CONSTRUCTION OF SUNNY NONEXPANSIVE RETRACTIONS IN BANACH SPACES

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Let $J$ be a commutative family of nonexpansive self-mappings of a closed convex subset $C$ of a uniformly smooth Banach space $X$ such that the set of common fixed points is nonempty. It is shown that if a certain regularity condition is satisfied, then the sunny nonexpansive retraction from $C$ to $F$ can be constructed in an iterative way.

1. INTRODUCTION

Let $C$ be a nonempty closed convex subset of a Banach space $X$ and let $D$ be a nonempty subset of $C$. A retraction from $C$ to $D$ is a mapping $Q : C \rightarrow D$ such that $Qx = x$ for $x \in D$. A retraction $Q$ from $C$ to $D$ is nonexpansive if $Q$ is nonexpansive (that is, $\|Qx - Qy\| \leq \|x - y\|$ for $x, y \in C$). A retraction $Q$ from $C$ to $D$ is sunny if $Q$ satisfies the property:

$$Q(Qx + t(x - Qx)) = Qx$$

for $x \in C$ and $t > 0$ whenever $Qx + t(x - Qx) \in C$.

A retraction $Q$ from $C$ to $D$ is sunny nonexpansive if $Q$ is both sunny and nonexpansive. It is known ([2]) that in a smooth Banach space $X$, a retraction $Q$ from $C$ to $D$ is a sunny nonexpansive retraction from $C$ to $F$ if and only if the following inequality holds:

$$(\text{1.1}) \langle x - Qx, J(y - Qx) \rangle \leq 0, \quad x \in C, \quad y \in D,$$

where $J : X \rightarrow X^*$ is the duality map defined by

$$Jx := \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad x \in X.$$

Hence a sunny nonexpansive retraction must be unique (if it exists).

If $C$ is a nonempty closed convex subset of a Hilbert space $H$, then the nearest point projection $P_C$ from $H$ to $C$ is the sunny nonexpansive retraction. This however is not true for Banach spaces since nonexpansivity of projections $P_C$ characterises Hilbert spaces.

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On the other hand, one sees by (1.1) that a sunny nonexpansive retraction can play a similar role in a Banach space as a projection does in a Hilbert space. So an interesting problem is this: In what kind of Banach spaces, does a sunny nonexpansive retraction exist? If it does exist, how can one find it? It is known ([2]) if \( C \) is a closed convex subset of a uniformly smooth Banach space \( X \) and there is a nonexpansive retraction from \( X \) to \( C \), then there exists a sunny nonexpansive retraction from \( X \) to \( C \). But Bruck's proof is not constructive.

The purpose of the present paper is to construct sunny nonexpansive retractions in a uniformly smooth Banach space in an iterative way. More precisely, we show that if \( F \) is the nonempty common fixed point set of a commutative family of nonexpansive self-mappings of a closed convex subset \( C \) of a uniformly smooth Banach space \( X \) satisfying certain regularity condition, then we are able to construct a sequence that converges to the sunny nonexpansive retraction \( Q \) of \( C \) to \( F \). This extends a result of Reich [7] where the case of a single nonexpansive mapping is dealt with.

2. Two Lemmas

**Lemma 2.1.** Let \((s_n)\) be a sequence of nonnegative numbers satisfying the condition:

\[
(2.1) \quad s_{n+1} \leq (1 - \alpha_n)(s_n + \beta_n) + \alpha_n \gamma_n, \quad n \geq 0,
\]

where \((\alpha_n), (\beta_n), (\gamma_n)\) are sequences of real numbers satisfying

(i) \( (\alpha_n) \subset [0,1], \lim_{n \to \infty} \alpha_n = 0, \) and \( \sum_{n=0}^{\infty} \alpha_n = \infty, \) or equivalently, \( \prod_{n=0}^{\infty}(1 - \alpha_n) = 0; \)

(ii) \( \limsup_{n \to \infty} \beta_n \leq 0; \)

(iii) \( \limsup_{n \to \infty} \gamma_n \leq 0. \)

Then \( \lim_{n \to \infty} s_n = 0. \)

