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A CLASS OF QUASI-NONEXPANSIVE MULTI-VALUED MAPS

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1. **Introduction.** Let (X, d) be a (nonempty) metric space. $bc(X)$ will denote the family of all nonempty bounded closed subsets of X endowed with the Hausdorff metric D induced by d [2, pp. 205]. Let f be a map of X into $bc(X)$. f is nonexpansive at a point x in X if $D(f(x), f(y)) \leq d(x, y)$ for all y in X. f is quasi-nonexpansive if the fixed point set $F_f = \{x \in X : x \in f(x)\}$ is nonempty and f is nonexpansive at each point in F_f. In this paper, we are interested in the following class of maps: f is Kannan if $D(f(x), f(y)) \leq \frac{1}{2}(d(x, f(x)) + d(y, f(y)))$ for all x, y in X. $(d(x, A) = \inf\{d(x, y) : y \in A\}, A \subseteq X, x \in X)$. It is easy to show that every Kannan map is nonexpansive at each of its fixed points (provided they exist). We refer to early history and results in this direction to [4] and [5]. In this paper, some fixed point theorems for the Kannan maps f are obtained by studying the nature of the function $d(x, f(x))$. Now let X be a weakly compact convex subset of a Banach space B and f be a Kannan map of X into the family $wcc(X)$ of all nonempty weakly compact convex subsets of X. If every f-invariant closed convex subset H of X is a convex body in itself (i.e. H has nonempty interior in the smallest flat which contains H), it is shown that f has a fixed point. Hence (i) f has a fixed point if each $f(x)$ is a convex body. (ii) f has a fixed point if B is finite dimensional. When B is one-dimensional, (ii) was proved in [5] by a different method.

2. **Kannan maps in a metric space.** First of all we note that if (X, d) is a metric space, $f: X \to bc(X)$ is a Kannan map, and a is a fixed point of f, then $f(a)$ is the fixed point set of f. Indeed, if b is also a fixed point of f, then $D(f(a), f(b)) \leq \frac{1}{2}(d(a, f(a)) + d(b, f(b))) = 0$ so that $b \in f(b) = f(a)$. On the other hand, if $x \in f(a)$, then $d(x, f(x)) \leq D(f(a), f(x)) \leq \frac{1}{2} d(x, f(x))$ so that $d(x, f(x)) = 0$ and hence $x \in f(x)$.

THEOREM 1. *Let (X, d) be a (nonempty) complete metric space and $f: X \to c(X)$ be a Kannan map, where $c(X)$ is the family of all nonempty compact subsets of X. If $\inf\{d(x, f(x)) : x \in X\} = 0$, then f has a fixed point say a, and for any sequence $\{x_n\}$ in X with $\{d(x_n, f(x_n))\}$ converges to 0, (a) a subsequence of $\{x_n\}$ converges to a fixed point of f, (b) each cluster point of $\{x_n\}$ is a fixed point of f and (c) $\{f(x_n)\}$ converges to $f(a)$.*

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Proof. Let $\{x_n\}$ be any sequence in X with $\{d(x_n, f(x_n))\}$ converging to 0. (Such a sequence exists since $\inf\{d(x, f(x)):x \in X\}=0$.) Since f is Kannan $\{f(x_n)\}$ is Cauchy in $(c(X), D)$. Since (X, d) is complete, it is well-known that $(c(X), D)$ is complete. Thus $\{f(x_n)\}$ converges to some A in $c(X)$. For each n, there exist $\bar{x}_n$ in $f(x_n)$, $\bar{a}_n$ in A such that $d(x_n, \bar{x}_n)=d(x_n, f(x_n))$ and $d(\bar{x}_n, \bar{a}_n)=d(\bar{x}_n, A)$. Since A is compact, a subsequence $\{\bar{a}_{h(n)}\}$ of $\{\bar{a}_n\}$ converges to some point a in A. Since for each n, $d(\bar{x}_n, \bar{a}_n)\leq D(f(x_n), A)$ which converges to 0, $\{\bar{x}_{h(n)}\}$ converges to a. Since

$$\begin{aligned} d(a, f(a)) &= \lim_{n\to\infty} d(\bar{x}_{h(n)}, f(a)) \\ &\leq \liminf_{n\to\infty} D(f(x_{h(n)}), f(a)) \\ &\leq \liminf_{n\to\infty}(\tfrac{1}{2}d(x_{h(n)}, f(x_{h(n)}))+\tfrac{1}{2}d(a, f(a))) \\ &= \tfrac{1}{2}d(a, f(a)), \end{aligned}$$

