INEQUALITIES FOR CERTAIN CYCLIC SUMS II

by J. C. BOARDER and D. E. DAYKIN

(Received 25th March 1972, revised 2nd August 1972)

1. Introduction

Let $k \geqq 2$ be an integer and each of $v_1, v_2, \ldots, v_k$ and $\delta_1, \delta_2, \ldots, \delta_k$ be 0 or 1. Then given any positive integer M and non-negative reals $a_1, a_2, \ldots, a_M$ we put

$$S = S_M = S_M(a_1, a_2, \ldots, a_M) = \sum_{i=1}^{M} t_i, \tag{1}$$

where

$$t_i = \frac{v_1 a_{i+1} + v_2 a_{i+2} + \ldots + v_k a_{i+k}}{\delta_1 a_{i+1} + \delta_2 a_{i+2} + \ldots + \delta_k a_{i+k}} \text{ for } 1 \leqq i \leqq M \tag{2}$$

and

$$a_{i+M} = a_i \text{ for all } i. \tag{3}$$

The object of our work is to evaluate inf S_M and sup S_M, where the inf and sup are evaluated over all choices of $a_1, a_2, \ldots, a_M$. When unable to find these we try to estimate $\inf_M \{\inf S_M/M\}$. It is (3) which gives the sum S its cyclic character. Of course, we do not allow zero denominators. Also we ignore the trivial cases $v_1 = v_2 = \ldots = v_k = 0$ and $v_j = \delta_j$ for $1 \leqq j \leqq k$. The cases with $k = 3$ were discussed in (**3**). Here we report some interesting facts discovered by considering the case $k = 4$ with the aid of a computer, complete details of the study are given in (**1**).

First we show that any given sup is either ∞ or obtainable from some inf. The best upper bounds we could find for inf S_M when $k = 4$ are summarised in Tables 1 and 2, and we are able to prove that some of them are best possible. Our study has led us to believe that one can get very close to inf S_M, for any given sum S_M, by evaluating S_M for a sequence $a_1, \ldots, a_M$ which is very simple, as simple for instance as those appearing in (5) and (6). Many of our examples are derived from the exponential sequence, and we discuss the local stability of this sequence in Section 8. Other examples are obtained by repeating a few numbers like 0, 4, 0, 3, 2. The nature of the sequences enables us to write down explicitly the value of S_M, and for a given sum S_M there are not too many kinds of sequence to choose between. We improve here on all earlier upper bounds for inf S_M, and in particular for Shapiro's sum. We hope we have in fact found $\lim_M \{\inf S_M\}$ for many sums, but we are by no means certain. The subject still abounds with open questions.

To simplify the notation we let

$$\Sigma ac/ab \text{ mean } \sum_{i=1}^{M} \frac{a_{i+1}+a_{i+3}}{a_{i+1}+a_{i+2}},$$

which is the case $v_1 = v_3 = \delta_1 = \delta_2 = 1$ and all other v_i, δ_i zero, and similarly for the other sums.

2. The supremum

We say that the numerator is contained in the denominator when $v_j = 1$ implies $\delta_j = 1$ for $1 \leqq j \leqq k$. In this case $S_M \leqq M$ because $t_i \leqq 1$ for every term t_i no matter how $a_1, a_2, \ldots, a_M$ are chosen. Moreover, there is another sum S' such that

$$\sup S = M - \inf S',$$

for example

$$\sup \Sigma b/abd = \Sigma abd/abd - \inf \Sigma ad/abd.$$

On the other hand, if the numerator is not contained in the denominator there is a j with $v_j = 1$ but $\delta_j = 0$, and then $\sup S = \infty$ because $t_1 \to \infty$ as $a_{1+j} \to \infty$. In view of these facts from now on we shall consider only inf S and not sup S.

