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A geometric interpretation of Ranicki duality

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Consider a commutative ring R and a simplicial map, $X \xrightarrow{\pi} K$, of finite simplicial complexes. The simplicial cochain complex of X with R coefficients, $\Delta^* X$, then has the structure of an (R, K) chain complex, in the sense of Ranicki . Therefore it has a Ranicki-dual (R, K) chain complex, $T\Delta^* X$. This (contravariant) duality functor $T : \mathcal{B}R_K \to \mathcal{B}R_K$ was defined algebraically on the category of (R, K) chain complexes and (R, K) chain maps.

Our main theorem, 8.1, provides a natural (R, K) chain isomorphism:

$$T\Delta^* X \cong C(X_K)$$

where $C(X_K)$ is the cellular chain complex of a CW complex X_K. The complex X_K is a (nonsimplicial) subdivision of the complex X. The (R, K) structure on $C(X_K)$ arises geometrically.

Keywords: Manifolds; Surgery; K-Theory

1. Introduction; description of results

This article is an addition to a theory of blocked surgery, pioneered by Ranicki, augmented by others in [**1**, **4**, **5**, **6**, **8**, **9**, **16**, **17**], and still in a developing state.

Let R be a commutative ring; let K be a finite simplicial complex. In [**16**] Ranicki introduced the category of (R, K) chain complexes and chain maps denoted $\mathcal{B}R_K$ here. He also defined algebraically, a contravariant functor $T : \mathcal{B}R_K \to \mathcal{B}R_K$.[1]

The simplest geometric example of an (R, K) chain complex arises from a K-*space* (X, π). This is a finite simplicial complex X and a simplicial map, $\pi : X \to K$. In that case, the simplicial cochains on X (with R coefficients) form an (R, K) chain complex denoted $\Delta^* X$.

At the same time, (X, π) specifies a regular CW complex X_K, which is a (nonsimplicial) subdivision of X. We show that the cellular chain complex (with R coefficients) of X_K forms a second (R, K) chain complex $C(X_K)$.

Our main theorem, theorem 8.1, exhibits a geometrically defined chain isomorphism between $C(X_K)$ and $T\Delta^* X$. Roughly put:

$$T\Delta^* X = C(X_K).$$

[1]The duality functor T for (R, K^{op}) complexes seems to play a lesser role at present in the geometric contexts of interest here.

It is also our aim to give a transparent definition of this duality functor T, a clear treatment of Ranicki's natural transformation $e : T^2 \to id.$ and a simple proof that $e_C : T^2C \to C$ is an (R, K) chain equivalence for all C.

Our larger goal is to facilitate applications of Ranicki's theory to geometric questions such as the topological rigidity of non-positively curved groups as in [**4**, **5**, **9**, **10**] when those groups have elements of finite order.

The vehicle for such applications would be a full blown *K-blocked surgery theory* of which there are only hints in [**16**]. This would start with a degree-one normal map between closed manifolds, $(f, b) : (M, \nu(M)) \to (X, \xi)$ (as in [**2**]) together with a reference map, $\pi : X \to K$ as above. It would seek an L-theoretic obstruction to finding a normal cobordism of (f, b) to a '*K-blocked homotopy equivalence*,' $M' \to X$. But we will not pursue this here or even define the terms precisely.

In the classical case ($K =$ point; [**2**, **12**, **18**, **19**, **20**]) one has the 'surgery obstruction' $\sigma_*(f, b) \in L_n(\mathbb{Z}[\pi_1(X)])$ to such a normal cobordism. But this functor $L_n()$, was generalized in [**16**] to yield obstruction groups $L_n(\mathcal{A})$ for any 'category-with-chain-involution' $(A, *, \epsilon)$. Here A is an additive category, $\mathcal{B}A \xrightarrow{*} \mathcal{B}A$ is a contravariant functor satisfying certain conditions, on the category $\mathcal{B}A$, of finite chain complexes in A, and $\epsilon : (*)^2 \to id$, is an equivalence in the homotopy category of $\mathcal{B}A$.

Ranicki, in [**16**], then starts with a finite complex K and a category with chain involution, $\mathcal{A} = (A, *, \varepsilon)$ as above. He then constructs the additive category A_K of K-blocked objects from A, and K-blocked A-maps. From $*$, and ε, he defines the *Ranicki Duality Functor* $T : \mathcal{B}(A_K) \to \mathcal{B}(A_K)$, and the natural transformation $e : T^2 \to id_{\mathcal{B}(A_K)}$. This construction allows one to define the surgery obstruction groups, $L_n(\mathcal{A}_K)$ where $\mathcal{A}_K = (A_K, T, e)$.

This seems to apply directly to a K-blocked normal map, $M^n \xrightarrow{(f,b)} X^n \xrightarrow{\pi} K$. Here the relevant category seems to be $A = A(R)$, the category of finitely generated free modules over a fixed commutative ring R. We write AR_K for $(AR)_K$ and $\mathcal{B}R_K$ for $\mathcal{B}(AR_K)$. Its objects are (R, K)-*chain complexes*. So the simplicial cochain complexes of X and M denoted Δ^*X and Δ^*M, and the simplicial chain complexes, $\Delta X'$ and $\Delta M'$, are (R, K)-chain complexes. (See § 3). Thus the L-groups of $(\mathcal{A}R_K, T, e)$ seem likely to be useful.

However, Ranicki's definition of $\mathcal{B}R_K \xrightarrow{T} \mathcal{B}R_K$ was only a starting point. Indeed his assertion in [**16**] of the *crucial* theorem that $(\mathcal{A}R_K, T, e)$ is a category with chain involution was only proved in 2018 (by Adams-Florou and Macko, [**1**]).

This paper interprets Ranicki's notions geometrically. Section 2 fixes chain-complex conventions. Section 3 reviews Ranicki's concepts concerning (R, K) complexes while attempting to simplify notation. In § 5 we introduce the (R, K) chain complex $C \otimes_K D$, defined if D is an (R, K) complex and C is an (R, K^{op}) complex. This complex $C \otimes_K D$ is a certain quotient of $C \otimes_R D$.

Our definition (see 6.1) of the Ranicki dual TC, of an (R, K) complex C, is:

$$TC = C^* \otimes_K \Delta^*K.$$

In § 7 we show, using work of M. Cohen [**3**], that each K-space (X, π) defines a certain regular CW-complex X_K, whose cellular chain complex has a natural

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(R, K) structure. Therefore from each K-space (X, π) we obtain three (R, K) chain complexes:

(1) $\Delta^* X$, the simplicial cochain complex of X (definition 4.1).

(2) $C(X_K)$, the cellular chain complex of the CW complex X_K (§ 8).

(3) $\Delta X'$, the simplicial chain complex of X', the barycentric subdivision of X (definition 7.2).

This paper shows that these three are closely related by T. Our main result, theorem 8.1, exhibits an isomorphism of (R, K) chain complexes:

$$\Phi_X : T\Delta^* X \cong C(X_K)$$

Then, using [**13**], we prove there are (R, K) chain homotopy equivalences:

$$T\Delta X' \simeq \Delta^* X; \quad C(X_K) \simeq \Delta X'.$$

When X is a pl-manifold, and $C = C(X_K)$, Poincare duality then becomes an n-cycle in the (R, K^{op}) complex, $Hom_{(R,K)}(TC, C)$.

This CW complex X_K is a subdivision of X, and X' is a simplicial subdivision of X_K. In fact, for each simplex S of X and each face σ of $\pi(S) \in K$, there is a single cell S_σ of X_K. Specifically, if $D(\sigma, \pi(S))$ is the dual cell of σ in $\pi(S)$:

$$S_\sigma = (\pi \mid S)^{-1} |D(\sigma, \pi(S))| \,.$$

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2. Chain complex conventions

Throughout this paper, R denotes a fixed commutative ring; AR is the additive category of finitely generated free R modules.