we must have $d(a,f(a))=0$ so that $a \in f(a)$. Since $d(x_n\, \bar{x}_n)=d(x_n, f(x_n))\to 0$, $\{x_{h(n)}\}$ also converges to a. This proves (a). Since $D(f(x_{h(n)}), f(a))\leq \frac{1}{2}d(x_{h(n)}, f(x_{h(n)}))\to 0$, $f(x_{h(n)})\to f(a)$. Thus $A=f(a)$ and hence $f(x_n)\to f(a)$. This proves (c). Finally, if x is a cluster point of $\{x_n\}$, then x is also a cluster point of $\{\bar{a}_n\}$ as $d(x_n, \bar{x}_n)\to 0$ and $d(\bar{x}_n, \bar{a}_n)\to 0$. Thus $x \in A=f(a)$ so that x is also a fixed point by the preceding remark. This proves (b).

3. **Kannan maps in a Banach space.** Let $(B, \|\ \|)$ be a Banach space and d be the metric on B induced by the norm $\|\ \|$ on B. For simplicity, B is assumed to be over the real field.

THEOREM 2. *Let K be a nonempty weakly compact subset of B and f be a Kannan map of K into the family $wc(K)$ of all nonempty weakly compact subsets of K. Then* (a) *there exists x_0 in K such that $d(x_0, f(x_0))=\inf\{d(x, f(x)):x \in K\}$, i.e. $x\mapsto d(x, f(x))$ attains its infimum, say r_0, on K.* (b) *$K_0=\{x \in K:d(x, f(x))=r_0\}$ is f-invariant.*

Proof. (a) For each $r\geq 0$, let $K_r=\{x \in K:d(x, f(x))\leq r\}$. Since K is bounded, the set $I=\{r\geq 0:K_r\neq\varnothing\}$ is nonempty. For each $r \in I$, let H_r be the weak closure $wcl(f(K_r))$ of $f(K_r)(=\bigcup_{x\in K_r} f(x))$. Then $\{H_r:r \in I\}$ is a family of weakly compact subsets of K which has the finite intersection property and therefore has nonempty intersection. It remains to show that $H_r\subset K_r$ for each $r \in I$. Let $r \in I$ and $x \in H_r$. Then there exists a net $\{x_\alpha\}_{\alpha\in\Gamma}$ in $f(K_r)$ which converges weakly to x. Thus for each $\alpha \in \Gamma$, $x_\alpha \in f(y_\alpha)$ for some y_α in K_r and $d(x_\alpha, z_\alpha)=d(x_\alpha, f(x))$ for some z_α in $f(x)$ Since $f(x)$ is weakly compact, by passing to a subnet, we may assume without

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loss of generality that $\{z_\alpha\}_{\alpha\in\Gamma}$ converges weakly to some point z in $f(x)$. Thus

$$\begin{aligned} d(x, f(x)) &\le d(x, z) \\ &\le \liminf_\alpha d(x_\alpha, z_\alpha) \\ &= \liminf_\alpha d(x_\alpha, f(x)) \\ &\le \liminf_\alpha D(f(y_\alpha), f(x)) \\ &\le \liminf_\alpha \tfrac{1}{2}(d(y_\alpha, f(y_\alpha))+d(x, f(x))) \\ &\le \tfrac{1}{2}r+\tfrac{1}{2}d(x, f(x)). \end{aligned}$$

Hence $d(x,f(x))\le r$ and therefore $x\in K_r$. Thus $H_r\subset K_r$ for each $r\in I$.

(b) From the proof of (a), $H_{r_0}\subset K_{r_0}$. Thus $f(H_{r_0})\subset f(K_{r_0})\subset H_{r_0}$.

The following result follows easily from Theorem 2.

THEOREM 3. *Let K be a nonempty weakly compact subset of B and f be a Kannan map of K into $wc(K)$. Then the following are equivalent:* (a) *f has a fixed point;* (b) $\inf\{d(x,f(x)):x\in K\}=0$; (c) *For each $x\in K$, if $d(x,f(x))>0$, then $d(y,f(y))<d(x,f(x))$ for some y in K.*

Theorems 2 and 3 were obtained in [5] for the case when K and each $f(x)$, $x\in K$, are weakly compact convex subsets of B.

