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ON KNOPP'S INEQUALITY FOR CONVEX FUNCTIONS

BY

J. E. PEČARIĆ AND P. R. BEESACK*

ABSTRACT. Knopp's inequality for convex functions ϕ on an interval $I = [m, M]$ states that

$$\int_0^1 \phi(g(t))dt - \phi\left(\int_0^1 g(t)dt\right) \leqq H(m, M; \phi)$$

for an explicit functional H, and all integrable $g : [0, 1] \to I$. In this paper we give results of this kind in which the integral operator, $\int$, is replaced by a general isotonic linear functional.

1. **Introduction**. In 1935, K. Knopp [5, Satz 1] proved a result which can be stated in the following equivalent form (see also, for example, T. Popoviciu [9, p. 34]):

Let ϕ be a convex function on $I = [m, M]$, $(-\infty < m < M < \infty)$, and let g be a real function on $[0, 1]$such that $m \leqq g(t) \leqq M$ for all $t \in [0, 1]$. Then

$$(1) \qquad \int_0^1 \phi(g(t))dt - \phi\left(\int_0^1 g(t)dt\right) \leqq \max_{x \in [m, M]} \left\{\frac{M - x}{M - m}\phi(m) + \frac{x - m}{M - m}\phi(M) - \phi(x)\right\}.$$

In case ϕ is strictly monotonic on I, the bound on the right hand side of (1) is attained for a single value of x, say $x = x_0$, where

$$(2) \quad x_0 = \lambda m + (1 - \lambda)M, \qquad \lambda = (M - m)^{-1}\left\{M - (\phi')^{-1}\left(\frac{\phi(M) - \phi(m)}{M - m}\right)\right\}.$$

If ϕ is concave, the direction of the inequality sign in (1) is reversed. In [5], only the strictly monotonic case of (1), (2) was stated explicitly. (In [7], (1) is (incorrectly) stated by requiring that g be nondecreasing.) The special case $\phi(x) = x^2$ of (1) gives the well-known inequality of G. Gruss [4]. See also D. S. Mitrinović [7, p. 70].

In this paper we shall give a generalization of this result in Section 2, and some applications or examples of the basic inequality in Section 3.

2. **Main result**. In the sequel E will denote a nonempty set, L a class of real functions on E containing the chararacteristic function 1_E and A a positive linear functional over L satisfying $A(1_E) = 1$. The following result was given in [3, Lemma 1].

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LEMMA 1. *Let* $\phi(t)$ *be convex on* $I = [m,M]$ $(-\infty < m < M < \infty$. *If* $g \in L$, $g(E) \subset I$ *and* $\phi(g) \in L$, *then*

$$A(\phi(g)) \leqq \{(M - A(g))\phi(m) + (A(g) - m)\phi(M)\}/(M - m). \tag{3}$$

REMARK 1. The right-hand side of (3) is a nondecreasing function of M and non-increasing function of m. This follows by writing this expression in either of the two forms

$$\phi(m) + (A(g) - m)\frac{\phi(M) - \phi(m)}{M - m} = \phi(M) - (M - A(g))\frac{\phi(M) - \phi(m)}{M - m},$$

and noting that $m \leqq A(g) \leqq M$, while $(\phi(M) - \phi(m))/(M - m)$ is a nondecreasing function of both M and m by the convexity of ϕ.

We now give our basic generalization of Knopp's inequality (1).

THEOREM 1. *Let* J *be an interval such that* $J \supset \phi(I)$. *If* $F(u, v)$ *is a real function defined on* $J \times J$, *non-decreasing in* u, *then*

$$\begin{aligned} F[A(\phi(g)), \phi(A(g))] \leqq \max_{x \in [m,M]} F\left[\frac{M - x}{M - m}\phi(m) + \frac{x - m}{M - m}\phi(M), \phi(x)\right] \\ \left(= \max_{\theta \in [0,1]} F[\theta\phi(m) + (1 - \theta)\phi(M), \phi(\theta m + (1 - \theta)M)]\right). \end{aligned} \tag{4}$$

The right-hand side of (4) *is a nondecreasing function of* M *and a nonincreasing function of* m.