For any additive category A we will write $\mathcal{B}A$ for the additive category of finite chain complexes, $C = \{C_q, \partial_q\}_{q\in\mathbb{Z}}$ and chain maps $f = \{f_q : C_q \to D_q\}_{q\in\mathbb{Z}}$ from A. (*Finite* means: $C_q = 0$ for all but finitely many q). We abbreviate $\mathcal{B}(AR)$ to $\mathcal{B}R$.

As usual two chain maps $f, g : C \to D$ are chain homotopic if there is a sequence of A maps, $h = \{h_q : C_q \to D_{q+1}\}$, for which $d^D_{q+1} h_q + h_{q-1} d^C_q = g_q - f_q \ \forall q$.

We regard A as the full subcategory of $\mathcal{B}A$ consisting of chain complexes concentrated in degree zero.

Let $C, D \in Ob(\mathcal{B}R)$. The complexes $C \otimes_R D$, and $Hom_R(C, D)$ in $Ob(\mathcal{B}R)$, are:

$$\boxed{(C \otimes_R D)_q = \sum_{r\in\mathbb{Z}} C_r \otimes_R D_{q-r}; \quad Hom_R(C, D)_q = \sum_{r\in\mathbb{Z}} Hom_R(C_r, D_{q+r})} \text{ and:}$$

$$\boxed{d^{C\otimes D}(x \otimes y) = d^C x \otimes y + (-1)^{|x|} x \otimes d^D y; \quad d^{Hom}\phi = d^D \circ \phi - (-1)^{|\phi|}\phi \circ d^C}$$

The *evaluation map*, $eval_{C,D} : Hom_R(C, D) \otimes_R C \longrightarrow D$ is the *R-chain map:*

$$eval_{C,D}(f \otimes x) = f(x).$$

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Note that $eval_{R,D} : Hom_R(R, D) \otimes_R R \cong D$.

Write $ev_C : C^* \otimes C \to R$ for $eval_{C,R}$.

The contravariant functor $\mathcal{B}R \xrightarrow{*} \mathcal{B}R$ is : $C^* = Hom(C, R);\ f^* = Hom(f, 1_R)$.

Therefore we have:

$$(C^*)_{-q} = Hom_R(C_q, R); \quad d^{C^*}_{-q} = (-1)^{q+1}(d^C_{q+1})^* : (C^*)_{-q} \to (C^*)_{-q-1}.$$

The functor $*$ comes with a natural equivalence, $\varepsilon : (*)^2 \to 1_{\mathcal{B}R}$. Specifically, the chain isomorphism $\epsilon_C : C^{**} \to C$ is characterized by the identity:

$$a(\varepsilon_C(\alpha)) = (-1)^q \alpha(a) \quad \forall \alpha \in (C^{**})_q,\ a \in (C^*)_{-q}.$$

3. Basic definitions for (R, K) chain complexes

Definition 3.1. Let K be a *finite poset with partial order* $\leqslant$.

K^{op} denotes the same set with the opposite partial order.

(Later we will specialize to the case when K is a finite simplicial complex).

(1) An *(R,K) module* is an ordered pair $M = (M(K), \{M(\sigma)\}_{\sigma \in K})$ such that:
   (a) $M(K)$ and each $M(\sigma)$ are R-modules in $Ob(AR)$;

   (b) $M(K) = \oplus_{\sigma \in K} M(\sigma)$.
   More generally, for any $S \subset K$ we write: $M(S) = \oplus_{\sigma \in S} M(\sigma)$.

(2) An (R, K) *map* $M \xrightarrow{f} N$ of (R, K) modules is a map $M(K) \xrightarrow{f} N(K)$ of R modules, whose components, $f(\tau, \sigma) : M(\sigma) \to N(\tau)$, satisfy:

$$f(\tau, \sigma) = 0 \text{ unless } \tau \geqslant \sigma.$$

(3) The additive category of (R, K) maps and modules is written AR_K.
   We abbreviate the category of chain complexes, $\mathcal{B}(AR_K)$, to $\mathcal{B}R_K$.

(4) An object $C = \{C_q, \partial_q\}_{q \in \mathbb{Z}}$ of $\mathcal{B}R_K$ is an *(R, K) chain complex*. We then write $C(K)$ for $\{C_q(K), \partial_q\}_{q \in \mathbb{Z}}$, an R-chain complex in $ob(\mathcal{B}R)$.
   Note: $C \in ob(\mathcal{B}R_K)$ is specified by specifying the R complex $C(K)$ and the required collection $\{C_q(\sigma)\}_{\sigma \in K, q \in \mathbb{Z}}$ of R submodules.

(5) Let $C, D \in ob(\mathcal{B}R_K)$. $Hom_{(R,K)}(C, D)$ is the (R, K^{op}) complex such that:
   (a) $Hom_{(R,K)}(C, D)(K)$ is the subcomplex of $Hom_R(C(K), D(K))$ given by those $f = \{f_q : C_q \to D_{q+|f|}\}_{q \in \mathbb{Z}}$ for which each f_q is an (R, K) map.

   (b) $Hom_{(R,K)}(C, D)_p(\sigma)$ is the set of $f \in Hom_{(R,K)}(C, D)(K)_p$ satisfying:

$$f_q \mid_{C_q(\tau)} = 0 \text{ if } \tau \neq \sigma,\ \forall\, q.$$

(6) We say a sequence of chain maps $0 \to C' \xrightarrow{i} C \xrightarrow{j} C'' \to 0$ in $\mathcal{B}R_K$ is *exact* if for each $\sigma \neq \tau$, $i(\sigma, \tau) = 0$, $j(\sigma, \tau) = 0$, and, for all q, the corresponding sequence, $0 \to C'_q(\sigma) \to C_q(\sigma) \to C''_q(\sigma) \to 0$. is an exact sequence in $\mathcal{A}R$. We then say i is an (R, K) *monomorphism* and j is an (R, K) *epimorphism*.

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(7) Note that $*$ specifies a contravariant functor, $\mathcal{B}R_K \xrightarrow{*} \mathcal{B}R_{K^{op}}$, provided that we define $(C^*)_q(\sigma)$ as $(C_{-q}(\sigma))^*$ and d^{C^*} as $d^{C(K)^*}$ for $C \in ob(\mathcal{B}R_K)$. $\mathcal{B}R_K \xrightarrow{*} \mathcal{B}R_{K^{op}}$ preserves exactness and homotopy. The transformation $\varepsilon_C : C^{**} \to C$ of § 2 is an (R, K) isomorphism, for all $C \in ob(\mathcal{B}R_K)$.

(8) We say $S \subset K$ is *full in* K if, whenever $\rho, \tau \in S$, then:

$$\{\sigma \in K \mid \rho \leqslant \sigma \leqslant \tau\} \subset S.$$

Let C be an (R, K) complex. Let S be a full subset of K. We define $\partial_q^{C(S)} : C_q(S) \to C_{q-1}(S)$ by:

$$\partial_q^{C(S)} x = \sum_{\tau \in S} \partial^C(\tau, \sigma)x, \quad \forall x \in C_q(\sigma), \forall \tau, \sigma \in S.$$

Then $C(S) := \{C_q(S), \partial_q^{C(S)}\}_{q \in \mathbb{Z}}$ is an R chain complex. But in many cases, it is neither a subcomplex nor a quotient complex of $C(K)$.

4. *K* spaces and their chain complexes

For the rest of this paper, K denotes a finite simplicial complex.

A simplicial complex K is a poset so the above definitions apply. In this case $\sigma \leqslant \tau$ means that the simplex σ is a face (not necessarily proper) of the simplex τ.

$\Delta_*(K; R) = \{\Delta_q(K; R), \partial_q\}_{q \in \mathbb{Z}}$ denotes the simplicial chain complex of K.

$\Delta^*(K; R) = Hom_R(\Delta_*(X; R), R)$ denotes the simplicial cochain complex of K.

