A NOTE ON NORMAL MATRICES

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1. Introduction. Let $U_n$ be an $n$-dimensional unitary space with inner product $(u, v)$. For vectors $u_1, \ldots, u_r \in U_n$, let $u_1 \wedge \ldots \wedge u_r$ denote the Grassmann exterior product of the $u_i$; it is a vector in $U_m$ where $m = \binom{n}{r}$. If also $v_1, \ldots, v_r \in U_n$, then $(u_1 \wedge \ldots \wedge u_r, v_1 \wedge \ldots \wedge v_r)$ is the determinant of the $r \times r$ matrix $((u_i, v_j)), 1 \leq i, j \leq r$. If $A$ is a linear transformation of $U_n$ to itself, the $r$th compound of $A$ is defined by

$$C_r(A)u_1 \wedge \ldots \wedge u_r = (Au_1) \wedge \ldots \wedge (Au_r).$$

For $1 \leq r \leq k \leq n$, denote by $Q_{k,r}$ the set of all $kC_r$ sequences $\omega = \{i_1, \ldots, i_r\}$ such that $1 \leq i_1 < \ldots < i_r \leq k$. For a set of vectors $x_1, \ldots, x_k \in U_n$ set

$$x_\omega = x_{i_1} \wedge \ldots \wedge x_{i_r},$$

$$g_{r}(x_1, \ldots, x_k) = \sum_{\omega \in Q_{k,r}} (C_r(A)x_\omega, x_\omega).$$

Let $E_r(a_1, \ldots, a_k)$ denote the elementary symmetric function of $a_1, \ldots, a_k$ of degree $r$ and let $\lambda_1, \ldots, \lambda_n$ denote the characteristic values of the linear transformation $A$. In (2) it was shown that if $A$ is Hermitian, then

$$\max g_r = E_r(\xi_1, \ldots, \xi_k),$$

$$\min g_r = E_r(\eta_1, \ldots, \eta_k),$$

where $\{\xi_1, \ldots, \xi_k\}$ and $\{\eta_1, \ldots, \eta_k\}$ are certain subsets of $\{\lambda_1, \ldots, \lambda_n\}$ and where the max and min are taken over all sets of $k$ orthonormal vectors $x_1, \ldots, x_k$ in $U_n$. In this note we offer the following generalization of this fact.

**Theorem 1.** If $A$ is normal and if $x_1, \ldots, x_k$ are orthonormal vectors in $U_n$, then $g_r(x_1, \ldots, x_k)$ lies in the convex hull $H_i$ of all the complex numbers

$$E_r(\lambda_{j_1}, \ldots, \lambda_{j_k}), \quad \{j_1, \ldots, j_k\} \in Q_{n,k}.$$

In Theorem 2 we identify the complex numbers which are of the form $g_r(x_1, \ldots, x_k)$ for orthonormal vectors $x_1, \ldots, x_k \in U_n$.

**Theorem 2.** If $A$ is normal, the set of all sums $(Ax_1, x_1) + \ldots + (Ax_k, x_k)$ for orthonormal vectors $x_1, \ldots, x_k \in U_n$ is $H_i$.

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We shall give an example to show that the analogue of Theorem 2 for \( r > 1 \) is, in general, false.

2. Proofs. Let \( |X| \) and \( ||X|| \) denote, respectively, the determinant and the absolute value of the determinant of the matrix \( X \). If \( \omega = \{i_1, \ldots, i_r\} \) and \( \tau = \{j_1, \ldots, j_r\} \) are in \( Q_{n,r} \), then \( X[\omega|\tau] \) denotes the submatrix of \( X \) which lies in the intersection of rows \( i_1, \ldots, i_r \) and columns \( j_1, \ldots, j_r \) of \( X \). Set \( \sigma(\tau) = j_1 + \ldots + j_r \) and let \( Q_{n,s} - \tau \) denote the set of sequences \( \{m_1, \ldots, m_s\} \in Q_{n,s} \) for which \( \{m_1, \ldots, m_s\} \cap \{j_1, \ldots, j_r\} \) is empty. If \( \omega \in Q_{n,s} \), then \( \omega' \) will denote the only element of \( Q_{n,n-s} - \omega \); \( \omega' \) contains the integers 1, \ldots, \( n \) which are not in \( \omega \).