For our next fixed point theorem, we need the following result which is of interest in itself. Let A be a nonempty subset of B. For each x in B, $d(x, A)$ is called the modulus of x with respect to A; thus $\|x\|$ is the modulus of x with respect to $\{0\}$.

THEOREM 4. (*Maximum modulus principle*). *Let A be a nonempty subset of a real normed space X and K be a nonempty weakly compact convex subset of X. Then the modulus function $g:x\mapsto d(x, K)$ on A does not attain its maximum value at the interior* $\operatorname{int}(A)$ *of A unless it is identically zero on A.*

Proof. Suppose that g is not identically zero on A. Then $r=\sup\{g(x):x\in A\}>0$. We need only to prove that $g(y)<r$ for each y in $\operatorname{int}(A)$. Suppose the contrary that there exists y in $\operatorname{int}(A)$ such that $g(y)=r$. Since K is weakly compact, $d(y, c)=r$ for some c in K. Let $B_r(y)=\{x\in B:\|x-y\|\le r\}$. Since $\operatorname{int}(B_r(y))$ and K are convex and $\operatorname{int}(B_r(y))\cap K=\varnothing$, by Hahn-Banach separation theorem, there is a non-zero continuous linear functional f on X such that $\sup\{f(x):x\in B_r(y)\}\le\inf\{f(x):x\in K\}$. Since $c\in B_r(y)\cap K$, $\sup\{f(x):x\in B_r(y)\}=f(c)=\inf\{f(x):x\in K\}$. As $y\in\operatorname{int}(A)$, $B_\delta(y)\subset A$ for some $\delta\in(0,r)$. Let $z=y+\delta(y-c)/r$. Then $z\in A$ and $\|y-z\|=\delta$. We shall show that

$$d(z, K)\ge r+\left(\frac{\delta}{r}\right)|f(y)-f(c)|/\|f\| \tag{1}$$

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Let $b \in K$. Then $f(b) \geq r > f(z)$. Let $\lambda = (f(b) - f(c))/(f(b) - f(z))$, $\beta = \delta/(r+\delta)$, $u = \lambda z + (1-\lambda)b$, $v = \beta z + (1-\beta)u$ and $w = v + y - a$. It can be checked that

$$f(u) = f(c) = f(w) \tag{2}$$

and

$$|f(u) - f(v)| = (\delta/r)\,|f(y) - f(c)| \tag{3}$$

From (2), $\|y-w\| \geq r$. Since b, z, u, v are collinear,

$$\begin{aligned} \|z-b\| &= \|z-v\| + \|v-u\| + \|u-b\| \\ &= \|y-w\| + \|v-u\| + \|u-b\| \\ &\geq r + |f(v) - f(u)|/\|f\| + \|u-b\| \\ &= r + (\delta/r)\,|f(y) - f(c)|/\|f\| + \|u-b\| \quad \text{(by (3))} \end{aligned}$$

so that $d(z, K) \geq r + (\delta/r)|f(y) - f(c)|/\|f\|$. Since $f(y) < f(c)$, $r \geq d(z, K) \geq r + (\delta/r)|f(y) - f(c)|/\|f\| > r$, which is a contradiction. Therefore $g(y) < r$ for each y in $\operatorname{int}(A)$.

Let A be a convex subset of a normed space. Then A is a convex body in itself if A has non-empty interior when it is considered as a topological subspace of the closure of the flat $\{\sum_{i=1}^{n} t_i x_i : \sum_{i=1}^{n} t_i = 1,\ t_i\text{'s are real},\ x_i \in A,\ n = 1, 2, \ldots\}$ spanned by A.

Before we prove the next theorem, we need the following result whose proof can be found in [5, Theorem 5].

