CONSTRUCTION OF SUNNY NONEXPANSIVE RETRACTIONS IN BANACH SPACES

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Let $J$ be a commutative family of nonexpansive self-mappings of a closed convex subset $C$ of a uniformly smooth Banach space $X$ such that the set of common fixed points is nonempty. It is shown that if a certain regularity condition is satisfied, then the sunny nonexpansive retraction from $C$ to $F$ can be constructed in an iterative way.

1. INTRODUCTION

Let $C$ be a nonempty closed convex subset of a Banach space $X$ and let $D$ be a nonempty subset of $C$. A retraction from $C$ to $D$ is a mapping $Q : C \to D$ such that $Qx = x$ for $x \in D$. A retraction $Q$ from $C$ to $D$ is nonexpansive if $Q$ is nonexpansive (that is, $\|Qx - Qy\| \leq \|x - y\|$ for $x, y \in C$). A retraction $Q$ from $C$ to $D$ is sunny if $Q$ satisfies the property:

$$Q(Qx + t(x - Qx)) = Qx \quad \text{for } x \in C \text{ and } t > 0 \text{ whenever } Qx + t(x - Qx) \in C.$$ 

A retraction $Q$ from $C$ to $D$ is sunny nonexpansive if $Q$ is both sunny and nonexpansive.

It is known ([2]) that in a smooth Banach space $X$, a retraction $Q$ from $C$ to $D$ is a sunny nonexpansive retraction from $C$ to $D$ if and only if the following inequality holds:

$$\langle x - Qx, J(y - Qx) \rangle \leq 0, \quad x \in C, \quad y \in D,$$

where $J : X \to X^*$ is the duality map defined by

$$Jx := \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad x \in X.$$

Hence a sunny nonexpansive retraction must be unique (if it exists).

If $C$ is a nonempty closed convex subset of a Hilbert space $H$, then the nearest point projection $P_C$ from $H$ to $C$ is the sunny nonexpansive retraction. This however is not true for Banach spaces since nonexpansivity of projections $P_C$ characterises Hilbert spaces.
On the other hand, one sees by (1.1) that a sunny nonexpansive retraction can play a similar role in a Banach space as a projection does in a Hilbert space. So an interesting problem is this: In what kind of Banach spaces, does a sunny nonexpansive retraction exist? If it does exist, how can one find it? It is known ([2]) if $C$ is a closed convex subset of a uniformly smooth Banach space $X$ and there is a nonexpansive retraction from $X$ to $C$, then there exists a sunny nonexpansive retraction from $X$ to $C$. But Bruck's proof is not constructive.

The purpose of the present paper is to construct sunny nonexpansive retractions in a uniformly smooth Banach space in an iterative way. More precisely, we show that if $F$ is the nonempty common fixed point set of a commutative family of nonexpansive self-mappings of a closed convex subset $C$ of a uniformly smooth Banach space $X$ satisfying certain regularity condition, then we are able to construct a sequence that converges to the sunny nonexpansive retraction $Q$ of $C$ to $F$. This extends a result of Reich [7] where the case of a single nonexpansive mapping is dealt with.

2. TWO LEMMAS

**Lemma 2.1.** Let $(s_n)$ be a sequence of nonnegative numbers satisfying the condition:

\[(2.1) \quad s_{n+1} \leq (1 - \alpha_n)(s_n + \beta_n) + \alpha_n\gamma_n, \quad n \geq 0,\]

where $(\alpha_n), (\beta_n), (\gamma_n)$ are sequences of real numbers satisfying

(i) $(\alpha_n) \subset [0, 1]$, $\lim \alpha_n = 0$, and $\sum_{n=0}^{\infty} \alpha_n = \infty$, or equivalently, $\prod_{n=0}^{\infty} (1 - \alpha_n) := \lim_{n \to \infty} \prod_{k=0}^{n} (1 - \alpha_k) = 0$;

(ii) $\limsup_{n \to \infty} \beta_n \leq 0$;

(iii) $\limsup_{n \to \infty} \gamma_n \leq 0$.

Then $\lim_{n \to \infty} s_n = 0$.