**Proof:** For any \( \varepsilon > 0 \), let \( N \geq 1 \) be an integer big enough so that

\[
(2.2) \quad \beta_n < \varepsilon/2, \quad \gamma_n < \varepsilon, \quad \alpha_n < 1/2, \quad n \geq N.
\]

It follows from (2.1) and (2.2) that, for \( n > N \),

\[
\begin{align*}
  s_{n+1} & \leq (1 - \alpha_n)(s_n + \beta_n) + \varepsilon \alpha_n \\
  & \leq (1 - \alpha_n)(1 - \alpha_{n-1})(s_{n-1} + \beta_{n-1}) + \varepsilon \left( 1 - (1 - \alpha_n)(1 - \alpha_{n-1}) + \frac{1}{2}(1 - \alpha_n) \right) \\
  & \leq (1 - \alpha_n)(1 - \alpha_{n-1})(s_{n-1} + \beta_{n-1}) + \varepsilon \left( 1 - \left( \frac{1}{2} - \alpha_n \right) \left( \frac{1}{2} - \alpha_{n-1} \right) \right).
\end{align*}
\]
Hence by induction we obtain
\[ s_{n+1} \leq \prod_{j=1}^{N} (1 - \alpha_j)(\beta_N + \epsilon) + \epsilon \prod_{j=N}^{n} (1 - \tilde{\alpha}_j), \quad n > N, \]
where \( \tilde{\alpha}_j := 1/2 + \alpha_j < 1, \ j \geq N. \) By condition (i) we deduce, after taking the limsup as \( n \to \infty \) in the last inequality, that \( \limsup_{n \to \infty} s_{n+1} \leq \epsilon. \)

**Lemma 2.2.** (The Subdifferential Inequality, see [3].) In a Banach space \( X \) there holds the inequality:
\[ \|x + y\|^2 \leq \|x\|^2 + 2\langle y, j \rangle, \quad x, y \in X, \ j \in J(x + y), \]
where \( J \) is the duality map of \( X \).

### 3. Iterative Processes

Let \( G \) be an unbounded subset of \( \mathbb{R}^+ \) such that \( s + t \in G \) whenever \( s, t \in G \). (Often \( G = \mathbb{N} \), the set of nonnegative integers or \( \mathbb{R}^+ \).) Let \( X \) be a uniformly smooth Banach space, \( C \) a nonempty closed convex subset of \( X \), and \( J = \{ T_s : s \in G \} \) a commutative family of nonexpansive self-mappings of \( C \). Denote by \( F \) the set of common fixed point of \( J \), that is, \( F = \{ x \in C : T_s x = x, \ s \in G \} \). Throughout this section we always assume that \( F \) is nonempty. Our purpose is to construct the sunny nonexpansive retraction \( Q \) from \( C \) to \( F \). We shall introduce two iterative processes to construct \( Q \). The first one is implicit, while the second one is explicit. But before introducing the iterative processes, we recall that the uniform smoothness of \( X \) is equivalent to the following statement:
\[ \lim_{\lambda \to 0^+} \frac{\|x + \lambda y\|^2 - \|x\|^2}{\lambda} = 2\langle y, Jx \rangle \quad \text{uniformly for bounded} \ x, y \in X. \]

Let \( u \in C \) be given arbitrarily and let \( (\alpha_s)_{s \in G} \) be a net in the interval \((0, 1)\) such that \( \lim_{s \to \infty} \alpha_s = 0 \). By Banach's contraction principle, for each \( s \in G \), we have a unique point \( z_s \in C \) satisfying the equation
\[ z_s = \alpha_s u + (1 - \alpha_s)T_s z_s. \]

**Theorem 3.1.** Let \( X \) be a uniformly smooth Banach space. Assume \( J \) is uniformly asymptotically regular on bounded subsets of \( C \); that is, for each bounded subset \( \tilde{C} \) of \( C \) and each \( r \in G \),
\[ \lim_{s \to \infty} \sup_{x \in \tilde{C}} \|T_r T_s x - T_s x\| = 0. \]
Then the net \( (z_s) \) defined in (3.1) converges in norm and the limit defines the sunny nonexpansive retraction \( Q \) from \( C \) to \( F \).
PROOF: First we observe that \((z_s)\) is bounded. Indeed, for \(p \in F\), we have

\[
\|z_s - p\| \leq \alpha_s \|u - p\| + (1 - \alpha_s)\|Tsz_s - p\| \leq \alpha_s \|u - p\| + (1 - \alpha_s)\|z_s - p\|.
\]