One can choose a basis, bK for $\Delta_*(K; R)$ consisting of one oriented q-simplex, $\sigma = \langle v_0 \ldots, v_q\rangle \in \Delta_q(K; R)$ for each q-simplex with vertices $v_0, \ldots v_q$, of K. Recall: $\langle v_0, \ldots, v_q\rangle = sgn(\pi)\langle v_{\pi(0)}, \ldots, v_{\pi(q)}\rangle$ for each $\pi \in S_{q+1}$. The oriented q-simplex $\sigma \in \Delta_q(K; R)$ defines a dual cochain $\sigma^* \in \Delta^*(K; R)_{-q}$ such that $\sigma^*(\tau) = 0$ for all $\tau \neq \pm\sigma$, and $\sigma^*(\sigma) = 1$.

One then defines $\sigma^{**} \in \Delta_q(K; R)^{**}$ by: $\varepsilon(\sigma^{**}) = \sigma$.

Each simplex $\sigma \in K$ defines subcomplexes, $\overline{\sigma}$ and $\partial\sigma$, and a subset $st(\sigma)$:

$$\overline{\sigma} = \{\tau \in K \mid \tau \leqslant \sigma\}; \quad \partial\sigma = \{\tau \in K \mid \tau < \sigma\}; \quad st(\sigma) = \{\tau \in K \mid \tau \geqslant \sigma\}$$

The incidence number $[\tau, \sigma] \in \{1, -1, 0\}$ is defined for any oriented simplices σ, τ of K. It satisfies: $\partial_q(\sigma) = \sum_{\tau \in bK}[\sigma, \tau]\tau$ for any basis, bK of oriented simplices of K. $[\sigma, \tau] \neq 0$ iff τ is a codimension-one face of σ.

Definition 4.1. (K-spaces, $\Delta^* X$ and ΔX)

Let K be a finite simplicial complex. A K-*space* is a pair (X, π) where X is a finite simplicial complex and $|X| \xrightarrow{\pi} |K|$ is a simplicial map, $X \to K$. A *map of K-spaces*, $(X, \pi_X) \to (Y, \pi_Y)$ is a simplicial map $f : |X| \to |Y|$ satisfying: $\pi_Y f = \pi_X$.

Let (X, π) be a K-space.

ΔX denotes the (R, K^{op}) complex for which $\Delta X(K) = \Delta_*(X; R)$. For each $\sigma \in K$, $(\Delta X)_p(\sigma)$ is the submodule generated by oriented p-simplices in $\Delta_p(X; R)$ whose underlying p-simplex, $S \in X$, satisfies $\sigma = \pi(S) \in K$.

By definition, $\Delta^* X = (\Delta X)^*$. Therefore $\Delta^* X(K) = Hom_R(\Delta_*(X; R), R) = \Delta^*(X; R)$, the simplicial cochain complex of X. For each $\sigma \in K$, $(\Delta^* X)_{-p}(\sigma)$ is

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therefore the submodule spanned by all S^* for which $S \in \Delta_p(X;R)$ is an oriented simplex and $\sigma = \pi(S) \in K$.

A map $f : X \to Y$ of K-spaces induces an (R,K) chain map $f^* : \Delta^* Y \to \Delta^* X$ and an (R, K^{op}) chain map $f_* : \Delta Y \to \Delta X$.

The next lemma will be used in § 6.

LEMMA 4.2. *Suppose $S \in K$ and there is no $\tau \in K$ for which $S < \tau$. The K-space $(\overline{S}, inclusion)$ specifies the (R,K) complex $\Delta^*\overline{S}$. Then $\Delta^*\overline{S}(st(\sigma))$ is a contractible R-complex for all $\sigma \in K$ such that $\sigma \neq S$. Also $\Delta^*\overline{S}(st(S)) = RS^*$.*

Proof. It is obvious that $\Delta^*\overline{S}(st(S)) = RS^*$ (after orienting S) and that $\Delta^*\overline{S}(st(\sigma)) = 0$ if σ is not a face of S. So we assume $\sigma < S$. Let τ be the complementary face of σ in S. Then the joins, $\overline{S} = \sigma * \tau$ and $\partial\sigma * \tau$ are contractible simplicial complexes. Note $st(\sigma) = \overline{S} - \partial\sigma * \tau$. Consequently, $\Delta^*\overline{S}(st(\sigma)) = \Delta^*(\sigma * \tau, \partial\sigma * \tau; R)$ is a contractible chain complex. □

5. $C \otimes_K D$ and the isomorphism $Hom_{(R,K)}(D, C^*) \cong (C \otimes_K D)^*$

Throughout this section, C denotes an (R, K^{op}) complex and D denotes an (R,K) complex.

We will first define two (R,K) complexes: $C \otimes_R D$ and a quotient of this, $C \otimes_K D$.

In K, the star of any simplex , $st(\sigma)$, as well as $K - st(\sigma)$ are *full* in K. Moreover the chain complex $C(K - st(\sigma))$ is a subcomplex of $C(K)$ and $C(st(\sigma))$ is a quotient complex. These fit into a short exact sequence of chain maps in $\mathcal{B}R$:

$$0 \to C(K - st(\sigma)) \xrightarrow{i_\sigma} C(K) \xrightarrow{p_{st(\sigma)}} C(st(\sigma)) \to 0$$

Here $C(K) \xrightarrow{p_{st(\sigma)}} C(st(\sigma))$ is defined by: $p_{st(\sigma)}|_{C_q(st(\sigma))} = 1_{C_q(st(\sigma))}$; and $p_{st(\sigma)}|_{C(K-st(\sigma))} = 0$.

(For the (R,K) complex D, we get $0 \to D(st(\sigma)) \to D(K) \to D(K - st(\sigma)) \to 0$).

DEFINITION 5.1. ($C \otimes_K D$, $C \otimes_R D$, and $C \otimes_R D \xrightarrow{\pi_{C,D}} C \otimes_K D$).

Let C be an (R, K^{op}) complex and D be an (R,K) complex.

(1) Let $C \otimes_R D$ be the (R,K) complex for which:

$$(C \otimes_R D)(K) = C(K) \otimes_R D(K);$$
$$(C \otimes_R D)_q(\rho) = (C(K) \otimes_R D(\rho))_q \quad \forall \rho \in K, q \in \mathbb{Z}$$

(2) Let $C \otimes_K D$ be the (R,K) complex for which:
  (a) $(C \otimes_K D)_q(K) = \sum_{\rho \in K} (C(st(\rho)) \otimes_R D(\rho))_q \ \forall q \in \mathbb{Z}$

  (b) $(C \otimes_K D)(\rho) = C(st(\rho)) \otimes_R D(\rho) \ \forall\ \rho \in K$

  (c) The map $C \otimes_R D \xrightarrow{\pi_{C,D}} C \otimes_K D$ is an (R,K) chain epimorphism, if we define $\pi_{C,D}$ by requiring that $\pi_{C,D}(\sigma, \rho) = 0$ for $\sigma \neq \rho$ and:

$$\pi_{C,D}(\rho, \rho) = p_{st(\rho)} \otimes_R 1_{D(\rho)} : C(K) \otimes_R D(\rho) \longrightarrow C(st(\tau)) \otimes_R D(\rho).$$

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Expicitly, for any $\rho \leqslant \tau$ and $x \otimes_R y \in C_r(\tau) \otimes_R D_{q-r}(\rho) \subset (C \otimes_K D)_q(\rho)$, we have

$$\boxed{d^{C \otimes_K D}(x \otimes y) = \sum_{\{\sigma | \rho \leqslant \sigma \leqslant \tau\}} d^C(\sigma, \tau) x \otimes y + (-1)^r x \otimes d^D(\sigma, \rho) y.} \tag{5.1}$$

We now show that $(C \otimes_K D)^*$ is a convenient expression for $Hom_{(R,K)}(D, C^*)$:

