In this paper we study the following problem: To what extent does the type of the Gauss map of a submanifold of $E^m$ determine the submanifold? Several results in this respect are obtained. In particular, submanifolds with 1-type Gauss map are characterized. Surfaces with 1-type Gauss map and minimal surfaces of $S^{m-1}$ with 2-type Gauss map are completely classified. Some applications are also given.

1. Introduction.

A compact submanifold $M$ of a Euclidean $m$-space $E^m$ is said to be of finite type if the immersion $x$ of $M$ in $E^m$ can be expressed as a finite sum of $E^m$-valued eigenfunctions of the Laplacian $\Delta$ of $M$, acting on $E^m$-valued functions. Minimal submanifolds of a hypersphere

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161
and equivariant immersions of a compact homogeneous space are the simplest and best known examples of finite type submanifolds (see [5, 7]).

Similarly, a smooth map \( \phi \) of a compact Riemannian manifold \( M \) into \( E^m \) is said to be of finite type if \( \phi \) is a finite sum of \( E^m \)-valued eigenfunctions of \( \Delta \). Some fundamental results on finite type maps are given in [9].

For an isometric immersion \( x : M \to E^m \) of a compact oriented \( n \)-dimensional Riemannian manifold \( M \) into \( E^m \), the Gauss map \( v : M \to G(n,m) \) of \( x \) is a smooth map which carries a point \( p \) in \( M \) into the oriented \( n \)-plane in \( E^m \) which is obtained from the parallel translation of the tangent space of \( M \) at \( p \) in \( E^m \) (where \( G(n,m) \) is the Grassmannian consisting of all oriented \( n \)-planes through the origin of \( E^m \)). Since \( G(n,m) \) is canonically imbedded in \( \mathbb{R}^n E^m = E^N \), \( N = \binom{m}{n} \), the notion of finite-type Gauss map is naturally defined.

The main purpose of this paper is to study the following problem: To what extent does the type of the Gauss map of a submanifold of \( E^m \) determine the submanifold?

For closed curves in \( E^m \), the type of a curve in \( E^m \) coincides with that of its Gauss map (Proposition 3.1). In contrast, for submanifolds of dimension \( \geq 2 \), the two notions are different.

A well-known result of Takahashi says that a compact submanifold of \( E^m \) is of \( I \)-type if and only if it is a minimal submanifold of a hypersphere. In Section 4 we study the following problem: Which submanifolds of \( E^m \) have \( I \)-type Gauss map? In this respect, we obtain a characterization theorem for submanifolds with \( I \)-type Gauss map. This result is then applied to obtain some classification theorems of such submanifolds. In Section 5, we show that a standard isometric immersion of an ordinary 2-sphere has \( 2 \)-type Gauss map if and only if it is not the first standard imbedding. The complete classification of flat minimal tori in \( S^{m-1} \) with \( 2 \)-type Gauss map is given in Section 6. In the last section, we give the complete classification of minimal surfaces of \( S^{m-1} \).
with 2-type Gauss map (Theorem 7.1).

2. Preliminaries.

Let $M$ be a compact Riemannian manifold and $\Delta$ the Laplacian of $M$ acting on the space $C^\infty(M)$ of smooth functions. Then $\Delta$ has an infinite discrete sequence of eigenvalues:

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_k < \ldots \rightarrow \infty.$$ 

For each $k = 0, 1, 2, \ldots$, the eigenspace $V_k = \{f \in C^\infty(M) \mid \Delta f = \lambda_k f\}$ is finite-dimensional. With respect to the inner product $(f, g) = \int fg\,dV$ on $C^\infty(M)$, the decomposition $\sum_k V_k$ is orthogonal and dense in $C^\infty(M)$.

Therefore, for any $f \in C^\infty(M)$, we have $f = f_0 + \sum_{t \geq 1} f_t$, where $f_0$ is a constant and $f_t$ is the projection of $f$ into $V_t$.

For any smooth map $\phi : M \rightarrow E^m$ of a Riemannian manifold $M$ into the Euclidean $m$-space $E^m$, we can apply the above decomposition to the $E^m$-valued function $\phi$:

\begin{equation}
\phi = \phi_0 + \sum_{t=1}^\infty \phi_t,
\end{equation}

where $\phi_0$ is a constant vector which is called the centre of gravity of $\phi$. The map $\phi$ is said to be of finite type if there exist only finitely many nonzero terms in the decomposition (2.1). More precisely, $\phi$ is said to be of $k$-type if there exist exactly $k$ nonzero $\phi_t$'s ($t \geq 1$) in the decomposition.

If the map $\phi$ is an isometric immersion, then $M$ is called a submanifold of finite type (or of $k$-type) if $\phi$ does.

The following result is known (see [5,7]).

**Theorem 2.1.** Let $x : M \rightarrow E^m$ be an isometric immersion of a compact Riemannian manifold $M$ into $E^m$ and let $H$ be the mean curvature vector of $M$ in $E^m$. Then we have

(i) $M$ is of finite type if and only if there is a nontrivial polynomial $Q(t)$ such that $Q(t)H = 0$. 


(ii) If $M$ is of finite type, there is a unique monic polynomial $P(t)$ of least degree with $P(A)H = 0$.

(iii) If $M$ is of finite type, then $M$ is of $k$-type if and only if $\deg P = k$.

The same results hold if $H$ is replaced by $x - x_0$.

For smooth maps, we have the following result analogous to Theorem 2.1, whose proof is the same as that of Theorem 2.1.

THEOREM 2.2. Let $\psi : M \to E^m$ be a smooth map from a compact Riemannian manifold $M$ into $E^n$ and let $\tau = \text{div}(d\psi)$ be the tension field of $\psi$. Then we have

(i) $\psi$ is of finite type if and only if there is a nontrivial polynomial $Q(t)$ such that $Q(\Delta)\tau = 0$.

(ii) If $\psi$ is of finite type, there is a unique monic polynomial $P(t)$ of least degree with $P(\Delta)\tau = 0$.

(iii) If $\psi$ is of finite type, then $\psi$ is of $k$-type if and only if $\deg P = k$.

The same results hold if $\tau$ is replaced by $\psi - \psi_0$.

The unique monic polynomial $P$ mentioned in Theorem 2.1 (respectively, in Theorem 2.2) is called the minimal polynomial of the finite type submanifold $M$ (respectively, of the finite type map $\psi$).

3. Gauss Map.

Let $V$ be an oriented $n$-plane in $E^m$. Denote by $e_1, \ldots, e_n$ an oriented orthonormal basis of $V$. Then $e_1 \wedge \ldots \wedge e_n$ is a decomposable $n$-vector of norm 1 and $e_1 \wedge \ldots \wedge e_n$ gives the orientation on $V$.

