) = m$ divides n, $o(w^2) = o(b) = n$. Therefore $o(w) = 2n$ is an index of G.

The proof for $L_{n,m}$ is almost identical, the only difference being that $o(b) = 3n$.

The index of a supersolvable M*-group has the form $2 \cdot 3^t$, by Corollary 2 to Theorem 1. We now show that for each possible index $q > 2$, there are infinitely many M*-groups.

THEOREM 3. *Let t be a positive integer and* $q = 2 \cdot 3^t$. *Then for each integer* $r \geq 2t$, *there is a supersolvable M*-group of order* $4 \cdot 3^r$ *and index q.*

Proof. First assume $r \geq 3t - 2$, and set $j = t - 1$, $m = 3^j$. Let $l = r - j$. If l is even, then let $i = (l-2)/2$ and $n = 3^i$. It is easy to see that $i \geq j$ so that m divides n. The group $L_{n,m}$ has order $36n^2m = 4 \cdot 3^r$ and index $6m = 2 \cdot 3^t$.

If on the other hand l is odd, let $i = (l-1)/2$ and $n = 3^i$. Again $i \geq j$. The group $K_{n,m}$ has order $12n^2m = 4 \cdot 3^r$ and index $6m = 2 \cdot 3^t$. Note that if $t \leq 2$, then $2t \geq 3t - 2$ and we do not need to consider any additional cases.

Next assume that $3t - 2 > r > 2t$ (and $t > 2$). Let $n = 3^t$, $j = r - 2t - 1$, and $m = 3^j$.

Note that $t > j \geq 0$. The group $K_{n,m}$ has order $12n^2m = 4\cdot 3^r$ and index $2n = 2\cdot 3^t$, using Proposition 6.

Finally assume that $r = 2t$. Let $i = t-1$ and $n = 3^i$. The group $L_{n,1}$ has order $36n^2 = 4\cdot 3^r$ and index $6n = 2\cdot 3^t$, again using Proposition 6.

An immediate consequence of Theorem 3 is the existence of many species of bordered surfaces with maximal symmetry and a supersolvable automorphism group.

THEOREM 4. *Let s and l be positive integers such that $s \geq l \geq (s+1)/2$. If $g = 3^s + 1$ and $k = 3^l$, then there is an orientable Klein surface of genus g with k boundary components that has maximal symmetry and supersolvable automorphism group.*

Proof. Let $t = s - l + 1$, so that $t \geq 1$. Since $2l \geq s+1$, we have $s + 1 \geq 2(s - l + 1)$. Hence $s + 1 \geq 2t$. Now by Theorem 3 there is a supersolvable M*-group G of order $4\cdot 3^{s+1}$ and index $q = 2\cdot 3^t$. Then G acts on a bordered surface X of genus g with maximal symmetry, where $12(g-1) = o(G)$. Thus $g = 3^s + 1$. Further, if X has k boundary components, then by (2.2) $4\cdot 3^{s+1} = 2(2\cdot 3^t)k$ so that $k = 3^{s-t+1} = 3^l$. Finally the surface X is orientable by Corollary 3 to Theorem 1.

Surprisingly, all the species with maximal symmetry and supersolvable automorphism group appear in Theorem 4, except of course those of genus 2. To establish this, we first show that an index of a supersolvable M*-group must be within a certain range in terms of the group's order.

Let G be a supersolvable M*-group with generators u, x, and z that satisfy the relations (2.1), and write $a = xz^{-3}$. Let P be the Sylow 3-subgroup of G, and let $\gamma_i = \gamma_i(P)$ be the terms of the lower central series of P. We need a rather technical result about this series. The key is a general theorem of Philip Hall [**6**, Th. 2.81].

LEMMA 1. *If for some integers d and j, $j \geq 2$, $z^d \in \gamma_j$, $z^d \notin \gamma_{j+1}$, and $[\gamma_j : \gamma_{j+1}] = 3$, then $[\gamma_{j+1} : \gamma_{j+2}] \leq 3$.*

Proof. By Proposition 2, $P = \langle a, z^2 \rangle$. Obviously γ_j is generated by γ_{j+1} and z^d. Then by Hall's theorem, γ_{j+1} is generated by γ_{j+2} together with the commutators $v = [a, z^d]$ and $[z^2, z^d]$. But $[z^2, z^d] = 1$ of course. Hence γ_{j+1} is generated by γ_{j+2} and v. In other words, the quotient group $\gamma_{j+1}/\gamma_{j+2}$ is cyclic. Now $[\gamma_{j+1} : \gamma_{j+2}] \leq 3$ by Proposition 1c).