3. Type *a* sums

An elementary result from calculus is

Lemma 1. *If r is an integer and* $x_i = x_{i+M} > 0$ *for all i then*

$$\inf \sum_{i=1}^{M} x_i/x_{i+r} = M.$$

It immediately yields a result in (**2**), namely

$$\inf S_M = (v_1+v_2+\ldots+v_k)M \text{ if } \delta_1+\delta_2+\ldots+\delta_k = 1,$$

so such sums need not be mentioned further. Also it gives

$$\inf \Sigma ab/cd = \inf \Sigma ac/bd = M,$$

and so on. Other sums can be changed so that the lemma applies, for example

$$\Sigma ad/bc = \Sigma ab/bc + \Sigma cd/bc - \Sigma bc/bc$$

so $\inf \Sigma ad/bc = M$, which is a neat proof of the main case of (**2**, Theorem 7). We say that the sums for which we can use Lemma 1 are of type *a*, they take their inf when $a_1 = a_2 = \ldots = a_M = 1$.

4. Results Table 1

In the last section we dismissed those sums with only one non-zero δ. For $k = 4$ all the remaining sums are considered in Table 1. The rows of the table correspond to the numerators and the columns to the denominators of the sums. Of course, if we reverse the order of the a's the v's and the δ's in a sum S we get

https://doi.org/10.1017/S0013091500009949 Published online by Cambridge University Press

TABLE 1

bc	$= M$	$\Sigma ab/ac$	$= 2$	$\leqq 3{\cdot}5$	$\leqq 6$	$= 2$	equal				
$abcd$	$= 2M$	$= 2M$	$M + \Sigma a/bcd$	$M + \Sigma b/acd$	$M + \Sigma bc/ad$	equal	$= 2M$				
$a\ d$	$\leqq \frac{2}{3}M^+$	$\leqq \frac{2}{3}M^+$	Table 2	$\leqq \frac{1}{3}M^+$	equal	Table 2	$= M$				
$ab\ d$	$M + \Sigma a/cd$	Table 2	Table 2	equal	$M + \Sigma b/ad$	Table 2	Table 2	Table 2	§ 7	$M + \Sigma a/bd$	Table 2
abc	$M + \Sigma a/bc$	$M + \Sigma b/ac$	equal	$\leqq 5$	$\leqq 9$	$= 3$	$M + \Sigma a/bc$	$\leqq 5{\cdot}5$	$= M$	Table 2	Table 2
$a\ c$	Table 2	equal	$= M - [\frac{1}{2}M]$	$\leqq 3{\cdot}5$	$\leqq 6$	$= 2$	$\Sigma ac/ab$	$\leqq 3$	Table 2	$= M$	Table 2
ab	equal	$= 4$	$= 2$	$\leqq 3$	$\leqq 6$	$= 2$	$= M$	$\leqq 4$	§ 7	Table 2	$= M$
b	$= 1$	$= 2$	$= 1$	$= 1$	$\leqq 3$	$= 1$	$= 1$	$\leqq 2$	$= 1$	$\Sigma a/ac$	$\Sigma a/bc$
a	$= 1$	2 cases	$= 1$	$= 1$	2 cases	$= 1$	Table 2	$= 1$	Table 2	$\leqq \frac{1}{3}M^+$	$\leqq \frac{1}{3}M^+$
	ab	$a\ c$	abc	$ab\ d$	$a\ d$	$abcd$	bc	$a\ cd$	bcd	$b\ d$	cd

another sum S' which is essentially the same as S, and in the table only one of S and S' appear in most cases. Also some further transformations are suggested in the table. For instance, reversing makes $\Sigma abcd/abd$ correspond to $M+\Sigma b/acd$. Also $\Sigma a/ad$ is the same as three sums or one sum of the form $\Sigma a/ab$ depending upon whether 3 divides M or not. The entry $\leqq 3$ in the table for sum $\Sigma b/ad$ means that 3 is the best upper bound we have found for inf $\Sigma b/ad$. The entry $= 2M$ for sum $\Sigma abcd/bc$ means that $\inf \Sigma abcd/bc = 2M$. The entry $\leqq \frac{2}{3}M^+$ for $\Sigma ad/ab$ means there is a (small) constant τ such that $\inf S_M \leqq \frac{2}{3}M+\tau$ for $M \geqq 1$. The same interpretation applies to the other sums. Those sums whose entry says " Table 2 " will be discussed in Section 7. Notice that all sums with $= M$ or $= 2M$ as their entry are of type a.