Conversely, for any decomposable $n$-vector of norm 1, it determines a unique oriented $n$-plane in $E^m$. Consequently, if we denote by $G(n,m)$ the Grassmannian of the oriented $n$-planes in $E^m$, then $G(n,m)$ can be identified naturally with the decomposable $n$-vectors of norm 1 in the $\binom{m}{n}$-dimensional Euclidean space $\Lambda^n E^m = E^N$. Let $S^{N-1}$, $N = \binom{m}{n}$, be the unit hypersphere in $\Lambda^n E^m = E^N$ centred at $0$. Then $G(n,m)$ is an
Gauss map of submanifolds

A \( n(m - n) \)-dimensional submanifold of \( S^{N-1} \). Thus, we have

\[
G(n,m) \subset S^{N-1} \subset E^N = \Lambda^n E^m
\]

Let \( x : M \to E^m \) be an isometric immersion of a compact, oriented, \( n \)-dimensional Riemannian manifold \( M \) into \( E^m \). For each vector \( X \) tangent to \( M \), we identify \( X \) with its image under \( dx \). Let \( e_1, \ldots, e_n \) be an oriented orthonormal frame on \( M \). Then the Gauss map

\[
v : M \to G(n,m) \subset S^{N-1} \subset E^N = \Lambda^n E^m
\]
is given by \( v(p) = (e_1 \wedge \ldots \wedge e_n)(p) \).

**LEMMA 3.1.** For a compact oriented submanifold \( M \) in \( E^m \), the Gauss map \( v : M \to E^N \) is mass-symmetric, that is, the centre of gravity \( v_0 \) coincides with the centre of the hypersphere \( S^{N-1} \) (that is, the origin) in \( E^N \).

**Proof.** Let \( x : M \to E^m \) be the isometric immersion and \( e_1, \ldots, e_n \) an oriented orthonormal local frame on \( M \). Denote by \( \omega^1, \ldots, \omega^n \), the dual frame of \( e_1, \ldots, e_n \). Then we have \( dx = e_1 \omega^1 + e_2 \omega^2 + \ldots + e_n \omega^n \).

By direct computation, we have

\[
dx \wedge \ldots \wedge dx = n!(e_1 \wedge \ldots \wedge e_n)\omega^1 \wedge \ldots \wedge \omega^n = n! \ast dV.
\]

Thus, we obtain

\[
n! \int_M v \ast dV = \int_M dx \wedge \ldots \wedge dx = \int_M d(x \wedge dx \wedge \ldots \wedge dx)^{n-1} = 0.
\]

This shows that the centre of gravity \( v_0 = \frac{\int v \ast dV}{\int dV} = 0 \).

If \( M \) is a closed curve in \( E^m \), we have

**PROPOSITION 3.1.** The Gauss map \( v \) of a closed curve \( C \) in \( E^m \) is of \( k \)-type if and only if \( C \) is of \( k \)-type in \( E^m \).
Proof. Let \( x : C \to E^m \) be the isometric immersion, \( s \) the arc length and \( e_1 = dx/ds \) the unit tangent vector. Then the Gauss map \( v \) is given by \( v = e_1 \in S^{m-1} = G(1, m) < \Lambda^1 E^m = E^m \). Assume \( C \) is of \( k \)-type and \( P \) is the minimal polynomial of \( C \). Then we have \( P(\Delta)(x - x_0) = 0 \). Thus

\[
P(\Delta)v = P(\Delta) \frac{d}{ds} (x - x_0) = \frac{d}{ds} P(\Delta)(x - x_0) = 0.
\]

Thus, by Theorem 2.2 and Lemma 3.1 we see that \( v \) is of \( h \)-type with \( h \leq k \).

Now, if \( v \) is of \( h \)-type with minimal polynomial \( \overline{P} \), then we have \( \overline{P}(\Delta)v = 0 \). Since \( d/ds \) commutes with \( \overline{P}(\Delta) \) and \( \Delta = -d^2/ds^2 \), we get \( \overline{P}(\Delta)H = 0 \), where \( H = de_1/ds \). Therefore, by Theorem 2.1, \( C \) is of \( l \)-type with \( l \leq h \). Combining these results, we obtain \( l = h = k \).

In the remaining part of this section, we compute the first Laplacian \( \Delta v \) of \( v \) for later use.

Let \( x : M \to E^m \) be an isometric immersion of an oriented, \( n \)-dimensional Riemannian manifold into \( E^m \). We choose an oriented orthonormal local frame \( e_1, \ldots, e_n, e_{n+1}, \ldots, e_m \) on \( M \) such that \( e_1, \ldots, e_n \) are tangent to \( M \) and hence \( e_{n+1}, \ldots, e_m \) are normal to \( M \). We shall make use of the following convention on the ranges of indices:

\[
1 \leq i, j, k, \ldots \leq n ; \quad n + 1 \leq r, s, t, \ldots \leq m.
\]

Let \( \nabla \) and \( \nabla' \) be the Levi-Civita connections on \( M \) and \( E^m \) respectively. Denote by \( \omega_A^B, A, B = 1, \ldots, m \), the connection forms. Then we have

\[
(3.1) \quad \nabla' e_i e_j = \omega_{i}^{k}(e_i)e_k + h_{ij}^{r} e_r,
\]

\[
(3.2) \quad \nabla' e_i e_r = -h_{ij}^{r} e_j + \omega_{r}^{s} e_s, \quad \nabla e_i e_r = \omega_{r}^{s} e_s,
\]

where \( D \) is the normal connection and \( h_{ij}^i \) the coefficients of the second fundamental form \( h \). The Einstein convention is used for repeated indices.
By regarding $v$ as an $E^N$-valued function on $M$, we have

$$(3.3) \quad e_i v = e_i (e_1 \wedge \ldots \wedge e_n) = h^r_{ij} e_1 \wedge \ldots \wedge e_r \wedge \ldots \wedge e_n.$$ 

Since

$$(3.4) \quad \Delta v = -e_i e_i v + (\nabla_{e_i} e_i) v,$$ 

by a direct computation we obtain

$$(3.5) \quad \Delta v = -h^r_{ij} e_1 \wedge \ldots \wedge e_r \wedge \ldots \wedge e_n$$

$$+ \|h\|^2 v,$$

where $\|h\|^2 = h^r_{ij} h^r_{ij}$ and

$$(3.6) \quad h^r_{jk,i} = e_i h^r_{jk} + h^t_{jk} \omega^r_{t}(e_i) - \omega^r_{j}(e_i) h^r_{ik} - \omega^r_{k}(e_i) h^r_{jl}.$$ 

By the Codazzi equation $h^r_{jk,i} = h^r_{ij,k}$, (3.5) yields

$$(3.7) \quad \Delta v = -n \sum_i e_i \wedge \ldots \wedge D_{e_i} H \wedge \ldots \wedge e_n$$

$$+ h^r_{ij} h^8_{ik} e_1 \wedge \ldots \wedge e_s \wedge \ldots \wedge e_r \wedge \ldots \wedge e_n + \|h\|^2 v,$$

where $H = (1/n) h^r_{ii} e_i$ is the mean curvature vector. We recall the following Ricci equation of $M$ in $E^m$:

$$(3.8) \quad R^D(e_j, e_k; e_r, e_s) = \langle [A_r, A_s] e_j, e_k \rangle = h^r_{ik} h^8_{ij} - h^r_{ij} h^8_{ik},$$

where $R^D$ is the normal curvature tensor and $A_r$ the Weingarten map at $e_r$. From (3.7) and (3.8) we obtain the following.

**Lemma 3.2.** Let $x : M \rightarrow E^m$ be an isometric immersion of an oriented $n$-dimensional Riemannian manifold $M$ into $E^m$. Then the Laplacian of the Gauss map $v : M \rightarrow G(n,m) \subset N E^m$ is given by
\[ \Delta \nu = -n \sum e_i \wedge \ldots \wedge D_{e_i}H \wedge \ldots \wedge e_n \]
\[ + R^D(e_j, e_i, e_r, e_k)e_1 \wedge \ldots \wedge e_s \wedge \ldots \wedge e_r \wedge \ldots \wedge e_n + \|h\|^2 \nu \]

Since the first term of the right-hand side of (3.9) is the only term tangent to \( G(n,m) \) and other two terms are normal to \( G(n,m) \), Lemma 3.2 implies the following result of [13].