THEOREM 5. *If G is a supersolvable M*-group with index q and order $4\cdot 3^r$, $r \geq 2$, then $q = 2\cdot 3^t$ for some integer t such that $1 \leq t \leq r/2$.*

Proof. (by induction on r). The result holds for $2 \leq r \leq 4$ by the main classification theorem of [**10**, p. 392]. Assume then that $r \geq 5$ and the result holds for all integers s such that $2 \leq s < r$. Let G be a supersolvable M*-group of order $4\cdot 3^r$ and index q. By Corollary 2 to Theorem 1 we may write $q = 2\cdot 3^t$ for some $t \geq 1$. We must prove $t \leq r/2$.

Suppose to the contrary that $t > r/2$. Then let G act (with index q) on the surface X with k boundary components. By (2.2) we have $4\cdot 3^r = 2(2\cdot 3^t)k$ so that $k = 3^{r-t}$. Also, since $r \geq 5$, $k \geq 9$ by Corollary 3 to Theorem 1.

We are assuming that the index q is large (and $q > k$). In this situation the approach in [**9**, §3] is quite useful. There is a positive integer $n \leq k$ such that n divides q and

$xz^n = z^nx$. Further the subgroup $N = \langle z^n \rangle$ is normal in G and the quotient group G/N is an M*-group with index n and order $2nk$ [**9**, p. 27]. G/N is supersolvable, of course, and since $k \geq 9$, $o(G/N)$ is not too small and 6 divides n, by Corollary 2 to Theorem 1 again. We may write $n = 2 \cdot 3^i$ for some $i \geq 1$. But since $n \leq k$, we also have $2 \cdot 3^i \leq 3^{r-t}$ so that $i \leq r-t-1$. But $t > r/2$, so $i < (r/2) - 1 < t-1$ and hence $i \leq t-2$. Thus $N = \langle z^n \rangle$ has order $q/n = 2 \cdot 3^t / 2 \cdot 3^i = 3^{t-i} \geq 9$. In particular, let $m = 2 \cdot 3^{t-2}$ and $M = \langle z^m \rangle$. Then $m \geq 6$ and n divides m. Thus $xz^m = z^mx$ and M is a normal subgroup of G of order 9. We rely heavily on the subgroup M and the inductive hypothesis (IH). We consider two cases.

First suppose r is even. Then $J = \langle z^{3m} \rangle$ is a normal subgroup of order 3. The quotient group G/J is a supersolvable M*-group of order $4 \cdot 3^{r-1}$ and index $3m = 2 \cdot 3^{t-1}$. By the IH, $t - 1 \leq (r-1)/2$. Since r is even, we have $t \leq r/2$, a contradiction.

Next assume that $r = 2l + 1$ is odd. Note that $l \geq 2$. Since $t > r/2$, $t \geq l + 1$. Using the IH and the subgroup J, it is easy to see that t cannot be larger than $l + 1$. Hence $t = l + 1$, the difficult case.

Let $c = \text{cl}(P)$. By Proposition 3, $c \geq t + 1 = l + 2 \geq 4$. The quotient group G/γ_c has order at most $4 \cdot 3^{r-1} = 4 \cdot 3^{2l}$ and hence index at most $2 \cdot 3^l = 2 \cdot 3^{t-1} = 3m$, using the IH again. Therefore $z^{3m} \in \gamma_c$. However, $z^m \notin \gamma_c$, since γ_c has exponent 3, by Proposition 1(c). Now the IH shows that $o(\gamma_c)$ cannot be larger than 3. Hence $o(\gamma_c) = 3$ and $\gamma_c = \langle z^{3m} \rangle$.

