ON THE NON-VANISHING OF POINCARE SERIES

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1.

Let $M = SL(2, \mathbb{Z})$ be the classical modular matrix group. One form of the Poincaré series on $M$ is

$$ g_q(z, m) = \frac{1}{2} m^{2q-1} \sum_{M \setminus M} (cz + d)^{-2q} \exp(2\pi imAz); \quad (1.1) $$

here $z \in \mathcal{H} = \{z = x + iy : y > 0\}$, $q \geq 2$ and $m \geq 1$ are integers, and the summation is over a complete system of matrices $(ab: cd)$ in $M$ with different lower row. The problem of the identical vanishing of the Poincaré series for different values of $m$ and $q$ goes back to Poincaré.

It is known that $g_q$ is a modular form of weight $q$ holomorphic in $\mathcal{H}$, i.e.,

$$ g_q(Az, m)(A'z)^q = g_q(z) \forall A \in M. \quad (1.2) $$

Since the translation $(11: 01) \in M$, $g_q$ has a Fourier series

$$ g_q(z, m) = \sum_{r=1}^{\infty} c_q(r, m) \exp(2\pi i rz), \quad (1.3) $$

the summation being over $r \geq 1$ because of $m \geq 1$.

From the Petersson scalar product formula, we deduce at once that $g_q(z, m) = 0$ if and only if $c_q(m, m) = 0$. On the other hand there is an explicit formula for $c_q$ as an infinite series involving Bessel functions and certain complicated number-theoretic sums (Kloosterman sums).

Very recently R. A. Rankin (3) made use of the above to prove the following result. For each $\epsilon > 0$ there is a $q_0(\epsilon) > 0$ such that $g_q(z, m) \neq 0$ if

$$ 1 \leq m \leq q^{2-\epsilon}, \quad q \geq q_0. \quad (1.4) $$

(Actually Rankin's result was a bit stronger than this.)

In the present note we wish to generalise the above theorem by replacing $M$ by an arbitrary fuchsian group $\Gamma$ that acts on $\mathcal{H}$ and has translations. Thus $\Gamma$ may be infinitely generated and it may be of the second kind. However, the inequality we obtain is weaker than (1.4); the exponent is only $\frac{2}{3}$.

We shall prove:

**Theorem.** Let $\Gamma$ be a fuchsian group acting on $\mathcal{H}$ and possessing translations. Let the
Poincaré series $g_q(z, m)$ be defined as in (1.1). Then there exist positive constants $q_0$ and $m$, depending only on $\Gamma$, such that $g_q(z, m) \neq 0$ for

$$q \geq q_0, m \leq m_1 q^{4/3}.$$ 

2.

Let $\Gamma$ be a fuchsian group acting on $H$ and possessing translations, of which the smallest is $z \rightarrow z + \lambda$, $\lambda > 0$. We represent $\Gamma$ as a group of matrices $G = \{A = (ab: cd), \ ad - bc = 1\}$ with real entries, and we assume without loss of generality that $-I = (-1 \ 0: 0 \ -1) \in G$, so that $G/\pm I \equiv \Gamma$. Denote by $G_\infty = <(1: \lambda: 0 \ 1)>$ the stabiliser of $\infty$.

The Poincaré series is defined as before by

$$g_q(z, m) = \sum_{G \in G} (cz + d)^{-2q} \exp (2\pi i mAz),$$

with the same conditions on $q$ and $m$ and the same meaning for $G_\infty \backslash G$. It has the Fourier expansion

$$g_q(z, m) = \sum_{r=1}^{\infty} c_q(r, m) \exp (2\pi i rz/\lambda),$$

which converges absolutely uniformly on compact subsets of $H$. Before writing down the formula for $c_q$ we introduce some further notation (1, 270–71, 297–98).

The set of positive numbers $|c|$ such that $(\ldots; c.) \in G$ is discrete: $0 < c_0 < c_1 < \ldots \rightarrow \infty$. For $c = c_n > 0$ let

$$D_c = \{d: (\ldots; \ cd) \in G, \ 0 \leq -d < c\lambda\};$$

this is a finite set. Now define the Kloosterman sum

$$W_c(M, \nu) = \sum_{d \in D_c} \exp \left\{2\pi i \frac{va + md}{c\lambda}\right\},$$

where $(ab: cd) \in G_\infty \backslash G$. Also let

$$J_r(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{r+2n}}{n! \Gamma (r+n+1)}$$

be the Bessel function of the first kind. Then

$$c_q(m, m) = m^{2q-1} \left\{1 + \frac{2\pi (-1)^q}{\lambda} \sum_{n=0}^{\infty} \frac{W_{c_n}(m, m)}{c_n} J_{2q-1} \left(\frac{4\pi m}{c_n \lambda}\right)\right\}.$$ 

The convergence of (2.2) and formula (2.4) are proved in (1, 276–7, 295–8) on the assumption that $\Gamma$ is an $H$-group, but examination of the proof shows this assumption to be unessential.

