FINITE INCIDENCE STRUCTURES WITH ORTHOGONALITY

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To Professor H. S. M. Coxeter on his sixtieth birthday

1. Introduction. An *incidence structure* consists of two sets of elements, called *points* and *blocks*, together with a relation, called *incidence*, between elements of the two sets. Well-known examples are inversive planes, in which the blocks are circles, and projective and affine planes, in which the blocks are lines. Thus in various examples of incidence structures, the blocks may have various interpretations. Very shortly, however, we shall impose a condition (Axiom A) which ensures that the blocks behave like lines. In anticipation of this, we shall refer to the set of blocks as the set of *lines*. Also, we shall employ the usual terminology of incidence, such as "lies on," "passes through," "meet," "join." etc.

A *projective plane* is an incidence structure satisfying the axioms:

I. Any two distinct points are incident with exactly one line.
II. Any two distinct lines are incident with exactly one point.
III. There exist four points, no three of which are collinear.

An *affine plane* is an incidence structure satisfying the axioms:

I. Any two distinct points are incident with exactly one line.
II′. Given a point P and a line q, not incident with P, there is exactly one line incident with P which does not meet q.
III′. There exist three non-collinear points.

Affine planes may be derived from projective planes by the familiar process of removing a line and the points on it. Conversely, projective planes may be obtained from affine planes by the introduction of "ideal" points and an "ideal" line. A finite projective or affine plane is said to be of *order n* if one line (and therefore every line) contains $n + 1$ points (for a projective plane) or n points (for an affine plane) (3, p. 392).

This paper is concerned with finite incidence structures $\mathfrak{J}$ subject to the condition:

A. *Any two distinct points are incident with exactly one line.*

In addition, we introduce a binary relation of *orthogonality* among the lines of $\mathfrak{J}$. Using the familiar symbol "$\perp$" to denote this relation, we impose the following postulates:

Received November 15, 1966.

B1. *If* $a \perp b$, *then* $b \perp a$.

B2. *If* $a \perp b$, *then* a *and* b *have at least one common point.*

B3. *Given a point* P *and a line* q, *there is at least one line* r *such that* $r \mathrel{\mathrm{I}} P$ (i.e. r is incident with P) *and* $r \perp q$. (Such a line will be called a *perpendicular* from P to q.)

B4. *Given a point* P *and a line* q *such that* $P \mathrel{\mathrm{I}} q$, *there is exactly one line* r *such that* $r \mathrel{\mathrm{I}} P$ *and* $r \perp q$.

B5. *There exist lines* a, b, c *such that* $a \perp b$, *but* $c \perp a$ *and* $c \perp b$ *do not hold, and* a, b, c *have no common point.*

The object of the paper is to prove the following result:

THEOREM 1. *Under the postulates* A *and* B1–B5, *any finite incidence structure* $\mathfrak{J}$ *is an affine plane of odd order.*

Axioms B1–B4 are the "orthogonality axioms" for a *metric plane* as defined by Bachmann (1, p. 24). Axiom B5 (cf. (1, p. 33, Axiom D)) is designed to eliminate trivial cases. Taken together, Axioms A and B1–B5 are easily seen to be equivalent to the "Incidence" and "Orthogonality" axioms for a metric plane (1, p. 24). But, as we shall indicate in §3, the "Axioms of Reflection," which are imposed on a metric plane, are not necessarily satisfied by our incidence structure $\mathfrak{J}$, so that $\mathfrak{J}$ is more general than a metric plane.

Bachmann has shown (1, p. 124) that a *finite* metric plane always satisfies the Euclidean parallel postulate, and is therefore an affine plane. Our Theorem 1 is another proof of this fact. Unlike in Bachmann's proof, however, no use of the powerful "Axioms of Reflection" is made here. Thus Bachmann's theorem is seen to hold in a more general context.

Theorems analogous to our Theorem 1 have been established by others beside Bachmann (cf. for example, (2, p. 91)) but they have involved different axioms; in particular more restrictive incidence axioms are usually employed.