Next the IH implies G/γ_{c-1} has index at most m, so that $z^m \in \gamma_{c-1}$. But $xz^m = z^mx$ so that $z^m \in Z(P)$, and $\gamma_{c-1} \not\subset Z(P)$ by (3.1). Hence $[\gamma_{c-1} : \gamma_c] \geq 9$. Now $z^{m/3} \notin \gamma_{c-1}$ by Proposition 1(c), and it follows from the IH that $[\gamma_{c-1} : \gamma_c] = 9$. At this point, we have

$$P \overset{9}{—} \gamma_2 \overset{3}{—} \gamma_3 — \cdots — \gamma_{c-1} \overset{9}{—} \gamma_c \overset{3}{—} 1.$$

Note that $c \neq 4$ since $o(P) \neq 3^6$. Hence $c \geq 5$. Also $[\gamma_3 : \gamma_{c-1}] = o(P) \cdot 3^{-6} = 3^{2l-5}$.

Since $\text{cl}(P) = c \geq l + 2$ is rather large, not all the factors γ_i/γ_{i+1}, $i = 3, \ldots, c-2$, can have order 9 or more. If that were so, then $[\gamma_3 : \gamma_{c-1}] \geq (3^2)^{c-4}$ but $2(c-4) > 2l - 5$. Hence for some i such that $3 \leq i \leq c-2$, we must have $[\gamma_i : \gamma_{i+1}] = 3$. Let j be the largest such integer, that is $[\gamma_j : \gamma_{j+1}] = 3$ and $[\gamma_i : \gamma_{i+1}] \geq 9$ if $j + 1 \leq i \leq c - 1$. In particular, note that $[\gamma_{j+1} : \gamma_{j+2}] \geq 9$ (even if $j = c-2$). Now $o(\gamma_j) \geq 3(3^2)^{c-(j+1)} \cdot 3 = 3^{2c-2j}$. Let $d = 2 \cdot 3^{t-1-(c-j)}$. A familiar argument with the IH shows that G/γ_j has index at most d. Thus z^d must be an element of γ_j. Note that since $j \geq 3$, $o(P)/27 = 3^{r-3} \geq o(\gamma_j)$, and it follows that $d \geq 6$. But $o(z^d) = q/d = 3^{c-j+1}$, and γ_{j+1} has exponent at most 3^{c-j} by Proposition 1(c). Therefore $z^d \notin \gamma_{j+1}$. Since $[\gamma_{j+1} : \gamma_{j+2}] \geq 9$, this now contradicts Lemma 1.

In each case, assuming $t > r/2$ leads to a contradiction. Hence $t \leq r/2$. This concludes the proof of Theorem 5.

Now we can easily establish the converse to Theorem 4.

THEOREM 6. *Let X be a bordered Klein surface of genus g with maximal symmetry and k boundary components. If the automorphism group of X is supersolvable, then X is orientable and there is an integer $s \geq 0$ such that the genus $g = 3^s + 1$. If $g > 2$, then further there is a positive integer l such that $s \geq l \geq (s+1)/2$ and $k = 3^l$.*

Proof. The surface X is orientable by Corollary 3 to Theorem 1. Let $G = A(X)$ have index q. By Theorem 1 $o(G) = 4 \cdot 3^r$ for some positive integer r. But $o(G) = 12(g-1)$. Set $s = r - 1$. Then $s \geqslant 0$ and $g = 3^s + 1$.

Now assume that $g > 2$ so that $r \geqslant 2$. We have $o(G) = 2qk = 4 \cdot 3^r$. By Theorem 5, $q = 2 \cdot 3^t$ for some integer t such that $1 \leqslant t \leqslant r/2$. Now $k = 3^{r-t}$, and $r - 1 \geqslant r - t \geqslant r/2$. Set $l = r - t$. Then $k = 3^l$ and $s \geqslant l \geqslant (s+1)/2$.

Together Theorems 4 and 6 neatly classify the species of bordered Klein surfaces with maximal symmetry that have a supersolvable automorphism group. Still remaining is the problem of classifying the supersolvable M*-groups. A piece of this problem seems approachable and definitely interesting. We conclude by calling attention to the following.

Problem. Classify the supersolvable M*-groups with partial presentation (2.1) and the additional relation $xz^6 = z^6x$.

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