5. Interval denominator sums

These are the sums for which $\delta_s = 1$ whenever $1 \leqq r \leqq s \leqq t \leqq k$ and $\delta_r = \delta_t = 1$. They include Shapiro's sum $\Sigma a/bc$ and Diananda's generalisations of it, $\Sigma a/bcd$, $\Sigma a/bcde$, etc.

Lemma 2. *If*

$$t_i = a_{i+j}/(a_{i+1}+a_{i+2}+\ldots+a_{i+k}) \tag{4}$$

where $0 \leqq j \leqq k+1$, *and* $a_M \leqq \min\{a_1, a_2, \ldots, a_{M-1}\}$ *then*

$$S_{M-1}(a_1, a_2, \ldots, a_{M-1}) \leqq S_M(a_1, a_2, \ldots, a_{M-1}, a_M).$$

Proof. First write out S_{M-1} and S_M and cancel terms which appear on both sides of the inequality $S_{M-1} \leqq S_M$. Then the remaining terms on the left pair off with terms on the right so that each left-hand term does not exceed its corresponding right-hand term. One term somewhere on the right-hand side is not paired off, but it will be non-negative. The other terms pair off in order of appearance. This completes our proof of the lemma.

The lemma immediately implies that $\inf S_{M-1} \leqq \inf S_M$ whenever $\delta_2 = \delta_3 = \ldots = \delta_{k-1} = 1$ in (1), and we feel that this is the case for all choices of the ν's and δ's.

Lemma 3. *Suppose* t_i *is as in* (4) *with* $0<j \leqq k+1$, *then*

$$S_{M+k}(a_1, a_2, \ldots, a_M, a_1, a_2, \ldots, a_k) = 1+S_M(a_1, a_2, \ldots, a_M).$$

This result is obvious upon expansion of S_{M+k}. It is interesting because it gives an upper bound on the rate of growth of the inf for many interval denominator sums, for example $\inf S_{M+4} \leqq 2+\inf S_M$ when $S = \Sigma ab/bcde$.

6. Type b sums

These are the sums with $\nu_1\nu_k = 0$ and $\delta_1\delta_k = 1$. It seems that they are the only ones for which $\inf S_M$ has an upper bound independent of M. We prove the existence of such a bound in

Lemma 4. *If* $v_1 v_k = 0$ *and* $\delta_1 \delta_k = 1$ *then inf* $S_M \leqq (k-1)^2$ *for* $1 \leqq M$.

Proof. We may assume $v_k = 0$ and let $S' = \Sigma t_i'$ be the sum with

$$t_i' = (a_{i+1} + a_{i+2} + \ldots + a_{i+k-1})/(a_{i+1} + a_{i+k}),$$

then $S_M \leqq S_M'$ for all a_i. Now $S_M' \to (k-1)^2$ as $\alpha \to \infty$ when $M \geqq k$ and the a_i are

$$\alpha, \alpha^2, \ldots, \alpha^N, \alpha^N, \ldots, \alpha^N \tag{5}$$

with $N = M - k + 2$. Also $S_M'(1, 1, \ldots, 1) = \frac{1}{2}(k-1)M$ for $1 \leqq M < k$, and the lemma follows.

Theorem 1. *If* $v_1 v_k = 0$ *and* $\delta_1 = \delta_2 = \ldots = \delta_k = 1$ *then*

$$\inf S_M = v_1 + v_2 + \ldots + v_k \textit{ for } M \geqq k.$$

Proof. We have $S_k = v_1 + \ldots + v_k$ whatever the values of $a_1, \ldots, a_k$. Also Lemma 2 shows that $\inf S_{M-1} \leqq \inf S_M$. Finally either $S_M(\alpha, \alpha^2, \ldots, \alpha^M)$ or $S_M(\alpha^M, \ldots, \alpha^2, \alpha)$ tends to $v_1 + \ldots + v_k$ as α tends to ∞. The theorem follows inductively.