**COROLLARY 3.1.** (Ruh and Vilms [13]). Let \( M \) be a submanifold of \( E^n \). Then the map \( \nu : M \rightarrow G(n,m) \) is harmonic if and only if \( M \) has parallel mean curvature vector in \( E^n \).

If we consider the map \( \overline{\nu} = i \cdot \nu : M \rightarrow G(n,m) \rightarrow S^{n-1} \) (\( i \) = the inclusion), then Lemma 3.2 gives

**COROLLARY 3.2.** Let \( M \) be a submanifold of \( E^n \). Then the map \( \overline{\nu} : M \rightarrow S^{n-1} \) is harmonic if and only if \( M \) has flat normal connection and parallel mean curvature vector.

**COROLLARY 3.3.** Let \( M \) be a compact submanifold of \( E^n \). If the map \( \overline{\nu} : M \rightarrow S^{n-1} \) is harmonic, then all of the Pontrjagin classes and the Euler class of the normal bundle \( T^1M \) vanish.

4. Submanifolds with 1-type Gauss Map.

From Theorem 2.2 and Lemma 3.2 we have the following.

**THEOREM 4.1.** Let \( x : M \rightarrow E^n \) be an isometric immersion of a compact, oriented Riemannian manifold \( M \) into \( E^n \). Then the Gauss map \( \nu : M \rightarrow N E_m \) is of 1-type if and only if \( M \) has constant scalar curvature, flat normal connection and parallel mean curvature vector in \( E^n \).

**Proof.** From Theorem 2.2 and Lemma 3.2 we see that \( \nu \) is of 1-type if and only if \( DH = 0 \), \( R^D = 0 \) and \( \|h\| \) is a constant. From Gauss' equation, the scalar curvature \( \tau \) of \( M \) satisfies \( n(n - 1)\tau = n^2|H|^2 - \|h\|^2 \). Since \( DH = 0 \) implies the constancy of the mean curvature, \( \|h\| \) is a constant and \( \tau \) is constant. Therefore, \( \nu \) is of 1-type.
curvature $|H|$, Theorem 4.1 follows.

If $M$ is a hypersurface of $E^m$, we have

**Theorem 4.2.** A compact hypersurface $M$ of $E^{n+1}$ has 1-type Gauss map $\nu : M \to \mathbb{R}^{n+1}$ if and only if $M$ is a hypersphere in $E^{n+1}$.

**Proof.** Let $M$ be a hypersurface of $E^{n+1}$. Then $M$ has flat normal connection. Thus, by Theorem 4.1, the Gauss map is of 1-type if and only if $M$ has constant mean curvature and constant scalar curvature. Since a compact hypersurface of $E^{n+1}$ has constant mean curvature and constant scalar curvature if and only if $M$ is a hypersphere (Corollary 6.1 of [7] which follows easily from Proposition 4.1 of [5, p. 271]), we conclude that $\nu$ is of 1-type if and only if $M$ is a hypersphere of $E^{n+1}$.

If $M$ is a compact hypersurface of a hypersphere $S^{n+1}$ of $E^{n+2}$, then the normal connection of $M$ in $E^{n+2}$ is also flat. Thus, Theorem 4.1 implies that $M$ has 1-type Gauss map if and only if $M$ has constant scalar curvature and constant mean curvature. Thus, by applying Theorem 2 of [6], we obtain the following.

**Theorem 4.3.** Let $M$ be a compact hypersurface of a hypersphere $S^{n+1}$ of $E^{n+2}$. Then $M$ has 1-type Gauss map if and only if $M$ is one of the following submanifolds:

(a) A mass-symmetric 2-type submanifold of $E^{n+2}$;

(b) A small hypersphere of $S^{n+1}$;

(c) A minimal hypersurface of $S^{n+1}$ with constant scalar curvature.

The following theorem classifies surfaces with 1-type Gauss map completely.

**Theorem 4.4.** Let $M$ be a compact surface in $E^m$. Then $M$ has 1-type Gauss map if and only if $M$ is one of the following surfaces:

(a) A sphere $S^2(r) \subset E^3 \subset E^m$; or

(b) The product of two plane circles $S^1(a) \times S^1(b) \subset E^4 \subset E^m$.
Proof. By Theorem 4.1 we see that both $S^2(a)$ and $S^1(a) \times S^1(b)$ have 1-type Gauss map.

Conversely, if $M$ is a compact surface in $E^m$ with 1-type Gauss map, then we have (i) $DH = 0$, (ii) $\tau$ is constant and (iii) $R^D = 0$. Since $M$ is compact, $H \neq 0$. Thus, Theorem 2.1 of [4, p. 106] shows that $M$ is either a minimal surface of a hypersphere $S^{m-1}$ of $E^m$, or it lies in $E^3 \subset E^M$ or in $S^3 \subset E^M$. If $M$ is a minimal surface of $S^{m-1}$, then by $R^D = 0$, $M$ lies in a $S^3 \subset E^M$ (Remark 2.1 of [4, p. 115]). Consequently, $M$ lies either in $E^3$ or in $S^3$. If $M$ lies in $E^3$, Theorem 4.2 shows that $M$ is a sphere $S^2(a) \subset E^3$. If $M$ lies in $S^3 \subset E^4$, Theorem 4.3 shows that $M$ is a sphere in $E^3$ or a minimal surface with constant Gauss curvature in $S^3$ or a 2-type surface in $S^3 \subset E^4$. If $M$ is a minimal surface of $S^3$ with constant Gauss curvature, then a result of [12] shows that $M$ is the product of two plane circles of the same radius. If $M$ is a 2-type surface in $S^3 \subset E^4$, Theorem 2 of [6] shows that $M$ is mass-symmetric in $S^3$. Thus a classification theorem of [5, p. 279] yields that $M$ is the product of two plane circles of different radius.

From Theorem 4.1, we obtain immediately the following.

COROLLARY 4.1. Let $x : M \rightarrow E^M$ be an isometric immersion of a compact oriented Riemannian manifold $M$ into $E^M$. If the Gauss map of $x$ is of 1-type, then all of the Pontrjagin classes and the Euler class of the normal bundle vanish.

Remark. In [1], Bleecker and Weiner had studied compact oriented submanifolds of $E^M$ whose Gauss map satisfies $\Delta v = \lambda v$ for some constant $\lambda$. They obtained results such as Theorems 4.1, 4.2 and 4.4.

5. Surfaces with 2-type Gauss Map.

The main purpose of this and the next two sections is to classify minimal surfaces of $S^{m-1}$ with 2-type Gauss map. In order to do so, we
need to compute $\Delta^2 v$.