**Lemma.** Let \( B = (b_{i,j}), 1 \leq i, j \leq n, \) be a unitary matrix. Let \( \tau = \{j_1, \ldots, j_r\} \) be a fixed member of \( Q_{n,r} \), and \( \mu = \{k+1, \ldots, n\} \) be a fixed member of \( Q_{n,n-k} \). Then, for \( 1 \leq r \leq k < n \),

\[
\sum_{\omega \in Q_{n,r}} ||B[\omega]|\tau||^2 = \sum_{\mu \in Q_{n,n-k}-\tau} ||B[\mu]|\rho||^2.
\]

**Proof.** For \( 1 \leq s < n \) let \( \gamma, \delta \) be two elements of \( Q_{n,s} \). Let \( B_{i,j} \) denote the cofactor of \( b_{i,j} \) in \( B \) and let \( B_{i,j} \) denote the \( n \times n \) matrix with \( B_{i,j} \) in row \( i \) and column \( j, 1 \leq i, j \leq n \). For any matrix \( B \) (not necessarily unitary) the following identity is known (3, Eq. 8.6):

\[
(B[\gamma,\delta]) = (-1)^{s(\rho)+s(\gamma)} |B[\gamma',\delta']| |B|^{|s|^{-1}}.
\]

If \( B \) is unitary, then \( B_{i,j}/|B| = b_{i,j} \), the complex conjugate of \( b_{i,j} \). Hence (2) becomes

\[
|B| |B[\gamma,\delta]| = (-1)^{s(\rho)+s(\gamma)} |B[\gamma',\delta']|.
\]

Let \( C = (c_{i,j}), 1 \leq i, j \leq n, \) where

\[
c_{i,j} = \begin{cases} 0 & \text{if } i \in \mu \text{ and } j \in \tau, \\ b_{i,j} & \text{otherwise.} \end{cases}
\]

Then, using (3),

\[
\sum_{\omega \in Q_{n,r}} ||B[\omega]|\tau||^2 = \sum_{\mu \in Q_{n,n-k}-\tau} ||B[\omega]|\rho||^2 = |B|^{-1} \sum_{\omega \in Q_{n,r}} \left(-1\right)^{s(\omega)+s(\tau)} |B[\omega]| |B[\omega'|\tau']|.
\]

But this expression, apart from the factor \( |B|^{-1} \), is just the Laplace expansion of \( |C| \) down columns \( j_1, \ldots, j_r \); the other terms that would normally appear in this Laplace expansion are all zero because of (4). Hence the left member of (1) is just \( |B|^{-1} |C| \).

On the other hand, if we expand \( |C| \) across rows \( k+1, \ldots, n \) and use (4) and (3),
\[ |B|^{-1}C = |B|^{-1} \sum_{\rho \in Q_{n,m-k-\tau}} (-1)^{s(\rho)+s(\rho')} |B[\mu|\rho]| |B[\mu'|\rho']| \]
\[ = \sum_{\rho \in Q_{n,m-k-\tau}} |B[\mu|\rho]| |B[\mu'|\rho']| \]
\[ = \sum_{\rho \in Q_{n,m-k-\tau}} |B[\mu|\rho]|^2. \]

**Proof of Theorem 1.** Throughout the rest of this paper, \( e_1, \ldots, e_n \) denotes an orthonormal set of characteristic vectors of \( A \) belonging to the characteristic values \( \lambda_1, \ldots, \lambda_n \), respectively. We are given orthonormal vectors \( x_1, \ldots, x_k \in U_n \). If \( k = n \), the result is clear since the vectors \( x_\omega \) for \( \omega \in Q_n \) form an orthonormal basis in \( U_m \) so that \( g_r = \text{trace} C_r(A) = \delta_r(\lambda_1, \ldots, \lambda_n) \).