This implies that \(\|z_s - p\| \leq \|u - p\|\) and \((z_s)\) is bounded. Thus \(\|z_s - Tsz_s\| = \alpha_s \|u - Tsz_s\| \to 0\) \((s \to \infty)\). Let \((s_n)\) be a subsequence of \(G\) such that \(\lim n s_n = \infty\). Next we define a function \(f\) on \(C\) by

\[
f(x) := \text{LIM}_n \|z_n - x\|^2, \quad x \in C,
\]

where \(\text{LIM}\) is a Banach limit and \(z_n := z_{s_n}\). We have for each \(r \in G\),

\[
f(T_r x) = \text{LIM}_n \|z_n - T_r x\|^2 \leq \text{LIM}_n \|T_sT_n z_n - T_r x\|^2 \leq \text{LIM}_n \|T_s z_n - x\|^2
\]

using (3.2). Therefore,

\[
(3.3) \quad f(T_r x) \leq f(x), \quad r \in G, \ x \in C.
\]

Let

\[
K := \{x \in C : f(x) = \min_C f\}.
\]

Then it can be seen that \(K\) is a closed bounded convex nonempty subset of \(C\). By (3.3) we see that \(K\) is invariant under each \(T_r\); that is, \(T_r(K) \subset K, \ r \in G\). By \(\text{LIM}\) \([5]\) the family \(\mathcal{J} = \{T_s : s \in G\}\) has a common fixed point, that is, \(K \cap F \neq \emptyset\). Let \(q \in K \cap F\).

Since \(q\) is a minimiser of \(f\) over \(C\) and since \(X\) is uniformly smooth, it follows that for each \(x \in C\),

\[
0 \leq \lim_{\lambda \to 0^+} \frac{1}{\lambda} \left[ f(q + \lambda(x - q)) - f(q) \right] = \text{LIM}_n \left( \lim_{\lambda \to 0^+} \frac{1}{\lambda} \left[ \|z_n - q\|^2 - \|z_n - q\|^2 \right] \right) = \text{LIM}_n \langle q - x, J(z_n - q) \rangle.
\]

Thus,

\[
\text{LIM}_n \langle x - q, J(z_n - q) \rangle \leq 0, \quad x \in C.
\]

In particular,

\[
(3.4) \quad \text{LIM}_n \langle u - q, J(z_n - q) \rangle \leq 0, \quad x \in C.
\]

On the other hand, by equation (3.1) we have for any \(p \in F\),

\[
z_n - p = (1 - \alpha_n)(T_n z_n - p) + \alpha_n(u - p),
\]
where $\alpha_n = \alpha_{s_n}$ and $T_n = T_{s_n}$. It follows that for $p \in F$,
\begin{align*}
\|z_n - p\|^2 &= (1 - \alpha_n)\langle T_n z_n - p, J(z_n - p) \rangle + \alpha_n \langle u - p, J(z_n - p) \rangle \\
&\leq (1 - \alpha_n)\|z_n - p\|^2 + \alpha_n \langle u - p, J(z_n - p) \rangle.
\end{align*}
Hence
\begin{equation}
\|z_n - p\|^2 \leq \langle u - p, J(z_n - p) \rangle.
\end{equation}
Combining (3.4) and (3.5) with $p$ replaced with $q$, we get
\begin{equation}
\lim_{n \to \infty} \|z_n - q\|^2 \leq 0.
\end{equation}
So we have a subsequence $(z_{n_j})$ of $(z_n)$ such that $z_{n_j} \to q$. Assume there exists another subsequence $(z_{m_k})$ of $(z_n)$ such that $z_{m_k} \to \tilde{q}$. Then (3.5) implies
\begin{equation}
\|q - \tilde{q}\|^2 \leq \langle u - q, J(q - \tilde{q}) \rangle.
\end{equation}
Similarly we have
\begin{equation}
\|\tilde{q} - q\|^2 \leq \langle u - q, J(q - \tilde{q}) \rangle.
\end{equation}
Adding up (3.6) and (3.7) obtains $q = \tilde{q}$. Therefore $(z_n)$ converges in norm to a point in $F$.

