A REMARK ON DIFFERENTIABLE STRUCTURES ON REAL PROJECTIVE $(2n-1)$-SPACES

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Dedicated to the memory of Professor TADASI NAKAYAMA

The main objective of this paper is to study the action of the group of differentiable structures Γ_{2n-1} on the $(2n-1)$-sphere S^{2n-1} on the diffeomorphism classes on the real projective $(2n-1)$-space P^{2n-1} by connected sum. This is done by considering universal covering spaces of the connected sum $P^{2n-1} \# \sum$, where $\sum$ is an exotic $(2n-1)$-sphere.

Throughout this paper all the manifolds considered are oriented, compact, and connected. Also the word *differetiable* is meant C^∞-*differentiable*.

1. Let M, N be n-dimensional differentiable manifolds. If there exists an orientation preserving diffeomorphism of M onto N, then we shall denote it by $M \approx N$. The manifold M with orientation reversed is denoted by $-M$.

Let $M_1 \# M_2$ be the connected sum of two n-manifolds M_1 and M_2. It is known that the connected sum operation is associative and commutative up to orientation preserving diffeomorphism. The sphere S^n serves as identity element (Cf. J. Milnor [2], M. Kervaire- J. Milnor [1]).

Let $\sum$ be a smooth combinatorial n-sphere, which is called an exotic sphere. Then $\sum \# (-\sum) \approx S^n$. Thus the set of all the orientation preserving diffeomorphism classes of the exotic spheres forms a group under connected sum, which is denoted by Γ_n. We shall denote the class of $\sum$ by $\{\sum\}$.

Let M be a differentiable n-manifold. Let $\sum$ be an exotic n-sphere such that $M \# \sum \approx M$. Let $\Delta(M)$ be the subset of Γ_n consisting of the classes of such $\sum$.

PROPOSITION 1. *$\Delta(M)$ is a subgroup of Γ_n, and $M \# \sum_1 \approx M \# \sum_2$, for exotic spheres $\sum_1$, $\sum_2$, if and only if $\{\sum_1\} - \{\sum_2\} \in \Delta(M)$.*

Proof. Let $\{\sum_1\}$, $\{\sum_2\} \in \Delta(M)$. Then

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$$M \# (\Sigma_1 \# \Sigma_2) \approx (M \# \Sigma_1) \# \Sigma_2 \approx M \# \Sigma_2 \approx M.$$

Also, $M \# (-\Sigma_1) \approx (M \# \Sigma_1) \# (-\Sigma_1) \approx M \# (\Sigma_1 \# (-\Sigma_1)) \approx M \# S^n \approx M.$ Thus $\Delta(M)$ is a subgroup of Γ_n.

Secondly, let $M \# \Sigma_1 \approx M \# \Sigma_2$. Then $M \# \Sigma_1 \# (-\Sigma_2) \approx M \# \Sigma_2 \# (-\Sigma_2) \approx M \# S^n \approx M$, that is $\{\Sigma_1 \# (-\Sigma_2)\} \in \Delta(M)$. Conversely, let $\{\Sigma_1 \# (-\Sigma_2)\} \in \Delta(M)$, then $M \approx M \# \Sigma_1 \# (-\Sigma_2)$. Adding Σ_2 from the right, we have $M \# \Sigma_2 \approx M \# \Sigma_1$.

Let $[G]$ be the order of a group G. Then we have the following.

Corollary. The action of Γ_n on the orientation preserving diffeomorphism classes of a manifold M by connected sum is completely determined by $\Delta(M)$. In particular, the number of the orientation preserving diffeomorphism classes of M obtained by connected sum with exotic spheres is equal to $[\Gamma_n/\Delta(M)]$.

2. Let M be a differentiable n-manifold such that the fundamental group $\pi_1(M)$ is a finite group of order $p = [\pi_1(M)]$. Let $\tilde{M}$ be the universal covering space of M. Let N be a simply connected differentiable n-manifold.

PROPOSITION 2. *Under the above assumption and $n \geqslant 3$, the universal covering manifold of $M \# N$ is $\tilde{M} \# N \cdots \# N$ (p-factors of N).*

Proof. Let D^n be the unit disc in the euclidean n-space R^n. Let $f : D^n \to M$, $g : D^n \to N$ be differentiable imbeddings such that f preserves orientation and g reverses orientation. Then $M \# N$ is obtained from the disjoint sum

$$(M - f(0)) + (N - g(0))$$

by identifying $f(tx)$ with $g((1-t)x)$ for each $x \in S^{n-1} = \partial D^n$ and each $0 < t < 1$, where 0 is the origin of R^n.

Let $\pi : \tilde{M} \to M$ be the natural projection of the universal covering manifold $\tilde{M}$ onto M. We can assume without loss of generality that $f(D^n)$ is contained in an open connected set U of M such that for each connected component U_i ($i = 1, 2, \ldots, p$) of $\pi^{-1}(U)$, $\pi_i = \pi | U_i : U_i \to U$ is a diffeomorphism onto. Let $f_i = \pi_i^{-1} \circ f : D^n \to U_i \subset \tilde{M}$. Then f_i is an orientation preserving diffeomorphism into.

Let $N_1, \ldots, N_p$ be p copies of N. We shall denote a point of N_i corresponding to $x \in N$ by x_i. Let $g_i : D^n \to N_i$ ($i = 1, 2, \ldots, p$) be copies of g, that is $g_i(x) = (g(x))_i$. Then $\tilde{M} \# N_1 \ldots \# N_p$ is obtained from the disjoint sum

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$$(\tilde{M}-\bigcup_{i=1}^{p} f_i(0))+(N_1-g_1(0))+\cdots+(N_p-g_p(0))$$

by identifying $f_i(tx)$ with $g_i((1-t)x)$ for each $x \in S^{n-1}$, each $0<t<1$, and $i=1, 2, \ldots, p$.