3.

In showing that $c_q \neq 0$ for certain values of $m$, Rankin was able to use very accurate estimates for the Kloosterman sum. But for the general fuchsian group only the trivial
estimate, obtained by setting each exponential equal to 1 in (2.3), is available. We shall employ the following

**Lemma.** Let \( \sum_{a, b} \) denote a sum over

\[
\alpha \leq c < \beta, \quad 0 \leq -d < c\lambda.
\]

Then

\[
\sum_{a, b} c^{-r} < m_1 \alpha^{2-\tau} - m_2 \beta^{2-\tau}, \quad r > 2
\]  \hspace{1cm} (3.1)

\[
\sum_{a, b} c^{-r} < m_3 \beta^{2-\tau} - m_4 \alpha^{2-\tau}, \quad r < 2
\]  \hspace{1cm} (3.2)

where \( m_1, \ldots \) are positive constants depending only on \( \Gamma \).

This result appears as a lemma in (2), p. 400. Again it is proved under the assumption that \( \Gamma \) is an \( H \)-group and again this hypothesis is unnecessary.

Some needed results on Bessel functions can be quoted directly from (3). Write

\[
x_\nu = (1 - \sigma^4)^{1/2}, \quad y_\nu = (1 + \sigma^4)^{1/2}, \quad \sigma = \nu^{-1/6},
\]

\( A_1, A_2, \ldots \) are absolute constants.

For all \( x \geq 0 \) and \( \nu \geq 1 \),

\[
| J_\nu(nx)| \leq (2\pi \nu^{-1/2})(ex/2)^\nu.
\]  \hspace{1cm} (3.3)

This is Lemma 4.1. From Lemma 4.3:

\[
| J_\nu(nx)| \leq A_2 (x^2 - 1)^{-1/4} \nu^{-1/2}, \quad x \geq y_\nu, \quad \nu \geq 15.
\]  \hspace{1cm} (3.4)

Finally, combining (4.9), (4.10) of (3) we get

\[
| J_\nu(nx)| \leq A_3 \nu^{-1/3}, \quad 2 \geq x \geq 2/e.
\]  \hspace{1cm} (3.5)

Let \( q \geq 8, \)

\[
S = \sum_{c>0} \frac{W_c(m, m)}{c} J_{2q-1}(4\pi(m/c\lambda)),
\]

and write

\[
\nu = 2q - 1 \geq 15, \quad Q = 4\pi m/\nu\lambda, \quad x = Q/c\lambda.
\]

Then

\[
| S | \leq \sum_{c>0} 1/c | J_\nu(Q/c) | \sum_{0 \geq -d < c\lambda} 1 = \sum_{0, \infty} 1/c | J_\nu(Q/c) | \]

\[
= \sum_{0, Q_1} + \sum_{0, \infty} + \sum_{Q_2, \infty} = S_1 + S_2 + S_3,
\]

with

\[
Q_1 = Q/2\lambda, \quad Q_2 = eQ/2\lambda.
\]
In $S_3$ we have $x = Q/c\lambda \geq 0$, so from (3.3) and (3.1) with $\beta = \infty$, we obtain

$$|S_3| \leq A_4 \nu^{-1/2} \sum_{Q_5, \infty} 1/c \left( \frac{eQ}{2c\lambda} \right)^{\nu} = A_4 \nu^{-1/2} \left( \frac{eQ}{2\lambda} \right)^{\nu} \sum_{c} \frac{1}{c^{\nu+1}}$$

$$\leq m_5 \nu^{-1/2} \left( \frac{eQ}{2\lambda} \right)^{\nu} \left( \frac{eQ}{2\lambda} \right)^{1-\nu} = m_6 \nu^{-1/2} Q$$

(3.6)

$$= m_7 m\nu^{-3/2}.$$ 

In $S_1$, $x > 2 > y$, so (3.4) and (3.2) give

$$|S_1| \leq A_2 \nu^{-1/2} (2^2 - 1)^{-1/4} \sum_{0, Q_1} c^{-1} \leq m_8 \nu^{-1/2} Q_1 = m_9 m\nu^{-3/2}. \quad (3.7)$$

Finally $2/e < x \leq 2$ in $S_2$; hence by (3.5) and (3.2),

$$|S_2| \leq A_3 \nu^{-1/3} \sum_{Q_1, Q_2} c^{-1} \leq m_10 \nu^{-1/3} \frac{eQ}{2\lambda} = m_{11} m\nu^{-4/3}.$$ 

It follows that $|S| < \lambda / 2\pi$ for $q \equiv q_0$, $m \equiv m_2 q^{4/3}$. Then (2.4) shows that

$$|c_q(m, M)|m^{1-2q} \geq 1 - \frac{2\pi}{\lambda} |S| > 0,$$

completing the proof of the THEOREM of Section 1.

REFERENCES


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