Now define $Q : C \to F$ by
\begin{equation}
Qu := \lim_{n \to \infty} z_n.
\end{equation}
Then $Q$ is a retraction from $C$ to $F$. Moreover, by (3.5) we get for $p \in F$,
\begin{equation}
\|Q u - p\|^2 \leq \langle u - p, J(Q u - p) \rangle \Rightarrow \langle u - Q u, J(p - Q u) \rangle \leq 0, \quad p \in F.
\end{equation}
Therefore $Q$ is a sunny nonexpansive retraction from $C$ to $F$.

Next we introduce an explicit iterative process to construct the sunny nonexpansive retraction $Q$ from $C$ to $F$.

Let $u \in C$ be arbitrary. Take a sequence $(r_n)$ in $G$ and a sequence $(\alpha_n)$ in the interval $[0,1]$. Starting with an arbitrary initial $x_0 \in C$ we define a sequence $(x_n)$ recursively by the formula:
\begin{equation}
x_{n+1} := \alpha_n u + (1 - \alpha_n)T_{r_n}x_n, \quad n \geq 0.
\end{equation}
The following is a convergence result for the process (3.8).

**Theorem 3.2.** Let $X$ be a uniformly smooth Banach space. Assume
\begin{enumerate}
\item $\alpha_n \to 0$, $\alpha_n/\alpha_{n+1} \to 1$, and $\sum_n \alpha_n = \infty$;
\end{enumerate}
(ii) \( r_n \to \infty; \)

(iii) \( \mathcal{J} \) is semigroup (that is, \( T_r T_s = T_{r+s} \) for \( r, s \in G \)) and satisfies the uniformly asymptotically regularity condition:

\[
\limsup_{r \to \infty, x \in \bar{C}} \|T_r T_r x - T_r x\| = 0 \quad \text{uniformly in } s \in G,
\]

where \( \bar{C} \) is any bounded subset of \( C \). Then the sequence \( (x_n) \) generated by (3.8) converges in norm to \( Qu \), where \( Q \) is the sunny nonexpansive retraction from \( C \) to \( F \) established in Theorem 3.1.

**Proof:** 1. First prove the sequence \( (x_n) \) is bounded. As a matter of fact, for \( p \in F \), we have

\[
\|x_{n+1} - p\| \leq \alpha_n \|u - p\| + (1 - \alpha_n) \|x_n - p\|.
\]

This together with an induction implies that

\[
\|x_n - p\| \leq \max\{\|u - p\|, \|x_0 - p\|\}, \quad \text{for all } n \geq 0.
\]

Thus \( (x_n) \) is bounded and it follows that

\[
\|x_{n+1} - T_{r_n} x_n\| = \alpha_n \|u - T_{r_n} x_n\| \to 0.
\]

2. Now prove \( \|x_{n+1} - x_n\| \to 0 \). Indeed we have

\[
\|x_{n+1} - x_n\| = \|(\alpha_n - \alpha_{n-1})(u - T_{r_{n-1}} x_{n-1})
\]

\[
+ (1 - \alpha_n)(T_{r_n} x_n - T_{r_n} x_{n-1}) + (1 - \alpha_n)(T_{r_n} x_{n-1} - T_{r_n} x_{n-1})\|
\]

\[
\leq |\alpha_n - \alpha_{n-1}| \|u - T_{r_n-1} x_{n-1}\| + (1 - \alpha_n) \|x_n - x_{n-1}\|
\]

\[
+ (1 - \alpha_n) \|T_{r_n} x_{n-1} - T_{r_n-1} x_{n-1}\|
\]

\[
\leq (1 - \alpha_n)\|x_n - x_{n-1}\| + \beta_n + \gamma_n,
\]

where \( \beta_n := \|T_{r_n} x_{n-1} - T_{r_n-1} x_{n-1}\| \) and \( \gamma_n := (\alpha_n^{-1}) |\alpha_n - \alpha_{n-1}| \|u - T_{r_n-1} x_{n-1}\| \). Since \( (x_n) \) is bounded, by condition (i), we have \( \gamma_n \to 0 \). It is easily seen that condition (iii) implies \( \beta_n \to 0 \). Indeed, if \( r_n > r_{n-1} \), since \( \mathcal{J} \) is a semigroup, we have \( \beta_n \to 0 \) by step 1 and (3.9). Interchanging \( r_n \) and \( r_{n-1} \) if \( r_n < r_{n-1} \) finishes the proof of \( \beta_n \to 0 \). Hence by Lemma 2.1 we get \( \|x_{n+1} - x_n\| \to 0 \).