LEMMA 5.2. *There is a natural isomorphism* Ψ *of functors, denoted,*

$$\Psi_{C,D} : Hom_{(R,K)}(D, C^*) \cong (C \otimes_K D)^* \tag{5.2}$$

for any $(C, D) \in Ob(\mathcal{B}R_{K^{op}} \times \mathcal{B}R_K)$*.*

Proof. Suppose f is in $Hom_{(R,K)}(D, C^*)_q(\sigma)$ for some $\sigma \in K$ and $q \in \mathbb{Z}$. Define an R-map, $\Psi(f) : C(st(\sigma)) \otimes D(\sigma)_{-q} \to R$, by the formula:

$$\Psi(f)(x \otimes y) = (-1)^{|x||y|} f(y)(x) \quad \text{for } x \otimes y \in (C \otimes_K D)_{-q}(\sigma).$$

The same formula yields 0, if $x \otimes y$ is in $(C \otimes_K D)(\tau)_{-q}$ for $\tau \neq \sigma$. One easily sees that this rule (i.e. $f \mapsto \Psi(f)$ gives an isomorphism,

$$\Psi_{C,D} : Hom_{(R,K)}(D, C^*) \xrightarrow{\cong} (C \otimes_K D)^*$$

of (R, K^{op}) complexes for all $(C, D) \in Ob(\mathcal{B}R_{K^{op}} \times \mathcal{B}R_K)$. Naturality is obvious. □

6. Ranicki Duality and the (R, K) chain equivalence $e : T^2 \to 1_{\mathcal{B}R_K}$

DEFINITION 6.1. Ranicki Duality is the contravariant functor $\mathcal{B}R_K \xrightarrow{T} \mathcal{B}R_K$ defined for a chain complex $C \in Ob(\mathcal{B}R_K)$ and a (R, K) chain map, $f : C \to D$ by:

$$TC = C^* \otimes_K \Delta^* K \quad Tf = f^* \otimes_K 1_{\Delta^* K}$$

$\Delta^* K$ comes from the K-space, $(K, 1_K)$. After examining [16], p. 75 and p. 26, lines -6 to -4 one can see that this is in agreement with the definition indicated there, up to isomorphism and differences in sign conventions. In particular compare our formula for $d^{C \otimes_K D}$ with that on p.26, line -5 of [16].

COROLLARY 6.2. *T is an exact homotopy functor.*

Proof. By lemma 5.2, $TC = C^* \otimes_K \Delta^* K$ is isomorphic to $Hom_{(R,K)}(\Delta^* K, C)^*$ (since $\varepsilon_C : C^{**} \cong C$ for all C). But $C \mapsto C^*$ and $C \mapsto Hom(\Delta^* K, C)$ are both exact homotopy functors. The result follows. □

We now want to show that $T^2 C$ and C are (R, K)-chain equivalent. See 6.5.

DEFINITION 6.3. (of $E_C : Hom_{(R,K)}(\Delta^* K, C) \otimes_K \Delta^* K \to C$).
Let C be an (R, K) complex.

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Consider the evaluation chain map, $eval_{A,B} : Hom_R(A,B) \otimes_R A \to B$, when $A = \Delta^*K(K)$ and $B = C(K)$. Its restriction to $(Hom_{(R,K)}(\Delta^*K, C) \otimes_R \Delta^*K)(K)$, denoted E'_C, is an (R,K) chain map,

$$E'_C : Hom_{(R,K)}(\Delta^*K, C) \otimes_R \Delta^*K \to C$$

(by definition of an (R,K) map). Moreover, for each $\sigma \in K$, E'_C annihilates $Hom_{(R,K)}(\Delta^*K, C)(K - st(\sigma)) \otimes_R \Delta^*K(\sigma)$. Therefore E'_C descends uniquely to an (R,K) chain map,

$$E_C : Hom_{(R,K)}(\Delta^*K, C) \otimes_K \Delta^*K \to C, \quad E_C(f \otimes \sigma^*) = f(\sigma^*).$$

satisfying: $E'_C = E_C \circ \pi_{H,\Delta^*K}$. Here $H = Hom_{(R,K)}(\Delta^*K, C)$ (see 5.1).

E is obviously natural in C.

For each (R,K) complex C, define

$$\Psi_{C^*} = \Psi_{C^*,\Delta^*K} : Hom_{(R,K)}(\Delta^*K, C^{**}) \xrightarrow{\cong} (C^* \otimes_K \Delta^*K)^*$$

In view of lemma 5.2. we have an (R,K) chain isomorphism:

$$\Psi_{C^*} \otimes 1_{\Delta^*K} : Hom_{(R,K)}(\Delta^*K, C^{**}) \otimes_K \Delta^*K \xrightarrow{\cong} (C^* \otimes_K \Delta^*K)^* \otimes_K \Delta^*K = T^2C.$$

Definition 6.4. For each (R,K) complex C define $e_C : T^2C \to C$ by

$$\begin{aligned} &e_C = \varepsilon_C \circ E_{C^{**}} \circ (\Psi_{C^*} \otimes 1_{\Delta^*K})^{-1} : \\ &\quad (C^* \otimes_K \Delta^*K)^* \otimes_K \Delta^*K \to Hom_{(R,K)}(\Delta^*K, C^{**}) \otimes_K \Delta^*K \to C^{**} \to C. \end{aligned}$$

Note e_C is an (R,K) chain epimorphism and e is a natural transformation.

Theorem 6.5. *$e_C : T^2C \longrightarrow C$ is an (R,K) chain equivalence, for each (R,K) complex C.*

Proof. By [**16**] (proposition 4.7), we need only prove that $e_C(\sigma,\sigma) : T^2C(\sigma) \to C(\sigma)$ is an R-chain equivalence, for all $\sigma \in K$. (No proof of this proposition appears in [**16**]. A brief proof appears in Appendix 2).

Case I: Assume there is a simplex $S \in K$ for which: $C(\sigma) = 0 \ \forall \ \sigma \neq S$.

We need only show $e_C(S,S)$ is a chain isomorphism, and $T^2C(\sigma)$ is contractible for $\sigma \neq S$. We compute, for all $\sigma \in K$, in view of the restriction on C:

$$\begin{aligned} TC(st(\sigma)) &= (C^* \otimes_K \Delta^*K)(st(\sigma)) = (C^* \otimes_R \Delta^*\overline{S})(st(\sigma)) \\ &= C^*(S) \otimes_R \Delta^*\overline{S}(st(\sigma)) \\ \text{So: } T^2C(\sigma) &\cong C^{**}(S) \otimes_R \Delta^{**}\overline{S}(st(\sigma)) \otimes_R R\sigma^* \end{aligned}$$

So for $\sigma \neq S$, $T^2C(\sigma)$ is contractible because $\Delta^{**}\overline{S}(st(\sigma))$ is contractible by 4.2.

Next we prove that the map

$$e_C(S,S) = \varepsilon_C(S,S) \circ E_{C^{**}}(S,S) \circ (\Psi^*_C \otimes 1_{\Delta^*K})^{-1}(S,S)$$

is an isomorphism, or equivalently that $E_C(S,S)$ is an isomorphism.

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Assume S has been oriented. Because $C(\sigma) = 0$ for $\sigma \neq S$,

$$E_C(S,S) : [Hom_{(R,K)}(\Delta^* K, C) \otimes_K \Delta^* K](S) \to C(S)$$

is simply: $eval_{RS^*,C(S)} : Hom_R(RS^*, C(S)) \otimes_R RS^* \to C(S)$.

This is a chain isomorphism as observed in § 2. So $e_C(\sigma,\sigma)$ is a chain isomorphism for $\sigma = S$ and a chain equivalence for $\sigma \neq S$. This completes the proof in Case I.