Theorem 2. *If* $v_j = \delta_j = \delta_{j+1} = 1$ *and all other v's are zero then* $\inf S_M = 1$

Proof. We have $1 \leqq \inf S_M$ by Theorem 1 and $S_M(\alpha, \alpha^2, \ldots, \alpha^M) \to 1$ as $\alpha \to \infty$.

It is possible to extend Theorem 2. For instance, $\inf \Sigma a/acd = 1$ by Theorem 1 and the example $\alpha^N, 0, \alpha^{N-1}, 0, \ldots$ with $N = [\frac{1}{2}(M+1)]$. It was by choosing sequences in this sort of way that we were able to establish the various bounds in Table 1 for the infs of the type *b* sums, full details are given in (**1**). Each sum with a constant entry in Table 1 is of type *b*.

7. Computer results

The sums in Table 2. In Table 2 we give the smallest values of S_M/M we could get for various sums S. The sequences $a_1, a_2, \ldots, a_M$ referred to in the table are

$$\ldots, \beta^4, 0, \beta^2, 0, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^{\theta M} \tag{6}$$

$$\ldots, \beta^8, \beta^7, 0, 0, \beta^4, \beta^3, 0, 0, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^{\theta M} \tag{7}$$

$$\underbrace{\ldots, \beta^6, 0, 0, \beta^3, 0, 0,}_{(1-\theta)M \text{ terms}} \underbrace{\alpha, \alpha^2, \alpha^3, \ldots, \alpha^{\theta M}}_{\theta M \text{ terms}} \tag{8}$$

and

$$1, 0, 0, 1, 0, 0, 1, 0, 0, \ldots \tag{9}$$

$$\ldots, 0, \alpha^8, 1, 0, \alpha^5, 1, 0, \alpha^2, 1 \tag{10}$$

$$\ldots, \alpha^9, 0, 1, \alpha^6, 0, 1, \alpha^3, 0, 1 \tag{11}$$

$$\alpha, 0, \alpha^3, 0, \alpha^5, 0, \ldots \tag{12}$$

https://doi.org/10.1017/S0013091500009949 Published online by Cambridge University Press

or their reverses. To see how we made these choices consider the sum $\Sigma abc/bd$. Examination of the computer output suggested that we might get a good value of S_M/M with sequence (7) when α, β, θ are suitably chosen. Now in S_M there are at most four terms t_i which involve both powers of α and powers of β, and we must ensure that these t_i are not too big. Therefore we would like α and β to be of the same order of magnitude, and similarly for $\beta^{(1-\theta)M}$ and $\alpha^{\theta M}$.

TABLE 2

Sum	S_M/M	Sequence	α	θ
$\Sigma a/bc$	0·49457	6 Reversed	1·10562	0·43493
$\Sigma ac/ab$	0·97801	6	1·21383	0·46034
$\Sigma abd/cd$	1·49814	6 reversed	1·03769	0·40000
$\Sigma abc/cd$	1·49811	6	1·03866	0·40000
$\Sigma abd/bc$	1·49732	6 reversed	1·05304	0·41441
$\Sigma abc/bd$	1·48408	7	1·19486	0·34900
$\Sigma abd/ac$	1·48306	7 reversed	1·18083	0·41074
$\Sigma a/bcd$	0·32598	8 reversed	1·11653	0·40640
$\Sigma ad/abc$	0·63755	8	1·22208	0·44910
$\Sigma ac/bcd$	0·65870	8 reversed	1·13410	0·38500
$\Sigma abd/abc$	0·96872	8	1·24974	0·43357
$\Sigma ad/abcd$	$\frac{1}{3}$	9		
$\Sigma ac/cd$	$\frac{2}{3}$	10	$\to 0$	
$\Sigma ab/bd$	$\frac{2}{3}$	11	$\to 0$	
$\Sigma abd/acd$	$\frac{1}{2}$	12	$\to\infty$	
$\Sigma abd/abcd$	$\frac{1}{2}$	12	$\to\infty$	