**Proof:** For any $\varepsilon > 0$, let $N \geq 1$ be an integer big enough so that

\[(2.2) \quad \beta_n < \varepsilon/2, \quad \gamma_n < \varepsilon, \quad \alpha_n < 1/2, \quad n \geq N.\]

It follows from (2.1) and (2.2) that, for $n > N$,

\[
s_{n+1} \leq (1 - \alpha_n)(s_n + \beta_n) + \varepsilon \alpha_n \\
\leq (1 - \alpha_n)(1 - \alpha_{n-1})(s_{n-1} + \beta_{n-1}) + \varepsilon \left(1 - (1 - \alpha_n)(1 - \alpha_{n-1}) + \frac{1}{2}(1 - \alpha_n)\right) \\
\leq (1 - \alpha_n)(1 - \alpha_{n-1})(s_{n-1} + \beta_{n-1}) + \varepsilon \left(1 - \left(\frac{1}{2} - \alpha_n\right)\left(\frac{1}{2} - \alpha_{n-1}\right)\right).\]
Hence by induction we obtain
\[ s_{n+1} \leq \prod_{j=N}^{n} (1 - \alpha_j) (s_N + \beta_N) + \varepsilon \left[ 1 - \prod_{j=N}^{n} (1 - \alpha_j) \right], \quad n > N, \]
where \( \alpha_j := 1/2 + \alpha_j < 1, \ j \geq N \). By condition (i) we deduce, after taking the limsup as \( n \to \infty \) in the last inequality, that \( \limsup_{n \to \infty} s_{n+1} \leq \varepsilon \).

**Lemma 2.2.** (The Subdifferential Inequality, see [3].) In a Banach space \( X \) there holds the inequality:
\[ \|x + y\|^2 \leq \|x\|^2 + 2\langle y, j \rangle, \quad x, y \in X, \ j \in J(x + y), \]
where \( J \) is the duality map of \( X \).

### 3. Iterative Processes

Let \( G \) be an unbounded subset of \( \mathbb{R}^+ \) such that \( s + t \in G \) whenever \( s, t \in G \). (Often \( G = \mathbb{N} \), the set of nonnegative integers or \( \mathbb{R}^+ \).) Let \( X \) be a uniformly smooth Banach space, \( C \) a nonempty closed convex subset of \( X \), and \( J = \{ T_s : s \in G \} \) a commutative family of nonexpansive self-mappings of \( C \). Denote by \( F \) the set of common fixed point of \( J \), that is, \( F = \{ x \in C : T_s x = x, \ s \in G \} \). Throughout this section we always assume that \( F \) is nonempty. Our purpose is to construct the sunny nonexpansive retraction \( Q \) from \( C \) to \( F \). We shall introduce two iterative processes to construct \( Q \). The first one is implicit, while the second one is explicit. But before introducing the iterative processes, we recall that the uniform smoothness of \( X \) is equivalent to the following statement:
\[ \lim_{\lambda \to 0^+} \frac{\|x + \lambda y\|^2 - \|x\|^2}{\lambda} = 2\langle y, Jx \rangle \quad \text{uniformly for bounded } x, y \in X. \]

Let \( u \in C \) be given arbitrarily and let \( (\alpha_s)_{s \in G} \) be a net in the interval \( (0, 1) \) such that \( \lim_{s \to \infty} \alpha_s = 0 \). By Banach’s contraction principle, for each \( s \in G \), we have a unique point \( z_s \in C \) satisfying the equation
\[ z_s = \alpha_s u + (1 - \alpha_s) T_s z_s. \quad (3.1) \]

**Theorem 3.1.** Let \( X \) be a uniformly smooth Banach space. Assume \( J \) is uniformly asymptotically regular on bounded subsets of \( C \); that is, for each bounded subset \( \tilde{C} \) of \( C \) and each \( r \in G \),
\[ \lim_{s \to \infty} \sup_{x \in \tilde{C}} \| T_r T_s x - T_s x \| = 0. \quad (3.2) \]