Case II (the general case): For any $C \neq 0$ in $\mathcal{B}R_K$ one can choose some $S \in K$ for which $C(S) \neq 0$, and an exact sequence $0 \to C' \xrightarrow{i} C \xrightarrow{j} C'' \to 0$ for which $i(S,S) : C'(S) \to C(S)$ is an isomorphism, and $C'(\sigma) = 0$ for $\sigma \neq S$. For example, choose S to be of maximum dimension among $\{\sigma \in K \mid C(\sigma) \neq 0\}$).

The argument is by induction on the number n, of $\sigma \in K$, for which $C(\sigma) \neq 0$.

If $n = 1$, Case I applies. If $n > 1$, by induction, $e_{C''}(\sigma,\sigma)$ and $e_{C'}(\sigma,\sigma)$ are R chain equivalences. Also the commuting diagram below has exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T^2C'(\sigma) & \longrightarrow & T^2C(\sigma) & \longrightarrow & T^2C''(\sigma) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle e_{C'}(\sigma,\sigma)} & & \downarrow{\scriptstyle e_C(\sigma,\sigma)} & & \downarrow{\scriptstyle e_{C''}(\sigma,\sigma)} & & \\ 0 & \longrightarrow & C'(\sigma) & \longrightarrow & C(\sigma) & \longrightarrow & C''(\sigma) & \longrightarrow & 0 \end{array}$$

Therefore $e_C(\sigma,\sigma)$ is an R-chain equivalence for all σ. This completes the proof. $\square$

Note: The first proof of the above theorem appeared in [**1**].

7. Construction of the ball complex X_K

The purpose of this section is to construct the complex X_K advertised in the introduction and establish its properties.

Definition 7.1. (of X'): Let X be a finite simplicial complex in a euclidean space, with vertex set V_X. Its underlying polyhedron is: $|X| = \cup\{\sigma \mid \sigma \in X\}$. For each $p \geqslant 0$, X_p denotes the set of p-simplices of X.

If $|X|$ is pl-homeomorphic to I^n we say $|X|$ or X is a pl n-ball and write ∂X for the subcomplex for which $|\partial X| = \partial|X|$.

Each p-simplex $\sigma \in X$ is the convex hull, $[v_0, v_1, \ldots, v_p]$, of its vertices in V_X. Its *barycenter* is $\hat{\sigma} := \frac{1}{p+1}\sum_{i=0}^{p} v_i \in \sigma^\circ$.

Choose a point $b\sigma \in \sigma^\circ$, the interior of σ, for each $\sigma \in X$.

The derived complex X' is defined as the unique simplicial subdivision of X for which $V_{X'} = \{b\sigma \mid \sigma \in X\}$. X' has one p-simplex, $[b\sigma_0, b\sigma_1 \ldots b\sigma_p]$, for each decreasing sequence of simplices $\sigma_0 > \cdots > \sigma_p$ of X.

If $\sigma_0 > \cdots > \sigma_p$, the ordered $p+1$ tuple $(b\sigma_0, b\sigma_1, \ldots, b\sigma_p)$ then specifies an oriented p-simplex in $\Delta_p(X'; R)$ *which we denote* $\langle\sigma_0, \sigma_1 \ldots, \sigma_p\rangle$ (suppressing the barycenters for concision).

These form a canonical basis for $\Delta_p(X'; R)$ (in contrast to $\Delta_p(X; R)$).

Because we want to use the McCrory cap product, we follow the orderings of [**13**] regarding simplices of X'.

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Definition 7.2. (of $\Delta X'$): Let (X,π) be a K-*space*. The *derived complexes of* (X,π) are the simplicial subdivisions X' of X, and K' of K whose vertex sets $\{b\sigma \mid \sigma\in K\}$ and $\{bS \mid S\in X\}$ are chosen as follows:

$$\text{If } \sigma\in K, \quad b\sigma := \hat{\sigma}\in\sigma^{\circ};$$

$$\text{If } S\in X \text{ and } \sigma=\pi(S), \quad bS := \text{ centroid of } (S\cap\pi^{-1}(\hat{\sigma}))\in S^{\circ}.$$

By construction, $\pi(V_{X'})\subset V_{K'}$. So π is also a simplicial map from X' to K', because π is linear on each simplex of X'.

X' provides a second geometric example, $\Delta X'$, of an (R,K) complex:
We define $\Delta X'$ by,

(1) $\Delta X'(K)=\Delta_*(X';R)$.

(2) For each $\sigma\in K, p\in\mathbb{Z}$, $(\Delta X')_p(\sigma)$ is the submodule of $\Delta_p(X';R)$ spanned by all $\langle Q^0,\dots Q^p\rangle$ in X' for which $\sigma=\pi(Q^p)$.

It is straightforward to see that $\Delta X'$ is an (R,K) complex.

The dual cone of a simplex $\sigma\in K$, denoted $D(\sigma,K)$, is a subcomplex of K' first defined in [**15**], § 7. It is a pl ball if K is a pl-manifold). It gives rise to several 'dual' subcomplexes in K' and X' which we define now.

Definition 7.3. Let (X,π) be a K-space. Suppose $\sigma,\tau\in K$, $T\in X$.

(1) $D(\sigma,K) := \{\langle\sigma_0,\sigma_1,\dots,\sigma_p\rangle\in K' \mid \sigma_p\geqslant\sigma\}$

(2) $D(\sigma,\tau) := \{\langle\sigma_0,\sigma_1,\dots,\sigma_p\rangle\in K' \mid \sigma_p\geqslant\sigma,\ \tau\geqslant\sigma_0\}$, the dual cell of σ in τ.

(3) $D_\sigma T := \{\langle S_0,S_1,\dots,S_p\rangle\in X' \mid \sigma\leqslant\pi(S_p),\ S_0\leqslant T\}$

(4) $T_\sigma := |D_\sigma T|$. (Therefore, $T_\sigma=(\pi\mid T)^{-1}|D(\sigma,\pi(T))|$).

Of course, $D(\sigma,\tau)=\emptyset$ unless $\sigma\leqslant\tau$, and $D_\sigma T=\emptyset$ unless $\sigma\leqslant\pi(T)$.
$D_\sigma T$ is a subcomplex of X'. $D(\sigma,K)$ and $D(\sigma,\tau)$ are subcomplexes of K'.

Lemma 7.4. *Let (X,π) be a K-space. Suppose $\sigma\in K$, $T\in X$, and $\sigma\leqslant\pi(T)$.*

(1) *$T_\sigma=|D_\sigma T|$ is a pl ball. $dim(T_\sigma)=dim(T)-dim(\sigma)$.*

(2) *$\partial D_\sigma T=\partial^i D_\sigma T\cup\partial^o D_\sigma T$, (the inner and outer boundaries) where:*

$$\partial^i D_\sigma T=\cup\{D_\rho T \ \mid \sigma<\rho\}; \quad \partial^o D_\sigma T=\cup\{D_\sigma S \ \mid S<T\}$$

(3) *Suppose $\sigma<\pi(T)$. Then $|\partial^i D_\sigma T|$ and $|\partial^o D_\sigma T|$ are pl balls of dimension $dim(D_\sigma T)-1$, and*

$$\partial(\partial^i D_\sigma T)=\partial(\partial^o D_\sigma T)=\partial^i D_\sigma T\cap\partial^o D_\sigma T\ .$$

Proof. of (1): For each vertex v of τ note that,

$$|D(v,\tau)| = \{x \in \tau \mid a_v(x) \geqslant a_w(x), \text{ for all vertices } w \text{ of } \tau\}.$$

where $a_v : |K| \to [0,1]$ denotes the barycentric coordinate function defined by the vertex v. This is a convex subset of τ. So

$$|D(\sigma,\tau)| = \cap_{v\in V(K)} |D(v,\tau)|$$

is also convex. Therefore $T_\sigma = (\pi_{|T})^{-1}(|D(\sigma,\tau)|)$ is also convex since $\pi_{|T} : T \to \tau$ is simplicial. So T_σ is a compact convex polyhedron and therefore a pl ball.