Therefore we make

$$\beta^{(1-\theta)} = \alpha^{\theta}, \tag{13}$$

and for large M the value of S_M/M is approximately

$$\tfrac{1}{4}(1-\theta)\left\{\frac{\beta^9+\beta^8+0}{\beta^8+0}+\frac{\beta^8+0+0}{0+\beta^5}+\frac{0+0+\beta^5}{0+\beta^4}+\frac{0+\beta^5+\beta^4}{\beta^5+0}\right\}+\theta\left\{\frac{\alpha+\alpha^2+\alpha^3}{\alpha^2+\alpha^4}\right\}.$$

Minimising this expression, subject to (13), over $0<\theta<1$ and $0<\alpha$ gave us the asymptotic result in Table 2 for $\Sigma abc/bd$. The other choices were made in a similar way.

https://doi.org/10.1017/S0013091500009949 Published online by Cambridge University Press

The sums $\Sigma a/bd$, $\Sigma a/cd$, $\Sigma ad/ab$, $\Sigma ad/ac$, $\Sigma ad/abd$. The computer indicated that these sums tend towards their smallest values as $\alpha \to \infty$ with $1, \alpha, 0, 1, \alpha, 0, \ldots$ or its reverse for the a_i. In any case it was these a_i which yielded the bounds for inf S_M for these sums in Table 1.

The sum $\Sigma abd/bcd$. Here the smallest value of S_M/M that we found was just less than 0·96, and was attained with M divisible by 5 and the a_i repetitions of a five-term sequence almost exactly 0, 4, 0, 3, 2. This sum and the next one are the only two we know which behave in this way.

The sum $\Sigma ab/bcd$. In this case our best result was $S_{11}/11 = 0{\cdot}65191$ with approximately 61, 0, 212, 0, 73, 184, 0, 146, 128, 0, 195 for the a_i. We found this sum to be particularly interesting because for $M = 100$ our computer program would consistently lead to stable choices of a_i which were not as good as the ones we could construct by repeating the above eleven terms.

The pair of sums $\Sigma abd/cd$ *and* $\Sigma abc/cd$. For reasons that we do not understand, for these sums our computer program led to identical values of S_M and the a_i, but with the a_i terms in reverse order. We got $S_{99}/99 = 1{\cdot}4985$ with the a_i as follows for the latter sum

18	18	19	20	20	21	21	23	22	24
23	25	26	25	30	24	36	19	45	12
53	4	58	0	56	0	53	0	50	0
47	0	44	0	41	0	39	0	36	0
34	0	32	0	30	0	28	0	26	0
25	0	23	0	22	0	20	0	19	0
18	0	17	0	16	0	14	1	12	3
11	5	9	7	9	8	9	9	9	10
10	11	10	11	11	12	12	12	13	13
13	14	14	15	15	16	16	17	17.	

Another sum with a very similar sequence is $\Sigma abd/bc$. In Table 2 we give better values of S_M/M for large M.

The turning value of M. For any sum S_M we define its turning value to be the largest integer M_0 such that $\inf S_M = S_M(1, 1, \ldots, 1)$ for $1 \leqq M \leqq M_0$. It was shown by P. Nowosad (5) that $M_0 \geqq 10$ for Shapiro's sum $\Sigma a/bc$. For each of the sums (1) with $k \leqq 4$ we found an upper bound for M_0 by the computer and they are given in (1). We only got $M_0 \leqq 39$ for $\Sigma abd/bc$ and this is surprising because we got $M_0 \leqq 22$ for all the other sums.

8. Local stability

A sequence $a_1, \ldots, a_M$ is locally stable at a_i when

$$\partial S/\partial a_i > 0 \text{ if } a_i = 0,$$

but

$$\partial S/\partial a_i = 0 \text{ and } \partial^2 S/\partial a_i^2 > 0 \text{ if } a_i > 0.$$