Let $x : M \to \mathbb{E}^m$ be an isometric immersion of a compact oriented surface into $\mathbb{E}^m$. Assume that $M$ lies in the unit hypersphere $S^{m-1}$ of $\mathbb{E}^m$ centred at the origin. Then the position vector $x$ is a unit normal vector. In the following, we choose an oriented orthonormal local frame $e_1, e_2, e_3, \ldots, e_m$ in such a way that $e_m = x$. Then we have

$$h^m_{ij} = -\delta_{ij} \quad \text{and} \quad \omega^m_r = 0.$$  \hfill (5.1)

In the following, we assume that $M$ is a minimal surface of $S^{m-1}$. Then the first normal space $\text{Im} \, h$ is of dimension $\leq 2$. Thus, we may also assume that $e_3, e_4$ lies in $\text{Im} \, h$. Then, with respect to the local frame chosen above, we have

$$A_5 = \ldots = A_{m-1} = 0, \quad A_m = -I.$$  \hfill (5.2)

Consequently, by Ricci's equation, we obtain

$$R^D(e_i, e_j; e_r, e_s) = 0 \quad \text{for} \quad r, s \neq 3, 4.$$  \hfill (5.3)

Because $DH = 0$, Lemma 3.2 gives

$$\Delta v = 2k^D e_3 \wedge e_4 + \|h\|^2 e_1 \wedge e_2,$$  \hfill (5.4)

where $k^D = R^D(e_1, e_2; e_3, e_4) = h^3_{20} h^4_{10} - h^3_{10} h^4_{20}.$

In the following, we shall make use of the following convention on the ranges of indices:

$$1 \leq i, j, k \leq 2; \quad 5 \leq \alpha, \beta, \gamma \leq m; \quad 3 \leq r, s, t, \leq m.$$

By a straightforward but lengthy computation, we may obtain

**LEMMA 5.1.** Under the hypothesis, we have

$$\Delta^2 v = 2(\Delta k^D + 2k^D(\|h\|^2 - 1) + \sum_\alpha \left( \|\omega_3^{\alpha} \|^2 + \|\omega_4^{\alpha} \|^2 \right) e_3 \wedge e_4$$

$$+ \left( \|h\|^2 + \|h\|^4 + 4(k^D) \right) e_1 \wedge e_2.$$
Now, we give some examples of compact minimal surfaces in $S^{m-1} \subset \mathbb{R}^m$ with 2-type Gauss map. More examples will be given in Section 6.

The first example is given by Veronese surface in $S^4$. We recall the Veronese surface as follows (see [5, 10]).

Let $(x, y, z)$ be the natural coordinate system in $E^3$ and $(u^1, u^2, u^3, u^4, u^5)$ the natural coordinate system in $E^5$. We consider the mapping defined by

\[
\begin{align*}
    u^1 &= \frac{1}{\sqrt{3}} yz, \\
    u^2 &= \frac{1}{\sqrt{3}} xz, \\
    u^3 &= \frac{1}{\sqrt{3}} xy, \\
    u^4 &= \frac{1}{2\sqrt{3}} (x^2 - y^2), \\
    u^5 &= \frac{1}{6} (x^2 + y^2 - 2z^2).
\end{align*}
\]

This defines an isometric minimal immersion of $S^2(\sqrt{3})$ into $S^4 = S^4(1)$. Two points $(x, y, z)$ and $(-x, -y, -z)$ of $S^2(\sqrt{3})$ are mapped into the same point of $S^4$ and this mapping defines an embedded real projective plane in $S^4$ which is called the Veronese surface. For the Veronese surface, we have

\[
\|h\|^2 = \frac{10}{3}, \quad K^D = \frac{2}{3}.
\]

Thus, Lemma 5.1 yields

\[
\Delta^2 \psi = -\frac{56}{9} e_3 \wedge e_4 + \frac{116}{9} e_1 \wedge e_2.
\]
From (5.4) we have

\[ \Delta \nu = -\frac{4}{3} e_3 \wedge e_4 + \frac{10}{3} e_1 \wedge e_2 . \]

Consequently, (5.7) and (5.8) give

\[ \Delta^2 \nu - \frac{14}{3} \Delta \nu + \frac{8}{3} \nu = 0 . \]

Therefore, from (5.4) and (5.9) and Theorem 2.2, we may conclude that the second standard immersion \( \psi_2 : S^2(\sqrt{3}) \rightarrow B^5 \) defined by (5.5) has 2-type Gauss map. Moreover, the order of the Gauss map is \([1,3]\) (with \( \lambda_1 = 2/3 \) and \( \lambda_3 = 4 \)).

In general, the \( k \)-th standard immersion \( \psi_k \) of a 2-sphere \( S^2 \) in \( S^{2k} \) can be defined as follows.

Let \((\theta, \phi)\) denote the spherical coordinates of \( S^2(r_k) \) of radius \( r_k = (k(k + 1)/2)^{1/2} \). Then the coordinates of \( S^2(r_k) \) in \( E^3 \) are given by

\[ \begin{align*}
x &= r_k \cos \phi, \\
y &= r_k \sin \phi \cos \theta, \\
z &= r_k \sin \phi \sin \theta .
\end{align*} \]

In terms of \((\theta, \phi)\), the \( k \)-th standard immersion \( \psi_k \) of \( S^2(r_k) \) into \( S^{2k} \) is given by

\[ \begin{align*}
u^0 &= (r_k/ \sqrt{2}) \cdot B^0_k \cdot P^0_k(\cos \phi) , \\
u^i &= r_k \cdot B^i_k \cdot P^i_k(\cos \phi) \cdot \cos(i\theta) , \quad i = 1, \ldots, k , \\
u^{k+i} &= r_k \cdot B^i_k \cdot P^i_k(\cos \phi) \cdot \sin(i\theta) ,
\end{align*} \]

where \((u^0, u^1, \ldots, u^{2k})\) is the Euclidean coordinate system of \( E^{2k+1} \).

Moreover,

\[ \rho^j_k(t) = (1 - t^2)^{j/2} \frac{d^{k+j}}{dt^{k+j}} [(1 - t^2)^k] , \quad j = 0, 1, \ldots, k , \]

are the Legendre functions and \( B^i_k \) are defined by
It is well-known that the $k$-th standard immersion is an isometric minimal immersion of $S^2(r_k^2)$ into $S^{2k}$. If $k$ is odd, it is an imbedding and if $k$ is even, it is a two-to-one map.

**THEOREM 5.1.** Let $x : S^2(r) \to S^{m-1} \subset \mathbb{E}^m$ be a minimal isometric immersion of a 2-sphere $S^2(r)$ into $S^{m-1} \subset \mathbb{E}^m$. If $x$ is not totally geodesic, then it has 2-type Gauss map.

**Proof.** Let $x : S^2(r) \to S^{m-1} \subset \mathbb{E}^m$ be a minimal isometric immersion of $S^2(r)$ into $S^{m-1}$. Then, by a well-known result of Calabi [3], $r = r_k$ for some natural number $k$ and the immersion $x$ is the $k$-th standard immersion $\psi_k$ of $S^2(r_k^2)$ into $S^{2k} \subset S^{m-1}$ (up to rigid motions of $S^{m-1}$). If $k = 1$, $x$ is a totally geodesic immersion. Thus, we obtain $k \geq 2$ from hypothesis.