Suppose \( k < n \) and choose \( x_{k+1}, \ldots, x_n \) so that \( x_1, \ldots, x_n \) is an orthonormal basis for \( U_n \). Let \( B = ((x_i, e_j)) \), \( 1 \leq i, j \leq n \). Then \( B \) is a unitary matrix. Now
\[ x_i = \sum_{j=1}^{n} (x_i, e_j)e_j, \quad 1 \leq i \leq k, \]
\[ Ax_i = \sum_{j=1}^{n} \lambda_j(x_i, e_j)e_j, \quad 1 \leq i \leq k. \]

Hence, using the multilinear and alternating properties of the Grassmann product, it follows that if \( \tau = \{j_1, \ldots, j_r\} \),
\[ x_\omega = \sum_{\tau \in Q_n} |B[\omega|\tau]|e_\tau, \]
\[ C_r(A)x_\omega = \sum_{\tau \in Q_n} \lambda_{j_1} \ldots \lambda_{j_r} |B[\omega|\tau]|e_\tau, \]
so that
\[ g_r = \sum_{\omega \in Q_n} \sum_{\tau \in Q_n} \lambda_{j_1} \ldots \lambda_{j_r} |B[\omega|\tau]|^2 \]
\[ = \sum_{\tau \in Q_n} \lambda_{j_1} \ldots \lambda_{j_r} \sum_{\omega \in Q_n} |B[\omega|\tau]|^2. \]
(5)

For \( \rho = \{m_1, \ldots, m_k\} \in Q_n,k \), let \( h_\rho = |B[\mu|\rho']| \), where \( \mu = \{k + 1, \ldots, n\} \). Then we claim that
\[ g_r = \sum_{\rho \in Q_n,k} |h_\rho|^2 E_r(\lambda_{m_1}, \ldots, \lambda_{m_k}). \]
(6)

To see this, note that the coefficient of
\[ \lambda_{j_1} \ldots \lambda_{j_r} \]
(7)
in (6) is
\[ \sum_{\rho' \in Q_{n,m-k-\tau}} |B[\mu|\rho']|^2. \]

By the Lemma, this is the same as the coefficient of (7) in (5).
The proof of Theorem 1 will now be complete if we can show that
\[ \sum_{\rho \in Q_{n,k}} |h_{\rho}|^2 = 1. \]
This is immediate since the $|B[4\rho']|$ for $\rho' \in Q_{n,n-k}$ are the co-ordinates of the unit vector $x_{k+1} \wedge \ldots \wedge x_n$ relative to the orthonormal basis $e_{\rho'}$ in the space $U_t$ with $t = \sum_{\rho \in Q_{n,k}} |h_{\rho}|^2$.

**Proof of Theorem 2.** Since any point $P \in H_1$ may be written as a convex combination of three of the vertices of $H_1$ and since the vertices of $H_1$ lie among the numbers
\[ \lambda_{j_1} + \ldots + \lambda_{j_k}, \quad \{j_1, \ldots, j_k\} \in Q_{n,k}, \]
it is enough to show that any point $P$ in the convex hull of three of the numbers (8) is of the form $P = \sum_{i} \lambda_i x_i$ for orthonormal vectors $x_1, \ldots, x_k \in U_n$.