Since $|D(\sigma,\tau)| \cap \tau^\circ \neq \emptyset$, this operator $(\pi_{|T})^{-1}$ preserves codimension:

$$dim(\tau) - dim(D(\sigma,\tau)) = dim(T) - dim(D_\sigma T).$$

Since $dim(D(\sigma,\tau)) = dim(\tau) - dim(\sigma)$, we get: $dim(D_\sigma T) = dim(T) - dim(\sigma)$. □

Proof. of (2): See [**3**], proposition 5.6(2), applied to $\pi_{|\overline{T}} : \overline{T} \to \pi(\overline{T})$. □

Proof. of (3): The equation in (3), and the fact that $|\partial^i D_\sigma T|$ and $|\partial^o D_\sigma T|$ are both pl manifolds, are proved in [**3**] [proposition 5.6 (3),(4)]. To show $|\partial^i D_\sigma T|$ is a pl ball, it suffices to note that it collapses to the vertex bT, and so $|\partial^i D_\sigma T|$ is a regular neighbourhood of bT in $|\partial D_\sigma T|$ (by 3.30 of [**14**]). Then by 3.13 of [**14**], $\partial^o D_\sigma T$ is also a pl ball. □

Definition 7.5. ([**14**] p.27) A *ball complex* is a finite collection $Z = \{B_i\}_{i\in I}$ of pl balls in a euclidean space, such that each point of $|Z| := \cup\{B \mid B \in Z\}$ lies in the interior of precisely one ball of Z, and the boundary of each $B \in Z$ is a union of balls of lesser dimension of Z. Therefore $(|Z|, Z)$ is a regular CW-complex.

Let Z and Y be ball complexes A pl map $f : |Z| \to |Y|$ is a *map of ball complexes* if for each ball B of Z, $f(B)$ is a ball of Y.

Definition 7.6. Let (X,π) be a K-space. We define

$$X_K = \{T_\sigma \mid \sigma \in K,\ T \in X,\ \sigma \leqslant \pi(T)\}$$

Theorem 7.7. *Let (X,π) be a K-space. Then X_K is a ball complex. Moreover X' is a simplicial subdivision of X_K. Also, X_K is a subdivision of X.*

Let $f : (X,\pi_X) \to (Y,\pi_Y)$ is a map of K-spaces. The induced map $f' : X' \to Y'$ of derived complexes is then a map of ball complexes, $f_K : X_K \to Y_K$.

Proof. (The induced map f' means the simplicial map $f' : X' \to Y'$ for which $f'(bS) = b(f(S))$ for each $S \in X$.) By lemma 7.4 the boundary of each T_σ is a

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union of balls of X_K with smaller dimension and

$$T_\sigma^\circ = \coprod \{A^\circ \ \mid A = \langle S_0, \ldots, S_p\rangle \in D_\sigma T,\ A \notin \partial^i D_\sigma T,\ A \notin \partial^o D_\sigma T\}.$$

This can be rewritten as:

$$T_\sigma^\circ = \coprod \{A^\circ \ \mid A = \langle S_0, \ldots, S_p\rangle \in X',\ \sigma = \pi(S_p),\ T = S_0\}, \tag{7.1}$$

By equation (7.1), for each $A \in X'$ there is a unique $T_\sigma \in X_K$ for which $A^\circ \subset T_\sigma^\circ$. Therefore : $|X'| = \coprod\{T_\sigma^\circ \ \mid \ T_\sigma \in X_K\} = |X_K|$.

This proves that X_K is a ball complex and that X' is a subdivision of X_K. Because $T_\sigma \subset T$, we see X_K is a subdivision of X.

Now let $f : (X, \pi_X) \to (Y, \pi_Y)$ be a map of K-spaces. For each simplex $S \in X$ we see $f(S) \in Y$ because f is simplicial. For each face σ of $\pi_X(S)$ in K, we see from the definitions that $f'(D_\sigma S) = D_\sigma f(S)$. So f' is a map of ball complexes, $f_K : X_K \to Y_K$. □

8. The isomorphism $\Phi_X : T\Delta^* X \cong C(X_K)$

Our main theorem is:

THEOREM 8.1. *For each K-space (X, π) the cellular chain complex of X_K with R coefficients, denoted $C(X_K)$, comes with a natural (R, K)complex structure. There is defined (below) an isomorphism of (R, K) chain complexes:*

$$\Phi_X : T\Delta^* X \cong C(X_K)\ .$$

For each map $f : (X, \pi_X) \to (Y, \pi_Y)$ of K-spaces, the square below commutes.

$$\begin{array}{ccc} T(\Delta^* X) & \xrightarrow{T(f^*)} & T(\Delta^* Y) \\ {\scriptstyle \Phi_X}\downarrow & & \downarrow{\scriptstyle \Phi_Y} \\ C(X_K) & \xrightarrow{f_K} & C(Y_K) \end{array}$$

Proof. Choose a basis bK of oriented cells for $\Delta_*(K; R)$. Choose next, a basis b_*X of oriented cells for $\Delta_*(X; R)$. But choose the orientations in b_*X so that if $T \in b_*X$ and $\sigma \in bK$ are both q-cells, and if $\pi_*(T) = \pm\sigma \in \Delta_q(K; R)$, then:

$$\pi_*(T) = (-1)^{dim(\sigma)}\sigma \in \Delta_q(K; R).$$

We call such a pair, (bK, b_*X) *an orientation for* (X, π).

Our first task is to construct the cellular chain complex $C_*(X_K; R)$ as the underlying R-complex of an (R, K) complex $C(X_K)$. Define

$$C(X_K) = \Delta X \otimes_K \Delta^* K; \quad C_*(X_K; R) = (\Delta X \otimes_K \Delta^* K)(K)$$

For each oriented simplex $\rho \in bK$ and oriented simplex $T \in b_*X$, define

$$[T_\rho] = T \otimes_K \rho^* \in C_{|T| - |\sigma|}(X_K; R) \quad (\text{where } |\sigma| = dim(\sigma)).$$

(The geometric intuition for this definition is the fact that, the map C_X, of corollary 9.3, takes $T \otimes_K \rho^*$ to a fundamental cycle, in $\Delta X'$, for the cell $D_\rho T$, whose underlying space is T_ρ).

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Define $bX_K = \{[T_\rho] \mid T \in b_*X,\ \rho \in bK,\ T_\rho \leqslant \pi(T)\}$. Then bX_K is an R-basis for $C_*(X_K; R)$ in bicorrespondence with the cells of X_K. Write ∂_q for the boundary map in $C_*(X_K; R)$, namely: $\partial_q = (d^{\Delta X \otimes_K \Delta^* K})_q$.

But to justify these definitions, we must check that $C_*(X_K; R)$ does compute the cellular homology of X_K. It suffices to check, for any $[T_\rho] \in bX_K$, that $\partial_q([T_\rho])$ is a sum with ± 1 coefficients of those $[S_\sigma] \in bX_K$ which are $(q-1)$-faces of T_ρ. (See [7], for example.)

All proper faces of T_ρ have the form T_σ, for $\rho < \sigma$, or S_ρ, for $S < T$.

Suppose $[T_\rho] \in bX_K$. So $T \in b_*X$, $\rho \in bK$. Set $\tau = \pi(T) \in K$. By (5.1):

$$\begin{aligned}\partial_q[T_\rho] &= d^{\Delta X \otimes_K \Delta^* K}(T \otimes_K \rho^*)\\ &= \sum_{\{\sigma \mid \rho \leqslant \sigma \leqslant \tau\}} \{(d^{\Delta X}(\sigma, \tau)T) \otimes \rho^* + (-1)^{|T|} T \otimes d^{\Delta^* K}(\sigma, \rho)\rho^*\}\\ &= \sum_{S<T} [T, S][S_\rho] + (-1)^{1+|T_\rho|} \sum_{\rho<\sigma} [\sigma, \rho][T_\sigma]\end{aligned}$$

which is as required.