Let $\pi' : M \# N_1 \# \ldots \# N_p \to M \# N$ be the map defined by

$$\pi'(x)=\pi(x) \quad \text{for } x \in (\tilde{M}-\bigcup_{i=1}^{p} f_i(0)) \text{ and}$$

$$\pi'(x_i)=x \quad \text{for } x_i \in N_i - g_i(0), \ (i=1, 2, \ldots, p).$$

Then π' is well defined and a local diffeomorphism onto $M \# N$. Thus, $\tilde{M} \# N_1 \# \ldots \# N_p$ is a covering manifold of $M \# N$.

On the other hand, $\tilde{M} \# N_1 \# \ldots \# N_p$ is simply connected as is seen by the following lemma. Therefore, the proposition is proved.

Lemma. Let M, N be simply connected n-manifolds and $n \geqslant 3$. Then $M \# N$ is simply connected.

Let $f : D^n \to M$, $g : D^n \to N$ be immbeddings as before. Then the connected sum $M \# N$ is obtained topologically from the disjoint sum $(M-f(\text{Int } D^n))+(N-g(\text{Int } D^n))$ by identifying $f(x)$ with $g(x)$ for $x \in S^{n-1}$, where Int D^n is the interior of D^n. Let K (resp. L) be the image of $(M-f(\text{Int } D^n))$ (resp. $N-g(\text{Int } D^n)$). Then $M \# N = K \cup L$ and $K \cap L$ is homeomorphic to S^{n-1} by f (or g). Thus $K \cap L$ is simply connected. Since K and $M-f(0)$ have the same homotopy type, K is simply connected by our assumption that M is simply connected and $n \geqslant 3$. By the same reason L is simply connected. It follows from the van Kampen's theorem that $M \# N = K \cup L$ is simply connected.

Let S^{2n-1} be the unit sphere in complex n-space C^n. That is a point of S^{2n-1} has a form $(z_1, \ldots, z_n)$, where z_i is a complex number and $\sum_{i-1}^{n} z_i \bar{z}_i = 1$. Let $\lambda = \exp(2\pi i/p)$, where p is a natural number. Let $T : S^{2n-1} \to S^{2n-1}$ be transformation defined by $T(z_1, \ldots, z_n)=(\lambda^{g1}z, \ldots, \lambda^{gn}z)$, where $q_1, \ldots, q_n$ are integers prime to p. Then T is a fixed point free transformation of period p. The orbit space $S^{2n-1}/T = L_p(q_1, \ldots, q_n)=L_p$ is an oriented differentiable manifold in a natural way. The natural projection $\eta : S^{2n-1} \to L_p$ is the universal covering. If $p=2$, $L_2 = P^{2n-1}$, the real projective $(2n-1)$-space. It is well known that $\pi_1(L_p)=Z_p$, the group of integers modulo p.

COROLLARY. *The universal covering space of $L_p \# \sum$, where $\sum$ is an exotic*

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sphere, is $S^{2n-1} \# \sum \# \cdots \# \sum \approx \sum \# \cdots \# \sum$ *(p-factors of* $\sum$*)* $(n \geqq 2)$.

In particular, the universal covering space of $P^{2n-1} \# \sum$ *is* $S^{2n-1} \# \sum \# \sum \approx \sum \# \sum$.

In conclusion, we have the following theorem.

THEOREM. *There exist at least* $[2\Gamma_{2n-1}]$ *distinct differentiable structures on* P^{2n-1} *up to orientation preserving diffeomorphism, where* $[2\Gamma_{2n-1}]$ *is the order of* $2\Gamma_{2n-1}$.

Also, there exist at least $[p\Gamma_{2n-1}]$ *distinct differentiable structures on a lens space* L_p.

Proof. Let $\sum$ be an exotic sphere such that $P^{2n-1} \# \sum \approx P^{2-1}$. Then the universal covering spaces must be diffeomorphic, that is $\sum \# \sum \approx S^{2n-1}$. Thus $\varDelta(P^{2n-1}) \subset {}_2\Gamma_{2n-1}$, where ${}_2\Gamma_{2n-1}$ is the subgroup of Γ_{2n-1} consisting of the elements of order two. Thus

$$[\Gamma_{2n-1}/\varDelta(P^{2n-1})] \geqq [\Gamma_{2n-1}/{}_2\Gamma_{2n-1}] = [2\Gamma_{2n-1}].$$

Our theorem follows from the fact that $P^{2n-1} \# \sum$ is homeomorphic to P^{2n-1} and the Corollary of Proposition 1.

The same argument shows the latter half of the theorem.

According to the work of Kervaire-Milnor [1], Γ_{4m-1} $(m \geqq 2)$ contains a cyclic group of order 2^{2m-2} $(2^{2m-1}-1)\cdot$numerator $(4B_m/m)$, where B_m denotes the m-th Bernoulli number. Therefore, there exist at least 2^{2m-3} $(2^{2m-1}-1)\cdot$ numerator $(4B_m/m)$ distinct differentiable structures on P^{4m-1} $(m \geqq 2)$. For example, 14 on P^7, 496 on P^{11}, 4064 on P^{15}, and etc..

REFERENCES

[1] M. Kervaire-J. Milnor, Groups of homotopy spheres, I, Ann. of Math., **77** (1963), 504–537.

[2] J. Milnor, Sommes de variétés differentiables et structures différentiables des sphères, Bull. Soc. Math. France **87** (1959), 439–444.

[3] M. Hirsch-J. Milnor, Some curious involutions of spheres, Bull. A.M.S. **70** (1964), 372–377.

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