Suppose we are given three sums (8), say $S_1, S_2, S_3$. With a proper choice of the notation we may assume that
\begin{align*}
S_1 &= (\lambda_1 + \ldots + \lambda_p) + (\lambda_{p+1} + \ldots + \lambda_{p+q}) + (\lambda_{p+q+1} + \ldots + \lambda_{p+q+r}) + (\lambda_{p+q+r+1} + \ldots + \lambda_{v}) + (\lambda_{v+1} + \ldots + \lambda_{w}), \\
S_2 &= (\lambda_1 + \ldots + \lambda_p) + (\lambda_{p+1} + \ldots + \lambda_{p+q}) + (\lambda_{p+q+1} + \ldots + \lambda_{p+q+r}) + (\lambda_{p+q+r+1} + \ldots + \lambda_{v}) + (\lambda_{w} + \ldots + \lambda_{w+s}), \\
S_3 &= (\lambda_1 + \ldots + \lambda_p) + (\lambda_{p+1} + \ldots + \lambda_{p+q}) + (\lambda_{p+q+1} + \ldots + \lambda_{p+q+r}) + (\lambda_{p+q+r+1} + \ldots + \lambda_{v}) + (\lambda_{w} + \ldots + \lambda_{w+s+t}),
\end{align*}
where, for brevity, we have let $w = p + q + r + s$. Here some of $p, q, r, s, t, u, v$ may be zero, in which case not all of the types of terms indicated need actually appear. We have
\begin{align*}
\lambda_{j_1} + \ldots + \lambda_{j_k} &= k, \\
p + q + r + t &= k, \\
p + q + s + u &= k, \\
p + r + s + v &= k.
\end{align*}
We may suppose that $t \geq u \geq v$. Let $\alpha, \beta, \theta$ be three real numbers with $\alpha^2 + \beta^2 + \theta^2 = 1$. We have to find orthonormal vectors $x_1, \ldots, x_k \in U_n$ such that
\[ (Ax_1, x_1) + \ldots + (Ax_k, x_k) = \alpha^2 S_1 + \beta^2 S_2 + \theta^2 S_3. \]
If $p > 0$, set $x_i = e_i$ for $1 \leq i \leq p$. Then
\[ (Ax_1, x_1) + \ldots + (Ax_p, x_p) = \lambda_1 + \ldots + \lambda_p. \]
If $v > 0$, set $x_{p+i} = \alpha e_{w+i} + \beta e_{w+t+i} + \theta e_{w+t+u+i}$ for $1 \leq i \leq v$. Then
\[ (x_{p+i}, x_{p+i}) = 1 \text{ and } \]
\[(Ax_{p+1}, x_{p+1}) + \ldots + (Ax_{k}, x_{k}) = \alpha^2(\lambda_{w+1} + \ldots + \lambda_{w+t}) + \beta^2(\lambda_{u+t+1} + \ldots + \lambda_{u+t+t}) + \theta^2(\lambda_{v+1} + \ldots + \lambda_{v+t+t})\]

From (10) and (11) it follows that \(r = q + (u - v)\); hence \(r > u - v\). If \(u > v\), let

\[x_{p+i+t} = \beta e_{w+t+i+t} + (\alpha^2 + \theta^2)\epsilon_{p+t+i} \quad \text{for } 1 \leq i \leq u - v.\]

Then \((x_{p+i+t}, x_{p+i+t}) = 1\) and

\[(Ax_{p+i+t}, x_{p+i+t}) + \ldots + (Ax_{p+i+u}, x_{p+i+u}) = \beta^2(\lambda_{w+t+i+t+1} + \ldots + \lambda_{w+t+i+u}) + (\alpha^2 + \theta^2)(\lambda_{p+t+i+1} + \ldots + \lambda_{p+t+i+u} - e).\]

It follows from (9) and (11) that \(s = q + (t - v)\). If \(t > v\), define

\[x_{p+i+t} = \alpha e_{w+t+i+t} + (\beta^2 + \theta^2)\epsilon_{p+t+i+t} \quad \text{for } 1 \leq i \leq t - v.\]

Then \((x_{p+i+t}, x_{p+i+t}) = 1\) and

\[(Ax_{p+i+t}, x_{p+i+t}) + \ldots + (Ax_{p+i+u+t-e}, x_{p+i+u+t-e}) = \alpha^2(\lambda_{w+t+i+1} + \ldots + \lambda_{w+t+i+u}) + (\beta^2 + \theta^2)(\lambda_{p+t+i+1} + \ldots + \lambda_{p+t+i+u+e}).\]