This completes the construction of the cellular chain complex of X_K, as an (R, K) complex, $C(X_K)$.

The (R, K) isomorphism, $\Phi_X : T\Delta^* X \cong C(X_K)$ is simply:

$$\Phi_X := (\varepsilon_{\Delta X} \otimes_K 1_{\Delta^* K}) : T\Delta^* X = \Delta^{**} X \otimes_K \Delta^* K \longrightarrow \Delta X \otimes_K \Delta^* K = C(X_K).$$

Naturality of Φ is obvious from the naturality of ε. $\square$

9. The McCrory cap product, $\Delta^* X$ and $\Delta X'$

We now use the work of McCrory [13] to construct, for any K-space, (X, π), an (R, K) chain monomorphism $C(X_K) \xrightarrow{C_X} \Delta X'$. serving two purposes.

First, it defines an (R, K) chain homotopy equivalence, $T\Delta^* X \simeq \Delta X'$.

Second, C_X identifies $C(X_K)$ with that (R, K) subcomplex of $\Delta X'$ which admits a basis consisting of one fundamental q-cycle, in $\Delta_q(D_\sigma T, \partial D_\sigma T) \subset \Delta_q(X')$, for each q-cell T_σ of X_K. (This will complete our geometric interpretation of T).

Let K be a finite simplicial complex. McCrory (see [13], and also [11]) defines a map, $c' : \Delta_*(K; R) \otimes_R \Delta^*(K; R) \to \Delta_*(K'; R)$ which he shows is chain homotopic to the composite,

$$\Delta_*(K; R) \otimes_R \Delta^*(K; R) \xrightarrow{\cap} \Delta_*(K; R) \xrightarrow{Sd} \Delta_*(K'; R)$$

where $\cap$ denotes the Whitney-Cech cap product. We will write c_K for c'. We repeat his definition here with appropriate sign changes because McCrory's sign conventions differ slightly from ours.

For any q-simplex, $Q = \langle Q^0, Q^1, \ldots Q^q \rangle$ of K' *in which each* Q_i *is oriented*, McCrory then defines

$$\varepsilon(Q) = [Q^0, Q^1][Q^1, Q^2] \ldots [Q^{q-1}, Q^q].$$

This is independent of the orientations on $Q_1, Q_2, \ldots Q_{q-1}$. If $q = 0$, set $\varepsilon(Q) = 0$.

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For any n-simplex τ and $(n-q)$-simplex σ of K, each simplex $Q = \langle Q^0, Q^1, \dots Q^q\rangle$ of $D(\sigma,\tau)_q$ satisfies: $Q^0 = \tau$; $Q^q = \sigma$. Therefore, $\varepsilon(Q)$ makes sense if τ and σ are oriented simplices chosen from some basis bK of oriented simplices for $\Delta_*(K;R)$ (but not if $\sigma = -\tau$).

The *McCrory Cap Product*, $\Delta_*(K;R) \otimes_R \Delta^*(K;R) \xrightarrow{c_K} \Delta_*(K';R)$ is the map defined by:

$$\boxed{c_K(\tau \otimes \sigma^*) = \sum_{Q \in D(\sigma,\tau)_q} (-1)^{dim(\sigma)} \varepsilon(Q) Q}$$

for any oriented simplices σ, τ in some basis bK. Here $q = dim(\tau) - dim(\sigma)$. Note this is zero unless $\sigma \leqslant \tau$. Note that c_K does not change if we change the basis.

c_K is a chain map. We reprove this in Appendix I, § A, because of the sign changes and because McCrory's proof, [**13**] p.155 lines 7-8, is only a sketch.

Now suppose (X, π) is a K-space.

Note that if T and σ are oriented simplices of X and K and $q = dim(T) - dim(\sigma) \neq 0$:

$$c_X(T \otimes_R \pi^*\sigma^*) = \sum_{Q \in (D_\sigma, T)_q} (-1)^{dim(\sigma)} \varepsilon(Q) Q \ \in \Delta_q X'(\sigma)$$

(because $D_\sigma T = \cup\{D(S,T) \mid S \in X, dim(S) = dim(\sigma), \pi(S) = \sigma\}$). This formula still makes sense and is true if $q = 0$ and $\pi_*(T) \neq -\sigma$).

In this way, $c_X \circ (1 \otimes \pi^*)$ defines an (R,K) chain map,

$$c_X \circ (1 \otimes \pi^*) : \Delta X \otimes_R \Delta^* K \longrightarrow \Delta X'$$

Proposition 9.1. *There is a unique (R,K) chain map*

$$C_X : C(X_K) = \Delta X \otimes_K \Delta^* K \longrightarrow \Delta X'$$

satisfying:

$$c_X \circ (1 \otimes \pi^*) = C_X \circ \pi_{\Delta X, \Delta^* K}$$

C_X is an (R,K) monomorphism. For all q-cells T_σ of X_K, with $q \neq 0$,

$$C_X(T \otimes_K \sigma^*) = \sum_{Q \in (D_\sigma T)_q} (-1)^{dim(\sigma)} \varepsilon(Q) Q.$$

For a 0-cell T_σ, of X_K, with $T \in \Delta_n(X;R), \sigma \in \Delta_n(K;R)$ oriented so that $\pi_*(T) = \sigma$, then

$$C_X(T \otimes_K \sigma^*) = (-1)^{dim(T)} \langle T \rangle, \quad (\langle T \rangle \text{ is the barycenter } bT \text{ of } T).$$

Proof. Note that $c_X(T \otimes_R \pi^*\sigma^*) = 0$ unless $\pi(T) \geqslant \sigma$. Also $c_X(T \otimes_R \pi^*\sigma^*) \in \Delta X'(\sigma)$ for all $\sigma \in K$ and $T \in X$ because each q-cell $Q \in D_\sigma T$ is in $\Delta_q X'(\sigma)$ if $q = dim(T_\sigma)$.

So $c_X \circ (1 \otimes \pi^*) : \Delta X \otimes_R \Delta^* K \to \Delta X'$ is an (R,K) chain map annihilating each $\Delta X(K - st(\sigma)) \otimes_R \Delta^* K(\sigma)$. Hence there is a unique (R,K) chain map monomorphism, $\Delta X \otimes_K \Delta^* K \xrightarrow{C_X} \Delta X'$ such that $c_X \circ (1 \otimes \pi^*) = C_X \circ \pi_{\Delta X, \Delta^* K}$. The calculation follows if $q \neq 0$. If $q = 0$, then $(\pi_{|T})^* \sigma^* = T^*$, so

$$C_X(T \otimes_K \sigma^*) = c_X(T \otimes T^*) = (-1)^{dim(T)} \sum_{Q \in D(T,T)_0} Q = (-1)^{dim(T)} \langle T \rangle$$

Clearly C_X is natural in (X, π). □

Remark 9.2. If we choose an orientation $(bK, b_* X)$ for X_K, then for each 0-cell $T_\sigma = T_{\pi(T)}$ of X_K, with $[T_\sigma] \in bX_K$, we have $C_X([T_\sigma]) = \langle T \rangle \in \Delta_0(X'; R)$.