PROOF. By (3) and the nondecreasing character of $F(., y)$ we have

$$\begin{aligned} F[A(\phi(g)), \phi(A(g))] &\leqq F\left[\frac{M - A(g)}{M - m}\phi(m) + \frac{A(g) - m}{M - m}\phi(M), \phi(A(g))\right] \\ &\leqq \max_{x \in [m,M]} d(x; m, M, \phi), \end{aligned}$$

where

$$d(x; m, M, \phi) = F[\{(M - x)\phi(m) + (x - m)\phi(M)\}/(M - m), \phi(x)],$$

proving the first part of (4). As in Remark 1 we have for $m \leqq x$, and $m < M' \leqq M$,

$$\begin{aligned} \{(M - x)\phi(m) + (x - m)\phi(M)\}/(M - m) \geqq \{(M' - x)\phi(m) \\ + (x - m)\phi(M')\}/(M' - m). \end{aligned}$$

Hence, by the nondecreasing character of $F(., y)$,

$$d(x; m, M, \phi) \geqq d(x; m, M', \phi), \qquad m \leqq x, \qquad m < M' \leqq M. \tag{5}$$

By (5) and the inclusion $[m, M'] \subset [m, M]$, it follows that

$$\max_{x \in [m,M]} d(x; m, M, \phi) \geqq \max_{x \in [m,M]} d(x; m, M', \phi) \geqq \max_{x \in [m,M']} d(x; m, M', \phi).$$

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Similarly we can prove that

$$\max_{x \in [m,M]} d(x; m, M, \phi) \leqq \max_{x \in [m',M]} d(x; m', M, \phi) \qquad \textit{if} \qquad m' \leqq m < M.$$

Finally, the second form of the right side of (4) follows at once from the change of variable $\theta = (M - x)/(M - m)$, so $x = \theta m + (1 - \theta)M$ with $0 \leqq \theta \leqq 1$.

In the same way (or more simply just by replacing F by $-F$ in the above theorem) we can prove

THEOREM 1′. *Under the same hypotheses as Theorem 1, except that F is nonincreasing in its first variable, we have*

$$\text{(4}'\text{)} \qquad F[A(\phi(g)), \phi(A(g))] \geqq \min_{x \in [m,M]} d(x; m, M, \phi) \Big(= \min_{\theta \in [0,1]} F[\theta\phi(m) + (1 - \theta)\phi(M), \phi(\theta m + (1 - \theta)M)]\Big).$$

The right-hand side of (4) *is a nonincreasing function of M and a nondecreasing function of m.*

3. **Some applications**. First, we shall show that Lemmas 2 and 3 from [3] are simple consequences of Theorems 1 and 1′.

COROLLARY 1. *Let $\phi(x)$ be convex on $I = [m, M]$ $(-\infty < m < M < \infty)$, such that $\phi''(x) \geqq 0$ with equality for at most isolated points of I (so that ϕ is strictly convex on I). Suppose that either (i) $\phi(x) > 0$ for all $x \in I$, or (ii) $\phi(x) < 0$ for all $x \in I$. If $g \in L$, $g(E) \subset I$ and $\phi(g) \in L$, then*

$$\text{(6)} \qquad A(\phi(g)) \leqq \lambda\phi(A(g))$$

holds for some $\lambda > 1$ in case (i) or $\lambda \in (0, 1)$ in case (ii).

PROOF. For case (i) we apply Theorem 1 and for case (ii) we apply Theorem 1′, both with $F(x, y) = x/y$, and $J = (0, \infty)$. We proceed only with case (i) since the proof in case (ii) is essentially the same. The inequality (4) becomes

$$\text{(7)} \qquad A(\phi(g))/\phi(A(g)) \leqq \max_{x \in [m,M]} f(x; m, M, \phi),$$

where

$$f(x) \equiv f(x; m, M, \phi) = \{(M - x)\phi(m) + (x - m)\phi(M)\}/(M - m)\phi(x).$$

Now, $f'(x) = G(x)/\phi(x)^2$, where $G(x) = \mu\phi(x) - (\phi(m) + \mu(x - m))\phi'(x)$. The equation $G(x) = 0$, i.e.