2. Proof of Theorem 1. We proceed to prove nine lemmas which, when combined, yield the proof of our theorem.

LEMMA 1. $\mathfrak{J}$ *contains four points, no three of which are collinear.*

Proof. Let a, b, c be the lines whose existence is postulated in Axiom B5. Let P be a common point of a and b (Axiom B2), and let x be a perpendicular from P to c (Axiom B3). Let Q be a common point of x and c, y a perpendicular from Q to a, R a common point of y and a, z a perpendicular from R to c, and S a common point of z and c. We show that no three of the four points P, Q, R, and S are collinear.

First, $Q \neq P$, since $Q \mathrel{\mathrm{I}} c$ and $P \not\mathrel{\mathrm{I}} c$ (Axiom B5). Next, $R \neq Q$, for

$$R = Q \Rightarrow a = x \Rightarrow a \perp c,$$

which is not so. Also, P, Q, and R are non-collinear, for

$$P, Q, R, \text{ collinear} \Rightarrow R = P \Rightarrow y = b \quad \text{(Axiom B4)}$$
$$\Rightarrow b \perp c \quad (\text{since } R = P \Rightarrow y = x)$$

which is false. Similar reasoning, chiefly involving appeals to Axioms B4 and B5, yields the remainder of the proof.

LEMMA 2. *If $x \perp y$, then $x \neq y$.*

Proof. Suppose that $x \perp y$ and $x = y$. By Lemma 1 there is a point $P \not\mathrm{I}\, x$. Let z be a perpendicular from P to x, and let Q be a common point of z and x. By Axiom B4, $z = y$. But $z \neq y$, since $P \mathrm{~I~} z$ and $P \not\mathrm{I}\, y$.

Lemma 2 is a strengthening of the statement of Axiom B2. The possibility of *isotropic* lines has been excluded, and we can now state that two perpendicular lines have precisely one common point.

LEMMA 3. *Through every point there pass an even number of lines.*

Proof. Let P be any point and let x be any line through P. Let y be the perpendicular to x through P. By Lemma 2, $y \neq x$, and by Axiom B4, y is the unique line through P which is perpendicular to x. Thus all lines through P occur in mutually exclusive pairs of distinct perpendicular lines.

LEMMA 4. *More than two distinct lines pass through each point.*

Proof. Axioms B3 and B5, with Lemma 3, assure that there are at least two lines through each point. Suppose, if possible, that there is a point X through which only two lines pass, and consider the quadrangle $PQRS$ constructed in the proof of Lemma 1. The lines XP, XQ, XR, XS must coincide in pairs; hence $X = PR \cap QS$, $PS \cap QR$, or $PQ \cap RS$. We consider each possibility separately.

(i) $X = PR \cap QS$. Now $PR = a$, $QS = c$, and by Axiom B5, $c \perp a$ does not hold. On the other hand, by Axiom B4, there must be a line t through $X = a \cap c$ such that $t \perp a$. But $t \neq c$, and $t \neq a$ by Lemma 2. Thus there are more than two lines through X, and we have a contradiction.

(ii) $X = PS \cap QR$. Let u be a perpendicular from X to QS. Then $u = PS$ or $u = QR$. But if $u = PS$, then, by Axiom B4, $PS = RS$, which is false; and if $u = QR$, then, again by Axiom B4, $PQ = QR$, which also is false. Thus we have a contradiction in this case as well.

(iii) $X = PQ \cap RS$. Let v be a perpendicular from X to PR. Then $v = PQ$ or $v = RS$. But

$$v = PQ \Rightarrow PQ = b \Rightarrow b \perp c$$

and

$$v = RS \Rightarrow RS = QR,$$

and both conclusions are false. This completes the proof of Lemma 4.