Then the net \((z_s)\) defined in (3.1) converges in norm and the limit defines the sunny nonexpansive retraction \( Q \) from \( C \) to \( F \).
PROOF: First we observe that \( (z_s) \) is bounded. Indeed, for \( p \in F \), we have
\[
\|z_s - p\| \leq \alpha_s \|u - p\| + (1 - \alpha_s) \|T_s z_s - p\| \leq \alpha_s \|u - p\| + (1 - \alpha_s) \|z_s - p\|.
\]
This implies that \( \|z_s - p\| \leq \|u - p\| \) and \( (z_s) \) is bounded. Thus \( \|z_s - T_s z_s\| = \alpha_s \|u - T_s z_s\| \to 0 \) \( (s \to \infty) \). Let \( (s_n) \) be a subsequence of \( G \) such that \( \lim_n s_n = \infty \). Next we define a function \( f \) on \( C \) by
\[
f(x) := \text{LIM}_n \|z_n - x\|^2, \quad x \in C,
\]
where \( \text{LIM} \) is a Banach limit and \( z_n := z_{s_n} \). We have for each \( r \in G \),
\[
f(T_r x) = \text{LIM}_n \|z_n - T_r x\|^2
\]
\[
= \text{LIM}_n \|T_r T_n z_n - T_r x\|^2
\]
\[
\leq \text{LIM}_n \|T_n z_n - x\|^2
\]
using (3.2). Therefore,
\[
(3.3) \quad f(T_r x) \leq f(x), \quad r \in G, \quad x \in C.
\]
Let
\[
K := \{x \in C : f(x) = \min_C f\}.
\]
Then it can be seen that \( K \) is a closed bounded convex nonempty subset of \( C \). By (3.3) we see that \( K \) is invariant under each \( T_r \); that is, \( T_r(K) \subset K \), \( r \in G \). By \( \text{Lim} \) \( [5] \) the family \( J = \{T_s : s \in G\} \) has a common fixed point, that is, \( K \cap F \neq \emptyset \). Let \( q \in K \cap F \). Since \( q \) is a minimiser of \( f \) over \( C \) and since \( X \) is uniformly smooth, it follows that for each \( x \in C \),
\[
0 \leq \lim_{\lambda \to 0^+} \frac{1}{\lambda} \left[ f(q + \lambda(x - q)) - f(q) \right]
\]
\[
= \text{LIM}_n \left( \lim_{\lambda \to 0^+} \frac{1}{\lambda} \left[ \|q_n - q\|^2 + \lambda \langle q_n - q, J(q_n - q) \rangle \right] \right)
\]
\[
= \text{LIM}_n \langle q - x, J(q_n - q) \rangle.
\]
Thus,
\[
\text{LIM}_n \langle x - q, J(q_n - q) \rangle \leq 0, \quad x \in C.
\]
In particular,
\[
(3.4) \quad \text{LIM}_n \langle u - q, J(q_n - q) \rangle \leq 0, \quad x \in C.
\]
On the other hand, by equation (3.1) we have for any \( p \in F \),
\[
z_n - p = (1 - \alpha_n)(T_n z_n - p) + \alpha_n(u - p),
\]
where $\alpha_n = \alpha_{s_n}$ and $T_n = T_{s_n}$. It follows that for $p \in F$,

$$\|z_n - p\|^2 = (1 - \alpha_n)\langle T_n z_n - p, J(z_n - p) \rangle + \alpha_n \langle u - p, J(z_n - p) \rangle \leq (1 - \alpha_n)\|z_n - p\|^2 + \alpha_n \langle u - p, J(z_n - p) \rangle.$$ 

Hence

$$\langle z_n - p, J(z_n - p) \rangle \leq \langle u - p, J(z_n - p) \rangle. \tag{3.5}$$

Combining (3.4) and (3.5) with $p$ replaced with $q$, we get

$$\lim n \|z_n - q\|^2 = 0.$$

So we have a subsequence $(z_{n_j})$ of $(z_n)$ such that $z_{n_j} \to q$. Assume there exists another subsequence $(z_{m_k})$ of $(z_n)$ such that $z_{m_k} \to \tilde{q}$. Then (3.5) implies

$$\|q - \tilde{q}\|^2 \leq \langle u - q, J(q - \tilde{q}) \rangle. \tag{3.6}$$

Similarly we have

$$\|q - \tilde{q}\|^2 \leq \langle u - \tilde{q}, J(q - \tilde{q}) \rangle. \tag{3.7}$$

Adding up (3.6) and (3.7) obtains $q = \tilde{q}$. Therefore $(z_n)$ converges in norm to a point in $F$.