Since the $k$-th standard immersion $\psi_k : S^2(r_k^2) \to S^{2k} \subset S^{m-1} \subset \mathbb{E}^m$ is isotropic (see Theorem 1 and Remark 1 of [6]), Lemma 3 of [6] implies that, with respect to a suitable orthonormal frame $e_1, e_2, e_3, \ldots, e_m$ so that $e_m = x$, we have

\begin{equation}
A_3 = \begin{pmatrix}
0 & c \\
c & 0
\end{pmatrix}, \quad A_4 = \begin{pmatrix}
c & 0 \\
0 & -c
\end{pmatrix}, \quad A_5 = \ldots = A_{m-1} = 0, \quad A_m = -I.
\end{equation}

Since $e_m = x$, we have

\begin{equation}
\omega_{2k+1} = 0, \quad r = 3, \ldots, m.
\end{equation}

Moreover, from (5.14) and equation of Gauss, we find

\begin{equation}
c^2 = (k - 1)(k + 2)/2k(k + 1).
\end{equation}

Let $D$ denote the normal connection of $S^2(r_k^2)$ in $\mathbb{E}^m$. Then, by (5.14), (5.15), (5.16), and Codazzi equation, we obtain

\begin{equation}
D e_1 e_3 + 2\omega_1^2 (e_1) e_4 = D e_2 e_4 - 2\omega_1^2 (e_2) e_3,
\end{equation}
(5.18) 
$$D_2 e_3 + 2\omega_1^2 (e_2) e_4 = -D_1 e_4 + 2\omega_1^2 (e_1) e_3 .$$

From (5.17) and (5.18) we get

(5.19) 
$$\omega_3^4 = -2\omega_1^2 ,$$

(5.20) 
$$\omega_3^3 (e_1) = \omega_4^3 (e_2) , \quad \omega_3^3 (e_2) = -\omega_4^3 (e_1) , \quad \alpha \geq 5 .$$

Moreover, from (5.14) and (5.16), we obtain

(5.21) 
$$\|h\|^2 = 4(k^2 + k - 1)/k(k + 1) ,$$

(5.22) 
$$K^D = (k - 1)(k + 2)/k(k + 1) .$$

Furthermore, (5.21) yields

(5.23) 
$$\sum_{\alpha} \|\omega_3^\alpha\|^2 = -2(\omega_3^\alpha \wedge \omega_4^\alpha) (e_1, e_2) .$$

On the other hand, by (5.14), (5.19) and structure equation, we have

(5.24) 
$$-\omega_3^\alpha \wedge \omega_4^\alpha = (2K + K^D) \omega_3^\alpha \wedge \omega_4^\alpha ,$$

where \( K = 2/k(k + 1) . \) Combining (5.20), (5.23) and (5.24), we find

(5.25) 
$$\sum_{\alpha} \|\omega_3^\alpha\|^2 = \sum_{\alpha} \|\omega_4^\alpha\|^2 = 2K + K^D = (k^2 + k + 2)/k(k + 1) .$$

From (5.14) and (5.20), we also get

(5.26) 
$$h_3^j \omega_3^\alpha (e_j) = h_3^j \omega_4^\alpha (e_j) \quad \text{for} \quad j = 1, 2 .$$

From the structure equations, we obtain

(5.27) 
$$(d\omega_4^\beta) (e_1, e_2) = -(\omega_3^\alpha \wedge \omega_4^\beta) (e_1, e_2) - (\omega_4^\beta \wedge \omega_4^\beta) (e_1, e_2) ,$$

(5.28) 
$$(d\omega_4^\beta) (e_1, e_2) = -(\omega_3^\alpha \wedge \omega_4^\beta) (e_1, e_2) - (\omega_4^\beta \wedge \omega_4^\beta) (e_1, e_2) .$$

Thus, by using (5.20), (5.27) and (5.28) we give

(5.29) 
$$(\varphi_{\omega_3^\alpha} e_3) e_4 = \omega_3^4 (e_3) \omega_4^\alpha (e_4) - \omega_3^\alpha (e_4) \omega_4^\alpha (e_3) ,$$

(5.30) 
$$(\varphi_{\omega_4^\beta} e_4) e_3 = \omega_3^\beta (e_3) \omega_4^\alpha (e_3) - \omega_3^\alpha (e_3) \omega_4^\beta (e_3) .$$
Consequently, (5.21), (5.22), (5.25), (5.26), (5.29), (5.30) and Lemma 5.1 yield
\[
\Delta^2 v = 4(K^D \|h\|^2 + 2K)e_3 \land e_4 \\
+ \{ \|h\|^4 + 4(K^D)^2 \} e_1 \land e_2.
\]
Since $\|h\|, K$ and $K^D$ are constant, (5.4), (5.31) and Theorems 2.2 and 4.4 imply that the Gauss map $v$ is of 2-type.

From the proof of Theorem 5.1 we have the following.

**COROLLARY 5.1.** Let $x : M \hookrightarrow S^{m-1} \subset \mathbb{E}^m$ be a minimal isometric immersion of a compact oriented surface $M$ into $S^{m-1}$. If $M$ is constant isotropic in $S^{m-1}$ (or in $\mathbb{E}^m$), then the Gauss map of $x$ is of either 1- or 2-type.

6. **Classification of Minimal Tori with 2-type Gauss Map.**

Let $(n, k, m)$ be a triple of integers with $n, k > 0$. Let $\Lambda$ be the lattice generated by
\[
\{(0, 2\sqrt{2/3} \pi n), (\sqrt{\pi} \pi, \sqrt{2/3} (2m - k)\pi)\}.
\]
Consider the map $\tilde{y}(n, k, m) : \mathbb{R}^2 \to \mathbb{E}^6$ defined by
\[
\tilde{y}(n, k, m)(s, t) = \frac{1}{\sqrt{3}} (\cos \frac{1}{\sqrt{2}} (s + \sqrt{3} t), \sin \frac{1}{\sqrt{2}} (s + \sqrt{3} t), \\
\cos \frac{1}{\sqrt{2}} (-s + \sqrt{3} t), \sin \frac{1}{\sqrt{2}} (-s + \sqrt{3} t), \cos \sqrt{2} s, \sin \sqrt{2} s).
\]
Then $\tilde{y}(n, k, m)$ is an isometric immersion and it induces a minimal isometric immersion of the flat torus $T(n, k, m) = \mathbb{R}^2/\Lambda$ into $S^5 \subset \mathbb{E}^6$ which is denoted by $y(n, k, m)$ so we have
\[
y(n, k, m) : T(n, k, m) \to S^5 \subset \mathbb{E}^6.
\]

The following result completely classifies minimal flat tori in $S^{m-1}$ with 2-type Gauss map.
THEOREM 6.1. (a) For any triple \((n,k,m)\) of integers with \(n,k > 0\), the minimal isometric immersion (6.3) has 2-type Gauss map. 

(b) Let \(y : T^2 \to S^{m-1} \subset E^m\) be an isometric minimal immersion of a flat torus \(T^2\) into \(S^{m-1}\). If the Gauss map of \(y\) is of 2-type, then

(b.1) \(T^2\) is isometric to the flat torus \(T(n,k,m)\) for some natural numbers \(k\) and \(n\) and integer \(m\);

(b.2) \(T^2\) is immersed fully in a totally geodesic 5-sphere \(S^5\) of \(S^{m-1}\); and

(b.3) up to rigid motions, \(y\) is given by the composition \(i\).

\(y(n,k,m) : T^2 \to S^5 \to S^{m-1} \subset E^m\), where \(i\) is the inclusion.

Proof. (a) Let \(y(n,k,m)\) be the isometric immersion of \(T(n,k,m)\) given by (6.3), induced from (6.2). Then, by a direct computation, we have \(\Delta y(n,k,m) = 2y(n,k,m)\). Thus, by a result of Takahashi, \(y(n,k,m)\) is a minimal immersion. Since the Gauss map is given by \(\nu = \partial / \partial s \wedge \partial / \partial t\), a straightforward computation yields

\(\Delta^2 \nu - 8\Delta \nu + 12\nu = 0\).