Corollary 9.3. *For each q-cell T_σ of X_K, $C_X(T \otimes \sigma^*)$ is a fundamental cycle, in $\Delta_q(D_\sigma T, \partial D_\sigma T; R)$ for the q-manifold $D_\sigma T$.*

Proof. $C_X(T \otimes_K \sigma^*)$ is a fundamental cycle in $\Delta_q(D_\sigma T, \partial D_\sigma T; R)$ since C_X is a chain map and since each $Q \in (D_\sigma T)_q$ appears with coefficient ± 1 in $C_X(T \otimes_K \sigma^*)$. □

Theorem 9.4. *For each K-space (X, π), the map $C(X_K) \xrightarrow{C_X} \Delta X'$ is an (R, K) chain homotopy equivalence.*

Proof. By 9.3, for all T_σ, C_X restricts to a homotopy equivalence,

$$C_*(T_\sigma, \partial T_\sigma; R) \to \Delta_*(D_\sigma(T), \partial D_\sigma(T); R)$$

and it takes chains on any subcomplex of X_K to chains on its subdivision. By an induction-excision argument on the number of cells in the subcomplex one sees C_X yields a homology equivalence and then a chain homotopy equivalence on each such subcomplex. So $C_X(\sigma, \sigma)$ is an R-chain equivalence for each σ. Therefore C_X is an (R, K) chain equivalence. □

Together, 9.4 and 8.1 clearly prove:

Corollary 9.5. *$T\Delta^* X \xrightarrow{C_X \Phi_X} \Delta X'$ is an (R, K) chain homotopy equivalence. Consequently $e_{\Delta^* X} \circ T(C_X \Phi_X)$ is an explicit (R, K) chain homotopy equivalence,*

$$T\Delta X' \simeq \Delta^* X.$$

Appendix A.

We must prove:

Proposition A.1. *$\Delta_*(K; R) \otimes_R \Delta^*(K; R) \xrightarrow{c_K} \Delta_*(K'; R)$ is a chain map. That is to say, for any oriented simplices σ, τ in some basis bK for ΔK, with $p = dim(\tau) - dim(\sigma)$,*

$$d^{K'} c_K(\tau \otimes \sigma^*) = c_K\{d^K \tau \otimes \sigma^* + (-1)^{dim(\tau)} \tau \otimes d^{\Delta^*(K)} \sigma^*\}$$

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where, by the definitions,

$$d^K\tau = \sum_{\rho\in b_K}[\tau,\rho]\rho, \quad d^{\Delta^*(K)}\sigma^* = (-1)^{dim(\sigma)+1}\sum_{\rho\in b_K}[\rho,\sigma]\rho^*$$

and for any p-simplex $Q = \langle Q^0, Q^1, \dots Q^p\rangle$ of K',

$$d^{K'}Q = \sum_{i=0}^{p}(-1)^i d^i(Q); \quad d^i(Q) = \langle Q^0, Q^1, \dots \hat{Q}^i \dots Q^p\rangle$$

Proof. We first prove: $d^o c(\tau\otimes\sigma^*) = c(d^K\tau\otimes\sigma^*)$, where $c = c_K$.

$$\begin{aligned} d^0c(\tau\otimes\sigma) &= (-1)^{dim(\sigma)}\sum_{Q\in D(\sigma,\tau)_p}\varepsilon(Q)\langle Q^1,\dots Q^p\rangle \\ &= (-1)^{dim(\sigma)}\sum_{\rho\in b_K}[\tau,\rho]\sum_{P\in D(\sigma,\rho)}\varepsilon(P)P \\ &= c(\sum_{\rho\in b_K}[\tau,\rho]\rho\otimes\sigma^*) = c(d^K\tau\otimes\sigma^*). \end{aligned}$$

Next we show: $(-1)^p d^p c(\tau\otimes\sigma^*) = (-1)^{dim(\tau)}c(\tau\otimes d^{\Delta^*(K)}\sigma^*)$:

$$\begin{aligned} (-1)^p d^p c(\tau\otimes\sigma^*) &= (-1)^{p+dim(\sigma)}\sum_{Q\in D(\sigma,\tau)_p}\varepsilon(Q)\langle\tau, Q^1\dots Q^{p-1}\rangle \\ &= (-1)^{p+1}c(\tau\otimes\sum_{\rho\in b_K}[\rho,\sigma]\rho^*) = (-1)^{dim(\tau)}c(\tau\otimes d^{\Delta^*(K)}\sigma^*) \end{aligned}$$

Finally we prove $d^i c(\tau\otimes\sigma^*) = 0$ for $0 < i < p$.

For such i and for $Q\in D(\sigma,\tau)$ note $d^iQ = \langle\tau,\dots\sigma\rangle \in D(\sigma,\tau) - \partial D(\sigma,\tau)$. So suppose P is a $p-1$ simplex of the form d^iQ in the p manifold $D(\sigma,\tau)$. Then there is exactly one other $S\in D(\sigma,\tau)_p$ having Q as a face. We can identify S by listing the vertices of τ as $v_0,\dots v_n$ so that $Q^j = [v_j,\dots v_n]$ for all j. Define $S^i = [v_0\dots v_{i-1}, v_{i+1}\dots v_n]$ and define $S^j = Q^j$ for $j\neq i$. Then $S := \langle S^0, S^1,\dots S^p\rangle$ in $D(\sigma,\tau)_p$ satisfies $d^iS = P$; $\varepsilon(S) = -\varepsilon(Q)$ so P must appear with zero coefficient in $d^ic(\tau\otimes\sigma^*)$ for all $p-1$ simplices P. So $d^ic(\tau\otimes\sigma^*) = 0$. □

Appendix B.

We must prove the following result of Ranicki and Weiss:

PROPOSITION B.1. *Let $i : A\to B$ be an (R,K) chain map in $\mathcal{B}R_K$ for some finite poset K. Then i is a chain equivalence in $\mathcal{B}R_K$ if and only if $i(\sigma,\sigma)$ is a chain equivalence in $\mathcal{B}R$ for all $\sigma\in K$.*

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LEMMA B.2. *[(The Contraction Principle)]: For any additive category A, with the split exact structure, and any exact sequence of chain complexes in $\mathcal{B}A$,*

$$0 \to C' \xrightarrow{f} C \xrightarrow{g} C'' \to 0$$

C'' is contractible if and only if f has a left inverse $r : C \to C'$ which is a chain homotopy inverse of f.

Proof. For any $h'' \in Hom_A(C'', C'')_1$ there is an $h \in Hom_A(C, C)_1$ such that $gh = h''g$ and $hf = 0$. Then h'' is a contraction of C'' iff h is a chain homotopy from 1_C to a chain map $\rho : C \to C$ for which $\rho = fr$ for some chain map $r : C \to C'$. r satisfies $rf = 1_{C'}$. So r is a left inverse of f and fr is chain homotopic to 1_C. □

Proof. of B.1: First assume i is a chain equivalence. Note, for each $\sigma \in K$, the functor $B \to B(\sigma)$ is an additive functor $AR_K \to A_R$. So it induces a homotopy functor $\mathcal{B}R_K \to \mathcal{B}R$. Therefore $i(\sigma, \sigma)$ is a chain equivalence for each $\sigma \in K$.

Conversely suppose $i(\sigma, \sigma)$ is a chain equivalence in $\mathcal{B}R$ for all $\sigma \in K$. We prove that i is a chain equivalence in $\mathcal{B}R_K$. Replacing B by the mapping cylinder of i if necessary, we can assume i fits into an exact sequence, $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$.

By B.2 then, each $C(\sigma)$ is contractible, and we have only to prove the claim that C is contractible. The proof is by induction on the number, $n(C)$, of $\sigma \in K$ for which $C(\sigma) \neq 0$. If $n = 0$ we are done. We can assume this claim is proved for complexes C' for which $0 \leqslant n(C') < n(C)$.

There is some $\rho \in K$ for which $C(\rho) \neq 0$, and an exact sequence of the form:

$$0 \to C' \xrightarrow{f} C \xrightarrow{g} C'' \to 0$$

for which $f(\rho, \rho)$ is an isomorphism, and $g(\sigma, \sigma)$ is an isomorphism for all $\sigma \neq \rho$. (For example pick ρ to be maximal in $\{\sigma \in K \mid C(\sigma) \neq 0\}$). C' is contractible because $C(\rho)$ is contractible. But C'' is contractible by induction, so that f is a chain equivalence, by B.2. So C is contractible as claimed. □

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