Up to this point \(p + u + t - v\) vectors \(x_i\) have been constructed; these vectors are automatically orthogonal because, when expressed in terms of the \(e_i\), no two \(x_i\) involve the same \(e_i\). There remain \(k - (p + u + t - v) = 2q\) vectors \(x_i\) to be constructed. Let \(G\) be the subspace of \(U_n\) spanned by

\[f_1 = e_{p+1}, \ldots, f_q = e_{p+q}, \quad f_{q+1} = e_{p+q+u-t+1}, \ldots, f_{2q} = e_{p+q+t},\]

and let \(\xi_1, \ldots, \xi_{2q}\) be the \(\xi_i\) belonging to \(f_1, \ldots, f_{2q}\). Let \(y_i = \theta f_i + \beta f_{q+i} + \alpha f_{2q+i}\) for \(1 \leq i \leq q\). Choose \(x_{k-2q+1}, \ldots, x_k\) such that \(y_1, \ldots, y_q, x_{k-2q+1}, \ldots, x_k\) is an orthonormal basis of \(G\). Then if we compute the trace of the restriction \(A_G\) of \(A\) to \(G\) we get:

\[
\text{trace } A_G = (\xi_1, x_1) + \ldots + (\xi_{2q}, x_{2q}) \\
= (Ay_1, y_1) + \ldots + (Ay_q, y_q) + (Ax_{k-2q+1}, x_{k-2q+1}) + \ldots + (Ax_k, x_k) \\
= \theta^2(\xi_1 + \ldots + \xi_q) + \beta^2(\xi_{q+1} + \ldots + \xi_{2q}) \\
+ \alpha^2(\xi_{2q+1} + \ldots + \xi_{3q}) + (Ax_{k-2q+1}, x_{k-2q+1}) + \ldots + (Ax_k, x_k).\]

Hence we find that

\[(Ax_{k-2q+1}, x_{k-2q+1}) + \ldots + (Ax_k, x_k) = (\alpha^2 + \beta^2)(\lambda_{p+1} + \ldots + \lambda_{p+q}) + (\alpha^2 + \theta^2)(\lambda_{p+q+u-t+1} + \ldots + \lambda_{p+q+u}) + (\beta^2 + \theta^2)(\lambda_{p+q+u+t-e+1} + \ldots + \lambda_{p+q+u+t+e}).\]

Then \(x_1, \ldots, x_k\) are orthonormal vectors in \(U_n\) such that

\[(Ax_1, x_1) + \ldots + (Ax_k, x_k) = \alpha^2 S_1 + \beta^2 S_2 + \theta^2 S_3.\]
We now give an example to show that the set of all numbers $g_r(x_1, \ldots, x_k)$ for orthonormal $x_1, \ldots, x_k$ need not be a convex set if $r > 1$. Let $r = k = 2$, $n = 4$, and take $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \lambda_4 = i = (-1)^{\frac{1}{4}}$. Let $\rho_{i,j} = |B[1, 2][i,j]|$, where $B$ is the matrix $((x_i, e_j))$, $1 \leq i \leq 2$, $1 \leq j \leq 4$. Then, from (5),

$$g_2(x_1, x_2) = |\rho_{1,1}|^2 - |\rho_{2,2}|^2 + \imath(|\rho_{1,1}|^2 + |\rho_{1,2}|^2 + |\rho_{2,1}|^2 + |\rho_{2,2}|^2).$$

Now $g_2(e_1, e_2) = 1$ and $g_2(e_3, e_4) = -1$. If $g_2(x_1, x_2) = 0$, then we must have $|\rho_{1,1}| = |\rho_{2,4}|$, $\rho_{1,3} = \rho_{1,4} = \rho_{2,3} = \rho_{2,4} = 0$. However, it is known (1) that $ho_{1,2}\rho_{2,4} = \rho_{1,2}\rho_{2,4} - \rho_{1,4}\rho_{2,3}$. Combining these facts, it follows that also $\rho_{1,2} = \rho_{2,4} = 0$. This is a contradiction, since

$$\sum_{1 \leq i < j \leq 4} |\rho_{i,j}|^2 = (x_1 \wedge x_2, x_1 \wedge x_2) = 1$$

if $x_1$ and $x_2$ are orthonormal.

References


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