LEMMA 5. *Every line contains the same number of points.*

https://doi.org/10.4153/CJM-1967-098-9 Published online by Cambridge University Press

Proof. Let r be a line with the property that no other line contains more points than r, and let s be any other line. Let n be the number of (distinct) points on r, and let these points be labelled P_i ($i = 1, 2, \ldots, n$). Let x_i be a perpendicular from P_i to s and let $Q_i = x_i \cap s$. Suppose that $Q_i = Q_j$ for some i, j such that $1 \leqslant i < j \leqslant n$. Then, by Axiom B4, $x_i = x_j$. Thus $P_j \mathrm{I} x_i$ and $r = P_i P_j = x_i$; so $r \perp s$. It follows that if $r \perp s$ does not hold, the n points Q_i are distinct and s has exactly n points.

Suppose that $r \perp s$ and let $U = r \cap s$. By Lemma 4, there is a line t through U which is distinct from r and s. Since $t \neq s$, $r \perp t$ does not hold and, by the above argument, t has exactly n points. Finally, since $t \neq r$, $t \perp s$ does not hold and, again by the above argument, s has exactly n points.

From now on, we shall let n be the number of points on a line of $\mathfrak{J}$.

LEMMA 6. *Let r be any line, and let s be any line perpendicular to r. Then r meets all lines of $\mathfrak{J}$, except possibly the lines which are perpendicular to s.*

Proof. Let x be any line which is not perpendicular to s. Let Y_i ($i = 1, \ldots, n$) be the points of x, and let y_i be any perpendicular from Y_i to s. As in the proof of Lemma 5, the y_i are all distinct, since $x \perp s$ does not hold. Hence there are n distinct lines y_i which are perpendicular to s. It follows from Lemma 5 and Axiom B4 that there are *only* n distinct lines which are perpendicular to s. Thus $r = y_j$ for some j ($1 \leqslant j \leqslant n$), so r meets x.

COROLLARY. *Two distinct lines either intersect or have a common perpendicular.*

Thus Axiom V* (1, p. 201) of Bachmann's book follows as a theorem (if we assume $\mathfrak{J}$ finite).

LEMMA 7. *Through every point there pass the same number of lines, and this number is n or $n + 1$, whichever is even.*

Proof. Let P be any point and let q be any line which does not contain P. Let t be a perpendicular from P to q. By Axiom A there are, including t, n lines which join P to points of q. Clearly these n lines are all distinct. If there is any other line x through P, then x cannot intersect q. Therefore, by Lemma 6, $x \perp t$. Thus there cannot be two lines through P which fail to meet q, since this would violate Axiom B4. The number of lines through P is therefore either n or $n + 1$; by Lemma 3 it is whichever of these numbers is even.

COROLLARY. *If n is odd, $\mathfrak{J}$ is an affine plane of order n. If n is even, $\mathfrak{J}$ is a projective plane of order $n - 1$.*

For if n is odd, then given a point P and a line $q \not\mathrm{I} P$, there must be exactly one line through P which fails to meet q, while if n is even, all lines through P meet q.

In order to complete the proof of Theorem 1 it remains only to show that n is odd.

https://doi.org/10.4153/CJM-1967-098-9 Published online by Cambridge University Press

LEMMA 8. *If n is even, then all lines perpendicular to a given line are concurrent.*

Proof. By the corollary to Lemma 7, $\mathfrak{J}$ is a finite projective plane of order $n-1$. Let a be any line, and let $x_1, x_2, \ldots, x_n$ be the perpendiculars to a, meeting a at the distinct points $P_1, P_2, \ldots, P_n$ respectively. Let $A_1, \ldots, A_{n-1}$ be the distinct points on x_1 which are distinct from P_1. Then the $n + (n-1)$ points $P_1, \ldots, P_n, A_1, \ldots, A_{n-1}$ are all distinct. Now every point of $\mathfrak{J}$ is on some line x_i $(i = 1, \ldots, n)$, for any perpendicular from a point to a must, by Axiom B4, be an x_i. Since x_1 and x_i $(\neq 1)$ intersect, there must be a point X_i on x_i such that $X_i = A_j$ for some j $(1 \leqslant j \leqslant n-1)$. If the $n-1$ points X_i $(2 \leqslant i \leqslant n)$ do not coincide, then suppose for definiteness that $X_3 \neq X_2$. Let $B_1, \ldots, B_{n-2}$ be the distinct points on x_2 which are distinct from P_2 and X_2. Then the points $P_1, \ldots, P_n, A_1, \ldots, A_{n-1}, B_1, \ldots, B_{n-2}$ are all distinct. Now x_2 and x_3 intersect, but not in the point X_2 (for then we would have $X_3 = X_2$). Therefore x_3 and x_2 intersect in one of the points B_k $(1 \leqslant k \leqslant n-2)$. Hence the number of points on x_3 which are distinct from the points P_i $(i = 1, \ldots, n)$, A_j $(j = 1, \ldots, n-1)$, B_k $(k = 1, \ldots, n-2)$ is less than $n-2$. The total number of distinct points in $\mathfrak{J}$ is therefore less than