We will show in Lemma 5 that exponential sequences like α, α^2, α^3, ... and 0, α^2, 0, α^4, ... are almost locally stable for all our sums. First we introduce some notation. Let

$$N_i = v_1 a_{i+1} + \ldots + v_k a_{i+k} \text{ and } D_i = \delta_1 a_{i+1} + \ldots + \delta_k a_{i+k}$$

so $t_i = N_i/D_i$. Then since a_k occurs only in $t_0, t_1, \ldots, t_{k-1}$ we have

$$\partial S/\partial a_k = +\{v_k D_0^{-1} + \ldots + v_1 D_{k-1}^{-1}\} - \{\delta_k N_0 D_0^{-2} + \ldots + \delta_1 N_{k-1} D_{k-1}^{-2}\} \tag{14}$$

$$\tfrac{1}{2}\partial^2 S/\partial a_k^2 = -\{v_k \delta_k D_0^{-2} + \ldots + v_1 \delta_1 D_{k-1}^{-2}\} + \{\delta_k N_0 D_0^{-3} + \ldots + \delta_1 N_{k-1} D_{k-1}^{-3}\} \tag{15}$$

and our result is

Lemma 5. *Suppose $\alpha > 0$ and p divides k and $M \geqq 2k$. If $a_{ip} = \alpha^{r+ip}$ when $1 \leqq i < 2k/p$ but $a_i = 0$ otherwise, then $\partial S_M/\partial a_k = 0$ when $a_k \neq 0$.*

Proof. Without loss of generality we assume $r = 0$. The choice of a_i makes the N's in (14) as follows

$$N_0 = v_p \alpha^p + v_{2p}\alpha^{2p} + \ldots + v_k \alpha^k$$
$$N_1 = v_{p-1}\alpha^p + v_{2p-1}\alpha^{2p} + \ldots + v_{k-1}\alpha^k$$

and so on. So $N_{i+p} = \alpha^p N_i$ and similarly for the D's in (14). Hence (14) becomes

$$\begin{aligned}
\partial S_M/\partial a_k &= \sum_{i=0}^{p-1} \{v_{k-i} D_i^{-1} + v_{k-i-p} D_{i+p}^{-1} + v_{k-i-2p} D_{i+2p}^{-1} + \ldots\} \\
&\quad - \sum_{i=0}^{p-1} \{\delta_{k-i} N_i D_i^{-2} + \delta_{k-i-p} N_{i+p} D_{i+p}^{-2} + \delta_{k-i-2p} N_{i+2p} D_{i+2p}^{-2} + \ldots\} \\
&= \sum_{i=0}^{p-1} \{v_{k-i}\alpha^{k-i} + v_{k-i-p}\alpha^{k-i-p} + v_{k-i-2p}\alpha^{k-i-2p} + \ldots\}\alpha^{i-k} D_i^{-1} \\
&\quad - \sum_{i=0}^{p-1} \{\delta_{k-i}\alpha^{k-i} + \delta_{k-i-p}\alpha^{k-i-p} + \delta_{k-i-2p}\alpha^{k-i-2p} + \ldots\}\alpha^{i-k} N_i D_i^{-2} \\
&= \alpha^{i-k} \sum_{i=0}^{p-1} (\{N_i\} D_i^{-1} - \{D_i\} N_i D_i^{-2}) = 0,
\end{aligned}$$

as stated.

In the lemma we checked $\partial S/\partial a$ only at a non-zero term a. Unfortunately at a zero term a of a sequence like 0, α^2, 0, α^4, ..., the sign of $\partial S/\partial a$ depends on the sum under consideration. The value of (15) always seems to depend on the sum S being considered. However each of the sequences in Table 2 are locally stable at all but a few of their terms a_i. This fact encourages us to hope that Table 2 gives the correct asymptotic value of inf S_M/M.

9. The computer program

This is much better than the one described in (2), and very simple. Given $a_1, a_2, \ldots, a_M$ and $\varepsilon > 0$ and j in $1 \leqq j \leqq M$ put

$$u_j^+, u_j^- = S_M(a_1, \ldots, a_{j-1}, a_j \pm \varepsilon, a_{j+1}, \ldots, a_M) - S_M(a_1, a_2, \ldots, a_M).$$