$$\text{(8)} \qquad \mu\phi(x) - \phi'(x)(\phi(m) + \mu(x - m)) = 0,$$

has exactly one solution since—in case (i)—

$$G'(x) = -\{(M - x)\phi(m) + (x - m)\phi(M)\}\phi''(x)/(M - m) < 0,$$

so that G is a decreasing function. Furthermore,

$$G(m)G(M) = \phi(m)\phi(M)(\mu - \phi'(m))(\mu - \phi'(M)) < 0,$$

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so $G(x) = 0$ holds for a unique $x = \bar{x}(m, M)$. Since ϕ is convex and positive, it follows that $f(x) \geqq 1$, with equality for $x = m$ and M. Hence, the maximum value on the right-hand side of (7) is attained for $x = \bar{x}$.

REMARK 2. More precisely, a value of λ (depending only on m, M, ϕ) for (6) may be determined as follows: set $\mu = (\phi(M) - \phi(m))/(M - m)$. If $\mu = 0$ let $x = \bar{x}$ be the unique solution of equation $\phi'(x) = 0$ $(m < \bar{x} < M)$; then $\lambda = \phi(m)/(\bar{x})$ suffices for (6). If $\mu \neq 0$, let $x = \bar{x}$ be the unique solution in (m, M) of the equation (8), then $\lambda = \mu/\phi'(\bar{x})$ suffices for (6).

COROLLARY 2. *If ϕ is differentiable and ϕ' is strictly increasing on I, then*

$$A(\phi(g)) \leqq \lambda + \phi(A(g)) \tag{9}$$

for some λ satisfying $0 < \lambda < (M - m)(\mu - \phi'(m))$, *where μ is defined as in Corollary 1.*

PROOF. In Theorem 1, take $F(x, y) = x - y$. Then (4) becomes

$$A(\phi(g)) - \phi(A(g)) \leqq \max_{x \in [m, M]} Y(x; m, M, \phi),$$

where

$$Y(x) \equiv Y(x; m, M, \phi) = \{(M - x)\phi(m) + (x - m)\phi(M)\}(M - m)^{-1} - \phi(x).$$

We have $Y'(x) = \mu - \phi'(x)$ strictly decreasing on I with $Y'(\bar{x}) = 0$ for a unique $\bar{x} \in (m, M)$. Hence $Y(x)$ has its maximum value for $x = \bar{x}$.

REMARK 3. More precisely, λ may be determined for (9) as follows: let $x = \bar{x}$ be the unique solution of the equation $\phi'(x) = \mu$ $(m < \bar{x} < M)$; then

$$\lambda = \phi(m) - \phi(\bar{x}) + \mu(\bar{x} - m)$$

suffices in (9).

REMARK 4. Corollaries 1 and 2 (i.e. Lemmas 2 and 3 from [3]) are generalizations of results from [2] and [8]. In the case of Corollary 1, the additional cases that either $\phi(m) = 0$ or $\phi(M) = 0$ were also dealt with in [3]. The result (1), (2) of Knopp is the special case $A(g) = \int_0^1 g\,dt$ of Corollary 2.

For our next result suppose that $\psi, \chi: I \to R$ are continuous and strictly monotonic and that $\psi(g), \chi(g) \in L$ for some $g \in L$. As in [3], we define the generalized mean with respect to the operator A and ψ, by

$$M_\psi(g; A) = \psi^{-1}(A(\psi(g))), \qquad g \in L.$$

COROLLARY 3. *Under the above assumptions we have*

$$\begin{aligned} F(M_\psi(g;A), M_\chi(g;A)) \\ \leqq \max_{\theta \in [0,1]} F[\psi^{-1}(\theta\psi(m) + (1-\theta)\psi(M)), \chi^{-1}(\theta\chi(m) + (1-\theta)\chi(M))] \end{aligned} \tag{10}$$

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provided ψ *is increasing,* $\psi \circ x^{-1}$ *is convex, and* $F(u, v)$ *is a real function defined on* $I \times I$, *nondecreasing in* u.