$$\begin{aligned} n + (n-1) + (n-1)(n-2) &= n^2 - n + 1 \\ &= (n-1)^2 + (n-1) + 1. \end{aligned}$$

But this is a contradiction, since $(n-1)^2 + (n-1) + 1$ is exactly the number of points in a finite projective plane of order $n-1$ (3, p. 392). Therefore all the lines $x_1, \ldots, x_n$ are concurrent, and the lemma is proved.

COROLLARY. *Given a point P and a line q not containing P, the number of lines through P which are perpendicular to q is* 1 *or* n.

LEMMA 9. *Suppose that n is even and let a and b be any two distinct lines. Let A and B be respectively the common points of the perpendiculars to a and the perpendiculars to b. Then A and B are distinct.*

Proof. Suppose to the contrary that $A = B$. Then every line through A is a perpendicular to a and also a perpendicular to b. In particular the line joining A to P, the intersection of a and b, is perpendicular to both a and b at P. But this contradicts Axiom B4.

COROLLARY. *Suppose that n is even, and let P be any point. Then there is exactly one line p which is perpendicular to all lines through P.*

For since $\mathfrak{J}$ is a projective plane in this case, there are just as many points as there are lines (3, p. 392).

We can now finish the proof of Theorem 1. Suppose that $\mathfrak{J}$ is even, and consider the mapping γ that maps points onto lines and lines onto points as follows:

(i) If P is any point, then P^γ, the image of P, is the line which is perpendicular to all lines through P.

https://doi.org/10.4153/CJM-1967-098-9 Published online by Cambridge University Press

(ii) If l is any line, then l^γ, the image of l, is the common point of all perpendiculars to l.

By Lemmas 8 and 9 and the corollary to Lemma 9, γ is a 1–1 onto mapping. Moreover, if X and y are any point and line respectively, it is easily seen that $X \mathrm{I} y$ if and only if $X^\gamma \mathrm{I} y^\gamma$. Thus γ is a *correlation*. If, for any point P, $P^\gamma = p$, then clearly $p^\gamma = P$. Thus γ is a correlation of period 2, i.e. a *polarity*.

An absolute (self-conjugate) point of any polarity π is a point P with the property that $P \mathrm{I} P^\pi$. By Axiom B4, the polarity γ can have no absolute points. But it is known (**2**, p. 82) that any polarity in a finite projective plane has absolute points. Thus we have a contradiction and n cannot be even.

3. Note on the generality of the axioms. Given a finite affine plane of odd order, it is easy to define a relation of orthogonality satisfying Axioms B1–B5. One simply sets up on the points of the ideal line (line at infinity) an involution (i.e. a 1–1 mapping of period 2) with no fixed points. Then two lines in the plane are perpendicular if and only if they meet the ideal line in corresponding pairs of the involution. The verification of Axioms B1–B5 for this relation is trivial.

Thus Axioms A and B1–B5, with the understanding that $\mathfrak{J}$ is finite, may be taken to be an alternative set of axioms for a finite affine plane of odd order. The axioms impose no further restrictions as to the nature of this plane; in particular, the plane is not necessarily Desarguesian. Bachmann's metric planes, however, are Desarguesian (**1**, §§5 and 6).

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