Now define $Q : C \to F$ by

$$Qu := s - \lim_{s \to \infty} z_s.$$

Then $Q$ is a retraction from $C$ to $F$. Moreover, by (3.5) we get for $p \in F$,

$$\|Qu - p\|^2 \leq \langle u - p, J(Qu - p) \rangle \Rightarrow \langle u - Qu, J(p - Qu) \rangle \leq 0, \quad p \in F.$$

Therefore $Q$ is a sunny nonexpansive retraction from $C$ to $F$. \qed

Next we introduce an explicit iterative process to construct the sunny nonexpansive retraction $Q$ from $C$ to $F$.

Let $u \in C$ be arbitrary. Take a sequence $(r_n)$ in $G$ and a sequence $(\alpha_n)$ in the interval $[0, 1]$. Starting with an arbitrary initial $x_0 \in C$ we define a sequence $(x_n)$ recursively by the formula:

$$x_{n+1} := \alpha_n u + (1 - \alpha_n)T_{r_n} x_n, \quad n \geq 0. \tag{3.8}$$

The following is a convergence result for the process (3.8).

**Theorem 3.2.** Let $X$ be a uniformly smooth Banach space. Assume

(i) $\alpha_n \to 0$, $\alpha_n/\alpha_{n+1} \to 1$, and $\sum \alpha_n = \infty$;

(ii) $\lim n \|x_n - p\|^2 = 0$ for all $p \in F$;
(ii) \( r_n \to \infty \);

(iii) \( \mathcal{J} \) is semigroup (that is, \( T_rT_s = T_{r+s} \) for \( r, s \in G \)) and satisfies the uniformly asymptotically regularity condition:

\[
\limsup_{r \to \infty} \sup_{x \in C} \|T_rT_s x - T_r x\| = 0 \quad \text{uniformly in } s \in G,
\]

where \( C \) is any bounded subset of \( C \). Then the sequence \((x_n)\) generated by (3.8) converges in norm to \( Qu \), where \( Q \) is the sunny nonexpansive retraction from \( C \) to \( F \) established in Theorem 3.1.

**Proof:**

1. First prove the sequence \((x_n)\) is bounded. As a matter of fact, for \( p \in F \), we have

\[
\|x_{n+1} - p\| \leq \alpha_n \|u - p\| + (1 - \alpha_n) \|x_n - p\|.
\]

This together with an induction implies that

\[
\|x_n - p\| \leq \max\{\|u - p\|, \|x_0 - p\|\}, \quad \text{for all } n \geq 0.
\]

Thus \((x_n)\) is bounded and it follows that

\[
\|x_{n+1} - T_{r_n}x_n\| = \alpha_n \|u - T_{r_n}x_n\| \to 0.
\]

2. Now prove \(\|x_{n+1} - x_n\| \to 0\). Indeed we have

\[
\|x_{n+1} - x_n\| = \|(\alpha_n - \alpha_{n-1}) (u - T_{r_{n-1}}x_{n-1})
\]

\[
+(1 - \alpha_n)(T_{r_n}x_n - T_{r_n}x_{n-1}) + (1 - \alpha_n)(T_{r_n}x_{n-1} - T_{r_n}x_{n-1})
\]

\[
\leq |\alpha_n - \alpha_{n-1}| \|u - T_{r_{n-1}}x_{n-1}\| + (1 - \alpha_n) \|x_n - x_{n-1}\|
\]

\[
+(1 - \alpha_n) \|T_{r_n}x_{n-1} - T_{r_n}x_{n-1}\|
\]

\[
\leq (1 - \alpha_n) \|x_n - x_{n-1}\| + \beta_n + \alpha_n \gamma_n
\]

where \(\beta_n := \|T_{r_n}x_{n-1} - T_{r_{n-1}}x_{n-1}\|\) and \(\gamma_n := (\alpha_n^{-1})|\alpha_n - \alpha_{n-1}| \|u - T_{r_{n-1}}x_{n-1}\|\). Since \((x_n)\) is bounded, by condition (i), we have \(\gamma_n \to 0\). It is easily seen that condition (iii) implies \(\beta_n \to 0\). Indeed, if \(r_n > r_{n-1}\), since \(\mathcal{J}\) is a semigroup, we have \(\beta_n = \|T_{r_n-r_{n-1}} T_{r_{n-1}}x_{n-1} - T_{r_n}x_{n-1}\| \to 0\) by step 1 and (3.9). Interchanging \(r_n\) and \(r_{n-1}\) if \(r_n < r_{n-1}\) finishes the proof of \(\beta_n \to 0\). Hence by Lemma 2.1 we get \(\|x_{n+1} - x_n\| \to 0\).