From Theorem 4.4, we know that \(\nu\) is not of 1-type. Thus, Theorem 2.2 implies that the Gauss map is of 2-type.

(b) Let \(y : T^2 \to S^{m-1} \subset E^m\) be an isometric minimal immersion of a flat torus \(T^2\) into \(S^{m-1}\) such that the Gauss map of \(y\) is of 2-type.

Assume that \(T^2 = \mathbb{R}^2 / \Lambda\), where \(\Lambda\) is a lattice in \(\mathbb{R}^2\) which defines the flat torus \(T^2\). Without loss of generality, we may assume that \(\Lambda\) is given by

\(\Lambda = \{ (2nu, 2nv + 2nw) \mid h,m \in \mathbb{Z} \}\),

where \(u, v, w\) are real numbers with \(u, v > 0\). The dual lattice of \(\Lambda\) is given by

\(\Lambda^* = \{ (\frac{k}{\pi n}, \frac{m}{\pi n}, \frac{n}{\pi n}) \mid k, n \in \mathbb{Z} \}\).
It is known that the spectrum of $T^2 = \mathbb{R}^2/\Lambda$ is given by
\[(6.7) \left\{ \left( \frac{k}{u} - \frac{n\omega}{v} \right)^2 + \left( \frac{n}{v} \right)^2 \mid k, n \in \mathbb{Z} \right\}.
\]
The eigenspace $V(\lambda)$ of $\Delta$ with eigenvalue $\lambda$ is given by
\[(6.8) \text{Span}\left\{ \cos\left( \frac{\xi u}{u} + \frac{nt}{v} \right)^2, \sin\left( \frac{\xi u}{u} + \frac{nt}{v} \right)^2 \mid \left( \frac{\xi}{u} \right)^2 + \left( \frac{n}{v} \right)^2 = \lambda \right\},
\]
where $\xi = k - \frac{n\omega}{v}$.

Since $y : T^2 \rightarrow S^{m-1} \subset E^m$ is minimal, $\Delta y = 2y$. Thus, every coordinate function of $y$ is an eigenfunction of $\Delta$ with eigenvalue 2. We put
\[(6.9) P = \left\{ (\xi_i, n_i) \mid \left( \frac{\xi_i}{u} \right)^2 + \left( \frac{n_i}{v} \right)^2 = 2 \right\},
\]
where $\xi_i = k_i - n_i \omega/v$ and $k_i, n_i \in \mathbb{Z}$. Let $\#P = l$ ($\#P$ denotes the cardinal number of $P$). For simplicity, we may assume $P = \{(\xi_i, n_i) \mid i \in I_l\}$, when $I_l = \{1, 2, \ldots, l\}$. Then the isometric immersion $y$ may assume to be of the following form:
\[(6.10) y(s, t) = \left( \mu_i \cos(\bar{\xi}_i s + \bar{n}_i t), \mu_i \sin(\bar{\xi}_i s + \bar{n}_i t) \right)_{i \in I},
\]
where $I$ is a subset of $I_l$, $\mu_i$ are positive constants and
\[(6.11) \bar{\xi}_i = \frac{\xi_i}{u}, \bar{n}_i = \frac{n_i}{v}, \bar{\xi}_i^2 + \bar{n}_i^2 = 2.
\]
If $\#I = 2$, then $T^2$ is a minimal flat torus in $S^3$. Thus, by a result of [12], $T^2$ is immersed as a Clifford torus. Thus, by Theorem 4.4, $y$ has $I$-type Gauss map which is a contradiction. Thus, we obtain $\#I \geq 3$.

Since $\cos(-\theta) = \cos \theta$ and $\sin(-\theta) = -\sin \theta$, without loss of generality we may put
\[(6.12) \bar{n}_i \geq 0 \text{ for } i \in I.
\]

Since $y$ is an isometric immersion of $T^2$ into $S^{m-1}$, we have
\[(6.13) \sum \bar{n}_i^2 = 1.
\]
By applying (6.10) we see that the nonzero coordinates of the Gauss map
\[ \nu : T^2 + \mathbb{R}^m = g^{m(m-1)/2} \]
are given by

(6.16) \[ \nu(s, t) = (u_{i,j}(-\cos(\bar{e}_i + \bar{e}_j)s + (\bar{n}_i + \bar{n}_j)t) + \]
\[ + \cos((\bar{e}_i - \bar{e}_j)s + (\bar{n}_i - \bar{n}_j)t)), \]
\[ - u_{i,j}(\sin(\bar{e}_i + \bar{e}_j)s + (\bar{n}_i + \bar{n}_j)t) + \]
\[ + \sin((\bar{e}_i - \bar{e}_j)s + (\bar{n}_i - \bar{n}_j)t)), \]
\[ - u_{i,j}(\sin(\bar{e}_i + \bar{e}_j)s + (\bar{n}_i + \bar{n}_j)t) - \]
\[ - \sin((\bar{e}_i - \bar{e}_j)s + (\bar{n}_i - \bar{n}_j)t)), \]
\[ u_{i,j}(\cos(\bar{e}_i + \bar{e}_j)s + (\bar{n}_i + \bar{n}_j)t) + \]
\[ + \cos((\bar{e}_i - \bar{e}_j)s + (\bar{n}_i - \bar{n}_j)t)) \] i < j,

where

(6.17) \[ u_{i,j} = \frac{1}{2} \frac{1}{\mu_{i,j}}(\bar{e}_i \bar{n}_j - \bar{e}_j \bar{n}_i). \]

By direct computation, we find

(6.18) \[ \Delta \nu = (u_{i,j}(-b_{i,j} \cos(\bar{e}_i + \bar{e}_j)s + (\bar{n}_i + \bar{n}_j)t) + \]
\[ + c_{i,j} \cos((\bar{e}_i - \bar{e}_j)s + (\bar{n}_i - \bar{n}_j)t)), \]
\[ - u_{i,j}(b_{i,j} \sin(\bar{e}_i + \bar{e}_j)s + (\bar{n}_i + \bar{n}_j)t) + \]
\[ + c_{i,j} \sin((\bar{e}_i - \bar{e}_j)s + (\bar{n}_i - \bar{n}_j)t)), \]
\[ - u_{i,j}(b_{i,j} \sin(\bar{e}_i + \bar{e}_j)s + (\bar{n}_i + \bar{n}_j)t) - \]
\[ - c_{i,j} \sin((\bar{e}_i - \bar{e}_j)s + (\bar{n}_i - \bar{n}_j)t)), \]
\[ u_{i,j}(b_{i,j} \cos(\bar{e}_i + \bar{e}_j)s + (\bar{n}_i + \bar{n}_j)t) + \]
\[ + c_{i,j} \cos((\bar{e}_i - \bar{e}_j)s + (\bar{n}_i - \bar{n}_j)t)) \] i < j,
(6.19) \[ \Delta^2 v = (v_{ij} - b_{ij}^2) \cos((\bar{e}_i + \bar{e}_j)s + (\bar{n}_i + \bar{n}_j)t) + \]
\[ + c_{ij}^2 \cos((\bar{e}_i - \bar{e}_j)s + (\bar{n}_i - \bar{n}_j)t) , \]
\[ - \mu_{ij} (b_{ij}^2 \sin((\bar{e}_i + \bar{e}_j)s + (\bar{n}_i + \bar{n}_j)t) + \]
\[ + c_{ij}^2 \sin((\bar{e}_i - \bar{e}_j)s + (\bar{n}_i - \bar{n}_j)t) , \]
\[ - \mu_{ij} (b_{ij}^2 \sin((\bar{e}_i + \bar{e}_j)s + (\bar{n}_i + \bar{n}_j)t) - \]
\[ - c_{ij}^2 \sin((\bar{e}_i - \bar{e}_j)s + (\bar{n}_i - \bar{n}_j)t)) , \]
\[ \mu_{ij} (b_{ij}^2 \cos((\bar{e}_i + \bar{e}_j)s + (\bar{n}_i + \bar{n}_j)t) + \]
\[ + c_{ij}^2 \cos((\bar{e}_i - \bar{e}_j)s + (\bar{n}_i - \bar{n}_j)t)) , \]
\[ i < j , \]