3. Next we show for each fixed \( s \in G \), \( \|T_s x_n - x_n\| \to 0 \). Indeed (3.10) and step 2 imply that \( \|x_n - T_{r_n} x_n\| \to 0 \). Let \( \bar{C} \) be any bounded subset of \( C \) which contains the sequence \( (x_n) \). It follows that

\[
\|T_s x_n - x_n\| \leq \|T_s x_n - T_s T_{r_n} x_n\| + \|T_s T_{r_n} x_n - T_{r_n} x_n\| + \|T_{r_n} x_n - x_n\|
\]

\[
\leq 2\|x_n - T_{r_n} x_n\| + \sup_{x \in \bar{C}} \|T_s T_{r_n} x - T_{r_n} x\|.
\]
So condition (iii) implies that \(\|T_s x_n - x_n\| \to 0\).

4. Now we prove \(\limsup_{n \to \infty} \langle u - q, J(x_n - q) \rangle \leq 0\), where \(q = Q(u)\). Recall that \(z_s\) satisfies (3.1) and we have shown that \(z_s \to q\). Writing

\[ z_s - x_n = \alpha_s (u - x_n) + (1 - \alpha_s) (T_s z_s - x_n) \]

and applying Lemma 2.2 we get

\[
\|z_s - x_n\|^2 \leq (1 - \alpha_s)^2 \|T_s z_s - x_n\|^2 + 2\alpha_s \langle u - x_n, J(z_s - x_n) \rangle \\
\leq (1 + \alpha_s^2) \|z_s - x_n\|^2 + M \|x_n - T_s x_n\| + 2\alpha_s \langle u - z_s, J(z_s - x_n) \rangle,
\]

where \(M\) is a constant such that \(\|x_n - T_s x_n\| + 2\|z_s - x_n\| \leq M\) for all \(n, s \in G\). It follows from the last inequality that

\[
\langle u - z_s, J(x_n - z_s) \rangle \leq \frac{\alpha_s}{2} \|z_s - x_n\|^2 + \frac{M}{2\alpha_s} \|T_s x_n - x_n\|.
\]

By step 3 we find that

\[
(3.11) \limsup_{n \to \infty} \langle u - z_s, J(x_n - z_s) \rangle \leq O(\alpha_s) \to 0 \quad \text{as } s \to \infty.
\]

Since \(X\) is uniformly smooth, the duality map \(J\) is norm-to-norm uniformly continuous on bounded subsets of \(X\), letting \(s \to \infty\) in (3.11) we obtain \(\limsup_{n \to \infty} \langle u - q, J(x_n - q) \rangle \leq 0\).

5. Finally we show \(x_n \to q\) in norm. Apply Lemma 2.2 to get

\[
\|x_{n+1} - q\|^2 = \|(1 - \alpha_n) (T_n x_n - q) + \alpha_n (u - q)\|^2 \\
\leq (1 - \alpha_n)^2 \|T_n x_n - q\|^2 + 2\alpha_n \langle u - q, J(x_{n+1} - q) \rangle \\
\leq (1 - \alpha_n) \|x_n - q\|^2 + 2\alpha_n \langle u - q, J(x_{n+1} - q) \rangle.
\]

Now using condition (i), step 4 and Lemma 2.1 \((\beta_n \equiv 0)\) we conclude that \(\|x_n - q\| \to 0\). \(\square\)

REMARKS. 1. The case where the family \(J\) consists of a single nonexpansive mapping was studied by Reich [7]. For the framework of a Hilbert spaces see also Halpern [4], Lions [6] and Wittmann [8].

2. For the case where \(J\) is a finite family of nonexpansive mappings and \(X\) is a Hilbert space, similar results involving with a periodic iteration process were proved in Lions [6] and Bauschke [1].

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