THEOREM. *Let K be a nonempty weakly compact convex subset of a Banach space B and T be a Kannan map of K into $wcc(K)$. Then*

(a) *There exists x_0 in K such that $d(x_0, T(x_0)) \leq d(x, T(x))$ for all x in K, i.e. the map $x \mapsto d(x, T(x))$ on K attains its infimum r_0.*

(b) *The set $A = \{x \in K : d(x, T(x)) = r_0\}$ is T-invariant, i.e. $T(A) = \bigcup_{x \in A} T(x) \subseteq A$.*

(c) *A contains a nonempty T-invariant closed convex subset of K.*

THEOREM 5. *Let K be a nonempty weakly compact convex subset of B and f be a Kannan map of K into $wcc(K)$. Suppose that each f-invariant closed convex subset of K is a convex body in itself. Then f has a fixed point.*

Proof. By Zorn's Lemma and weak compactness of K, there exists a minimal nonempty closed convex subset H of K which is f-invariant. By Theorem 2 (or the above Theorem, part (a)), there exists x_0 in H such that $d(x_0, f(x_0)) = \inf\{d(x, f(x)) : x \in H\}$. Let $r = d(x_0, f(x_0))$. Suppose $r > 0$. By the above Theorem, part (c), $H_1 = \{x \in H : d(x, f(x)) = r\}$ contains a closed convex and f-invariant subset. Thus $H_1 = H$, by the minimality of H. By hypothesis, we may assume that H has nonempty interior. Note that the closed convex hull $\overline{Co}(f(H))$ of $f(H)$ is also f-invariant, so that $H = \overline{Co}(f(H))$. Let $x_1, x_2 \in H$. We shall first prove that

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$d(x_1, f(x_2)) \leq r$. Let $\varepsilon > 0$, then there exists z in $Co(f(H))$ such that $\|x_1 - z\| < \varepsilon$. Thus $z = \sum_{i=1}^n t_i z_i$ for some z_i in $f(H)$ and t_i in $[0, 1]$ with $\sum_{i=1}^n t_i = 1$. But then $z_i \in f(h_i)$ for some $h_i \in H$, $i = 1, \ldots, n$. Now

$$\begin{aligned} d(x_1, f(x_2)) &\leq d(x_1, z) + d(z, f(x_2)) \\ &< \varepsilon + d\left(\sum_{i=1}^n t_i z_i, f(x_2)\right) \\ &\leq \varepsilon + \sum_{i=1}^n t_i d(z_i, f(x_2)) \\ &\leq \varepsilon + \sum_{i=1}^n t_i D(f(h_i), f(x_2)) \\ &\leq \varepsilon + \sum_{i=1}^n t_i \tfrac{1}{2}(d(h_i, f(h_i)) + d(x_2, f(x_2))) \\ &= \varepsilon + r. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $d(x_1, f(x_2)) \leq r$. Thus the modulus function $x \to d(x, f(x_2))$ on H attains its supremum at x_2. By the maximum modulus principle, x_2 is not in the interior of H. Since x_2 in H is arbitrary, H has empty interior, which is a contradiction. Hence we must have $r = 0$ so that x_0 is a fixed point of f.

COROLLARY 1. *Let K be a weakly compact convex body in B (whcih is necessarily reflexive) and f be a Kannan map of K into the family of all weakly compact convex bodies in B which are contained in K. Then f has a fixed point.*

Since every nonempty closed convex subset of a finite dimensional Banach space is a convex body in itself, we have the following.

COROLLARY 2. *If B is finite dimensional, K is a nonempty compact convex subset of B and $f: K \to cc(K)$ is a Kannan map, then f has a fixed point.*

If B is infinite dimensional, a bounded closed convex subset of B may not be a convex body in itself. In [3], the author made a stronger assertion that (*) "for any Banach space B, every closed convex set K in B is a convex body in the closure of its linear span" [3, pp. 79 and pp. 93] and used this assertion to prove the important theorem for monotone operators.

THEOREM. *Let C be a closed convex subset of a reflexive Banach space X and $T: C \to X^*$ a monotone hemicontinuous and coercive mapping. Then for each u_0 in X^*, there exists an x_0 in C such that $(T(x_0) - u_0)(x - x_0) \geq 0$, for all x in C.*

F. E. Browder [1] proved the above theorem for the case when $0 \in C$. The following counterexample to the above theorem can be found in [6].

EXAMPLE. Let X be the two-dimensional Euclidean space. Let C be the closed convex subset $\{(x, y) \in X : x + y = 1\}$ of X. Let T be the map on C such that

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$T((x,y))=(x^2, x^2)$ for all $(x, y) \in C$. Then T satisfies the hypothesis but not the conclusion of the above theorem.

It can be easily seen that (*) is true if we make the restriction that $0 \in K$ and B is finite dimensional. In fact, every bounded closed convex subset K of a Banach space B is a convex body in itself if and only if B is finite dimensional. This fact limits the generality of Corollary 2 at the present stage.

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