where

(6.20) \[ b_{ij} = 4 + 2\bar{e}_i \bar{e}_j + 2\bar{n}_i \bar{n}_j , \]

(6.21) \[ c_{ij} = 4 - 2\bar{e}_i \bar{e}_j - 2\bar{n}_i \bar{n}_j . \]

If \( b_{ij} = c_{ij} \) for all \( i < j \), then

(6.22) \[ \bar{e}_i \bar{e}_j = -\bar{n}_i \bar{n}_j , \quad i < j . \]

This implies that either \( \bar{e}_j = \bar{n}_i \) for all \( j \in I \) or \( \bar{n}_j = \bar{e}_i \) for all \( j \in I \). Thus, by (6.9), we obtain

(6.23) \[ \bar{n}_j^2 = 2/(1 + c^2) \quad \text{or} \quad \bar{e}_j^2 = 2/(1 + c^2) \quad \text{for} \quad j \in I . \]

This gives \( |I| \leq 2 \) which contradicts the assumption. Consequently, there is a pair \((i,j) \quad (i < j)\) such that \( b_{ij} \neq c_{ij} \). Without loss of generality, we may assume that \( b_{12} \neq c_{12} \). This is equivalent to

(6.24) \[ \bar{e}_1 \bar{e}_2 \neq -\bar{n}_1 \bar{n}_2 . \]

Thus, from (6.16), (6.18), (6.19) and Theorem 2.2, we find

(6.25) \[ \{b_{ij}, c_{ij} \mid i, j \in I, i < j\} = \{b_{12}, c_{12}\} . \]

We put
(6.26) $\beta_{ij} = \frac{1}{2} b_{ij} - 2$, $\gamma_{ij} = -\beta_{ij}$.

From (6.9) and (6.12) we have

(6.27) $\bar{n}_j = (2 - \bar{\varepsilon}_j^2)^{-1}$.

If $b_{1j} = b_{12}$, then (6.20), (6.26) and (6.27) give

(6.28) $\bar{\varepsilon}_j = \frac{1}{2} \left( \beta_{12} \bar{\varepsilon}_1 \pm \bar{n}_1 \sqrt{4 - \beta_{12}^2} \right)$.

If $b_{1j} = c_{12}$, we have

(6.29) $\bar{\varepsilon}_j = -\frac{1}{2} \left( \beta_{12} \bar{\varepsilon}_1 \pm \bar{n}_1 \sqrt{4 - \beta_{12}^2} \right)$.

From (6.27), (6.28) and (6.29) we get $\#I \leq 5$.

If $\beta_{12}^2 = 4$, then $\#I = 2$, which is impossible. Therefore, we have $\beta_{12}^2 < 4$. This condition is equivalent to the condition $\bar{\varepsilon}_1 \bar{n}_2 \neq \bar{\varepsilon}_2 \bar{n}_1$.

Without loss of generality, we may assume

(6.30) $\bar{\varepsilon}_1 \bar{n}_2 < \bar{\varepsilon}_2 \bar{n}_1$.

From (6.20), (6.26) and (6.30) we find

(6.31) $\bar{\varepsilon}_2 = \frac{1}{2} \left( \beta_{12} \bar{\varepsilon}_1 + \bar{n}_1 \sqrt{4 - \beta_{12}^2} \right)$.

If we put

(6.32) $\bar{\varepsilon}_3 = \frac{1}{2} \left( \beta_{12} \bar{\varepsilon}_1 - \bar{n}_1 \sqrt{4 - \beta_{12}^2} \right)$,

then we have

(6.33) $\{\bar{\varepsilon}_i\}_{i \in I} < \{\bar{\varepsilon}_1, \bar{\varepsilon}_2, -\bar{\varepsilon}_2, \bar{\varepsilon}_3, -\bar{\varepsilon}_3\}$.

It is clear that $1, 2 \in I$. Moreover, we have $\#I \geq 3$.

If $\bar{\varepsilon}_3, -\bar{\varepsilon}_3 \notin \{\bar{\varepsilon}_i\}_{i \in I}$, then $\#I = 3$ and $\{\bar{\varepsilon}_i\}_{i \in I} = \{\bar{\varepsilon}_1, \bar{\varepsilon}_2, -\bar{\varepsilon}_2\}$.

If $\bar{\varepsilon}_3$ or $-\bar{\varepsilon}_3$ belongs to $\{\bar{\varepsilon}_i\}_{i \in I}$, then by (6.9) and (6.32) we may find

$|\bar{\varepsilon}_3| = |\bar{\varepsilon}_2|$. Consequently, we always have
Without loss of generality, we may assume that $\overline{\varepsilon}_2 > 0$. Let us simply denote $\overline{\varepsilon}_2$ by $\varepsilon$ and denote $\overline{n}_2$ by $n$. Then from (6.34) we have

$$\{(\overline{\varepsilon}_i, \overline{n}_i)\}_{i \in I} = \{(-\varepsilon, n), (\varepsilon, n), (\overline{\varepsilon}_i, \overline{n}_i)\}, \quad \varepsilon > 0, \quad n > 0.$$  

If we apply our argument of deriving (6.31) to (6.35), we find

$$\overline{\varepsilon}_1 = \pm \varepsilon(3 - 2\varepsilon^2), \quad \overline{\varepsilon}_1 = \pm \varepsilon.$$

Therefore, by using (6.27), we get

$$\overline{n}_1 = \{(2 - \varepsilon^2)(1 - 2\varepsilon^2)^{-\frac{1}{2}}\}, \quad \overline{n}_1 \neq n.$$

Therefore, (6.20), (6.21), (6.35), (6.36) and (6.37) yield

$$\{(b_{i,j}, \sigma_{i,j})\}_{i < j} = \{4\varepsilon^2, 4n^2, (\varepsilon + \varepsilon_1)^2 + (\overline{n} + \overline{n}_1)^2, (\varepsilon - \varepsilon_1)^2 + (\overline{n} - \overline{n}_1)^2, (\varepsilon + \varepsilon_1)^2 + (\overline{n} + \overline{n}_1)^2, (\varepsilon + \varepsilon_1)^2 + (\overline{n} + \overline{n}_1)^2\}.$$