PROOF. Suppose first that χ is increasing on I. Set $F_1(x, y) = F(\psi^{-1}(x), \psi^{-1}(y))$, $\phi_1(x) = \psi(\chi^{-1}(x))$, $g_1 = \chi \circ g$, $m_1 = \chi(m)$, $M_1 = \chi(M)$. Then the conclusion follows from Theorem 1 applied to F_1, ϕ_1, g_1. If χ is decreasing on I, we need only define $m_1 = \chi(M)$ and $M_1 = \chi(m)$. Then (4) now implies

$$\begin{aligned} &F(M_\psi(g; A), M_\chi(g; A)) \\ &\quad \leqq \max_{\theta \in [0,1]} F(\psi^{-1}(\theta\psi(M) + (1-\theta)\psi(m)), \chi^{-1}(\theta x(M) + (1-\theta)x(m))] \end{aligned}$$

and this is equivalent to (10).

REMARK 5. The special case $F(x,y) = x - y$, $\chi(x) \equiv x$, and $A(g) = \int_0^1 g dt$ of (10) yields the inequality

$$(11) \qquad \begin{aligned} \psi^{-1}\left(\int_0^1 \psi(g)dt\right) - \int_0^1 g dt \leqq \max_{\theta \in [0,1]} &(\psi^{-1}(\theta\psi(m) + (1-\theta)\psi(M)) \\ &- (\theta m + (1-\theta)M)). \end{aligned}$$

This inequality is a companion inequality to (1) and was also proved by K. Knopp [5, Satz 2] under the assumptions $\psi' > 0$, $\psi'' > 0$ (or $\psi' < 0$, $\psi'' < 0$) on $I = [m, M]$. In this case, the maximum value on the right hand side of (11) is attained for the value

$$\theta = [\psi(M) - \psi(m)]^{-1}\left\{\psi(M) - \psi\left[(\psi')^{-1}\left(\frac{\psi(M) - \psi(m)}{M - m}\right)\right]\right\},$$

as was shown in [5].

REMARK 6. Corollary 3 is a generalization of a result of E. Beck [1], who considered quasiarithmetic mean values $M_\phi(x; a) = \phi^{-1}(\Sigma_1^n a_i\phi(x_i))$. See also [6, pp. 135–136]. For $F(x,y) = x/y$ or $x - y$, Corollary 3 also gives generalizations of some results for means of Specht, Cargo and Shisha, and Mond and Shisha. See, for example Beck [1], [6, pp. 103–111], or [7, pp. 79–81]. Also, Corollary 3 is a generalization of inequalities of Schweitzer, Pólya and Szego, Kantorovič, and Greub and Rheinboldt. See [7, pp. 59–61].

REFERENCES

1. E. Beck, *Komplementare Ungleichungen bei vergleichbaren Mittelwerten*, Montash. für Math. **73** (1969), pp. 289–308.

2. P. R. Beesack, *On inequalities complementary to Jensen's*, Can. J. Math. **35** (1983), pp. 324–338.

3. P. R. Beesack and J. E. Pečarić, *On Jessen's inequality for convex functions*, J. Math. Anal. Appl. **110** (1985), pp. 536–552.

4. G. Grüss, *Über das Maximum des absoluten Betrages von* $(b-a)^{-1}\int_a^b f(x)g(x)dx - (b-a)^{-2}\int_a^b f(x)dx \int_a^b g(x)dx$, Math. Zeit. **39** (1935), pp. 215–226.

5. K. Knopp, *Über die maximalen Abstńde und Verhältnisse verschiedener Mittelwerte*, Math. Zeit. **39** (1935), pp. 768–776.

6. D. S. Mitrinović, P. S. Bullen and P. M. Vasić, *Sredine i sa njima povezane nejednakosti*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 600 (1977), pp. 1–232.

7. D. S. Mitrinović, *Analytic Inequalities*, Berlin-Heidelberg-New York, 1970.

8. D. S. Mitrinović and P. M. Vasić, *The centroid method in inequalities*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 498-541 (1975), pp. 3–16.

9. T. Popoviciu, *Les Fonctions Convexes*, Paris, 1944.

FACULTY OF CIVIL ENGINEERING
UNIVERSITY OF BELGRADE
BULEVAR REVOLUCIJE 73
11000 BELGRADE, YUGOSLAVIA

CARLETON UNIVERSITY
OTTAWA, ONT., CANADA

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