3. Next we show for each fixed \(s \in G\), \(\|T_s x_n - x_n\| \to 0\). Indeed (3.10) and step 2 imply that \(\|x_n - T_{r_n}x_n\| \to 0\). Let \(\widetilde{C}\) be any bounded subset of \(C\) which contains the sequence \((x_n)\). It follows that

\[
\|T_s x_n - x_n\| \leq \|T_s x_n - T_{r_n}x_n\| + \|T_{r_n}x_n - x_n\| + \|T_{r_n}x_n - n\|
\]

\[
\leq 2\|x_n - x_n\| + \sup_{x \in \widetilde{C}} \|T_{r_n}x_n - x_n\|.
\]
So condition (iii) implies that \( \|T_s x_n - x_n\| \to 0. \)

4. Now we prove \( \limsup_{n \to \infty} \langle u - q, J(x_n - q) \rangle \leq 0 \), where \( q = Q(u) \). Recall that \( z_s \) satisfies (3.1) and we have shown that \( z_s \xrightarrow{s} q \). Writing

\[
 z_s - x_n = \alpha_s (u - x_n) + (1 - \alpha_s) (T_s z_s - x_n)
\]

and applying Lemma 2.2 we get

\[
 \|z_s - x_n\|^2 \leq (1 - \alpha_s)^2 \|T_s z_s - x_n\|^2 + 2\alpha_s \langle u - x_n, J(z_s - x_n) \rangle
\]

\[
 \leq (1 + \alpha_s^2) \|z_s - x_n\|^2 + M \|x_n - T_s x_n\| + 2\alpha_s \langle u - z_s, J(z_s - x_n) \rangle,
\]

where \( M \) is a constant such that \( \|x_n - T_s x_n\| + 2\|z_s - x_n\| \leq M \) for all \( n, s \in G \). It follows from the last inequality that

\[
 \langle u - z_s, J(x_n - z_s) \rangle \leq \frac{\alpha_s}{2} \|z_s - x_n\|^2 + \frac{M}{2\alpha_s} \|T_s x_n - x_n\|.
\]

By step 3 we find that

\[
 (3.11) \quad \limsup_{n \to \infty} \langle u - z_s, J(x_n - z_s) \rangle \leq O(\alpha_s) \to 0 \quad \text{as} \quad s \to \infty.
\]

Since \( X \) is uniformly smooth, the duality map \( J \) is norm-to-norm uniformly continuous on bounded subsets of \( X \), letting \( s \to \infty \) in (3.11) we obtain \( \limsup_{n \to \infty} \langle u - q, J(x_n - q) \rangle \leq 0 \).

5. Finally we show \( x_n \to q \) in norm. Apply Lemma 2.2 to get

\[
 \|x_{n+1} - q\|^2 = \|(1 - \alpha_n)(T_n x_n - q) + \alpha_n (u - q)\|^2
\]

\[
 \leq (1 - \alpha_n)^2 \|T_n x_n - q\|^2 + 2\alpha_n \langle u - q, J(x_{n+1} - q) \rangle
\]

\[
 \leq (1 - \alpha_n) \|x_n - q\|^2 + 2\alpha_n \langle u - q, J(x_{n+1} - q) \rangle.
\]

Now using condition (i), step 4 and Lemma 2.1 (\( \beta_n \equiv 0 \)) we conclude that \( \|x_n - q\| \to 0. \)

**Remarks.** 1. The case where the family \( \mathcal{J} \) consists of a single nonexpansive mapping was studied by Reich [7]. For the framework of a Hilbert spaces see also Halpern [4], Lions [6] and Wittmann [8].

2. For the case where \( \mathcal{J} \) is a finite family of nonexpansive mappings and \( X \) is a Hilbert space, similar results involving with a periodic iteration process were proved in Lions [6] and Bauschke [1].

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