Since the Gauss map is of 2-type, $\#\{(b_{i,j}, \sigma_{i,j}) \mid i < j\} = 2$. Thus, by (6.36), (6.37) and $\varepsilon, n > 0$, we obtain $\overline{n}_1 = 0$. Therefore, by (6.37), we obtain $\varepsilon = 2$ or $1/2$. If $\varepsilon = 2$, we obtain from (6.27) that $\overline{n} = 0$ which yields $\#I = 2$ by virtue of (6.35). Hence, we find

$$\varepsilon = \sqrt{2}, \quad \overline{\varepsilon}_1 = \pm \sqrt{2} = \pm \sqrt{2}, \quad \overline{n} = \frac{\sqrt{2}}{2}.$$

Since $\overline{n}_1 = 0$, we may choose $\overline{\varepsilon}_1 = \sqrt{2}$. Consequently, we obtain

$$\{(\overline{\varepsilon}_i, \overline{n}_i)\}_{i \in I} = \left\{(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(\sqrt{2}, 0\right)\right\}.$$  

Substituting (6.40) into (6.13), (6.14) and (6.15) we get

$$u_1^2 = u_2^2 = u_3^2 = 1/3.$$  

Therefore, we find that the nonzero coordinates of $y : T^2 \to S^{m-1} \subset E^m$ are given by the following functions:
Gauss map of submanifolds

\[ y_1 = \frac{1}{\sqrt{3}} \cos \frac{1}{\sqrt{2}} (s + \sqrt{3} t) , \quad y_2 = \frac{1}{\sqrt{3}} \sin \frac{1}{\sqrt{2}} (s + \sqrt{3} t) \]

\[ y_3 = \frac{1}{\sqrt{3}} \cos \frac{1}{\sqrt{2}} (-s + \sqrt{3} t) , \quad y_4 = \frac{1}{\sqrt{3}} \sin \frac{1}{\sqrt{2}} (-s + \sqrt{3} t) \]

\[ y_5 = \frac{1}{\sqrt{3}} \cos \frac{1}{\sqrt{2}} s , \quad y_6 = \frac{1}{\sqrt{3}} \sin \frac{1}{\sqrt{2}} s . \]

Because \( y_1, \ldots, y_6 \) are functions on \( T^2 = \mathbb{R}^2 / \Lambda \), they are invariant under the action of \( \Lambda \). From this we see that \((hu + \sqrt{3}(mv + hw)) / \sqrt{2}\), \((-hu + \sqrt{3}(mv + hw)) / \sqrt{2}\) and \(\sqrt{2} hu\) are integers for any integers \( h, m \). In particular, we have

\[ u = k / \sqrt{2} , \quad v = \sqrt{2 / 3} n , \quad \omega = (2m - k) / \sqrt{6} \]

for some integer \( m \) and natural numbers \( k \) and \( n \). Therefore, we find that the lattice \( \Lambda \) is generated by

\[ \{(0, 2\sqrt{2 / 3} mn), (\sqrt{2} k n, \sqrt{2 / 3}(2m - k) n / \sqrt{6})\} . \]

It is easy to verify that the functions \( y_\alpha \) are invariant under the action of \( \Lambda \). Thus, we complete the proof of (b).

7. Classification of Surfaces with 2-type Gauss Map.

We give the following.

THEOREM 7.1 (Classification). Let \( x : M \rightarrow S^{m-1} \subset \mathbb{E}^m \) be a minimal isometric immersion of a compact oriented surface \( M \) into \( S^{m-1} \). Then \( x \) has 2-type Gauss map if and only if either (1) \( M \) is a 2-sphere \( S^2(r_k) \) with radius \( r_k = \sqrt{k(k + 1) / 3} \) for some integer \( k \geq 2 \) and \( x \) is given by the \( k \)-th standard immersion \( \psi_k \) of \( S^2(r_k) \) or (2) \( M \) is the flat torus \( T(n, k, h) = \mathbb{R}^2 / \Lambda \) for some integers \( n, k, h \) with \( n, k > 0 \), where \( \Lambda \) is the lattice generated by

\[ \{(0, 2\sqrt{2 / 3} mn), (\sqrt{2} k n, \sqrt{2 / 3}(2h - k) n / \sqrt{6})\} , \]

and the immersion \( x \) is induced from the isometric immersion \( \tilde{x} : \mathbb{R}^2 \rightarrow S^5 \subset \mathbb{E}^6 \subset \mathbb{E}^m \) defined by
\[ \overline{x}(s, t) = \frac{1}{\sqrt{3}} \left( \cos \frac{1}{\sqrt{2}} (s + \sqrt{3} t), \sin \frac{1}{\sqrt{2}} (s + \sqrt{3} t) \right), \]
\[ \cos \frac{1}{\sqrt{2}} (-s + \sqrt{3} t), \sin \frac{1}{\sqrt{2}} (-s + \sqrt{3} t), \cos \sqrt{2} s, \sin \sqrt{2} s, 0, \ldots, 0 \), \]
up to rigid motions of \( S^{m-1} \).

Proof. Let \( x : M \to S^{m-1} \subset \mathbb{E}^m \) be a minimal isometric immersion of a compact oriented surface into \( S^{m-1} \). If the Gauss map is of 2-type, then, by Theorem 2.2, there exist two constants \( b \) and \( c \) such that the Gauss map \( \nu \) of \( x \) satisfies
\[ A^2 \nu + b \Delta \nu + c \nu = 0. \] (7.3)
By looking at \( \nu = e_1 \wedge e_2 \), at equation (5.4) and at Lemma 5.1, we find
\[ (e_i \| h \|^2) (h_m^1 e_1 \wedge e_2 + h_m^2 e_1 \wedge e_m) = 0. \] (7.4)
Since \( A_m = -I \), (7.4) implies that \( \| h \| \) is constant. Similarly, by looking at the coefficients of \( e_1 \wedge e_2 \) of (5.4) and using Lemma 5.1 and (7.3) we obtain
\[ \| h \|^4 + 4(K_D)^2 + b \| h \|^2 + c = 0. \] (7.5)
Because \( \| h \|, b \) and \( c \) are constant, (7.5) shows that \( K_D \) is also constant. If \( K_D = 0 \), then, by the constancy of \( \| h \| \) and minimality of \( M \) in \( S^{m-1} \), we conclude from Theorem 4.1 that the Gauss map is of 1-type which is a contradiction. Thus, \( K_D \) is a nonzero constant. Since \( M \) is minimal in \( S^{m-1} \) and \( \| h \| \) is constant, \( M \) has constant Gauss curvature. Therefore, by applying a result of [2], we may conclude that \( M \) is either an ordinary 2-sphere \( S^2(r) \) of radius \( r \) or a flat torus. If \( M \) is \( S^2(r) \), we conclude from Theorem 4.4 and a result of [3] that \( r = r_k = \sqrt{k(k+1)/2} \) for \( k \geq 2 \) and \( x \) is the \( k \)-th standard immersion \( \psi_k \). If \( M \) is a flat torus, then we conclude from Theorem 6.1 that \( M \) is given by \( R^2/\Lambda \) for some lattice generated by (7.1) where \( n,k,h \) are...
Gauss map of submanifolds

integers with \( n, k > 0 \). Moreover, by Theorem 6.1, we also see that \( x \) is induced by the isometric immersion \( \bar{x} \) of \( \mathbb{R}^2 \) into \( \mathbb{E}^m \) defined by (7.2) up to rigid motions.

The converse of this was given in Theorems 5.1 and 6.1.

References


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