Very little work is required to evaluate u_j^+, u_j^- as we only need to consider those terms of S_M which involve a_j. Since the a_j must remain non-negative when $a_j-\varepsilon<0$ the computer considers $a_j-\varepsilon$ to be equal to 0. If, however, this condition gives a zero denominator the computer in fact only evaluates u_j^+ and puts $u_j^- = +\infty$. The program consists of repeating the following process. With $j = 1, 2, \ldots, M$ in turn, (i) if $0 \leqq u_j^+$ and $0 \leqq u_j^-$ do nothing, (ii) if $u_j^+<0$ and $u_j^+ \leqq u_j^-$ replace a_j by $a_j+\varepsilon$, (iii) if $u_j^-<0$ and $u_j^-<u_j^+$ replace a_j by $a_j-\varepsilon$ when $a_j-\varepsilon>0$ but by 0 when $a_j-\varepsilon \leqq 0$. In case it turned out that $0 \leqq u_j^+, u_j^-$ for $j = 1, 2, \ldots, M$ so that there were no replacements then halve ε before the next repetition. The program stops when ε is sufficiently small.

Clearly the program reduces S_M at each replacement and leads to a sequence $a_1, \ldots, a_M$ which is locally stable. We have already remarked that it may not yield inf S_M from all initial values of $a_1, \ldots, a_M$. An improved version of the program only increments j when it enters case (i) above. When it enters cases (ii) or (iii) it re-evaluates u_j^+, u_j^- with the same j but the new value of a_j and obeys (i), (ii) or (iii) again.

We stopped using the method described in (3), even though it is potentially very fast, because it tends to produce an oscillating sequence of a_i's and we could not get the machine to take over the smoothing of the sequence which was previously done by human interpolation.

10. Diananda's sums

The generalisations in (16) below of Shapiro's sum $\Sigma a/bc$ were considered by P. H. Diananda (4). In particular he proved

Theorem 3.

$$\Sigma a_i/(a_{i+1}+a_{i+2}+\ldots+a_{i+k}) \geqq M/k \tag{16}$$

$$\left.\begin{array}{l} \textit{if } k \textit{ divides } M+2 \textit{ or } 2M \textit{ or } 2M+1 \textit{ or } 2M+2 \\ \textit{or } k \equiv 5 (\mathrm{mod}\ 8) \textit{ or } k \equiv 6 (\mathrm{mod}\ 9) \textit{ or } k \equiv 8, 9 (\mathrm{mod}\ 12). \end{array}\right\} \tag{17}$$

To conclude this paper we report some computer results obtained by K. Y. Choong and one of us (D. E. D.) at the University of Malaya in 1968. Examples of $a_1, \ldots, a_M$ were sought for which (16) is false. In other words, an effort was made to determine the turning value of M for small fixed values of k. The results indicated that conditions (17) are the best possible for k sufficiently large, and enough examples were found to establish that this is the case for $k = 11, 23, 24$. In the cases of (k, M) listed below inequality (16) is not proved true by (17) or Nowosad (5) but no example was found which made it false. For $k \leqq 12$ the list contains all cases (k, M) for which it is not yet known whether (16) holds for all a_i.

(2, 11)	(2, 13)	(2, 15)	(2, 17)	(2, 19)	(2, 21)	(2, 23)	(2, 12)
(3, 10)	(3, 13)	(3, 16)	(3, 19)	(3, 9)	(3, 12)	(4, 13)	(4, 17)
(4, 12)	(4, 16)	(5, 16)	(5, 15)	(6, 19)	(6, 18)	(7, 22)	(7, 11)
(7, 21)	(8, 25)	(8, 24)	(9, 27)	(10, 15)	(10, 30)	(12, 18)	(12, 9).

https://doi.org/10.1017/S0013091500009949 Published online by Cambridge University Press

REFERENCES

(1) J. C. Boarder, M.Phil. Thesis, University of Reading, (1972).

(2) D. E. Daykin, Inequalities for functions of a cyclic nature, *J. London Math. Soc.* (2) **3** (1971), 453-462.

(3) D. E. Daykin, Inequalities for certain cyclic sums, *Proc. Edinburgh Math. Soc.* (2) **17** (1971), 257-262.

(4) P. H. Diananda, Extensions of an inequality of H. S. Shapiro, *Amer. Math. Monthly*, **66** (1959), 489-491.

(5) P. Nowosad, Isoperimetric eigenvalue problems in algebra, *Comm. Pure Appl Math.* **21** (1968), 401-465.

University of Reading
Berkshire