GRAVITATIONAL CAPTURE TO RESONANCE ROTATION OF THE EARLY MOON IN GENERAL RELATIVITY AND GRAVITATION

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1. Introduction

The capture of the Moon represents a very unique event in the history of the solar system (Singer, 1968; Goldreich *et al.*, 1975; Goldreich, 1966). The Moon formed separately in a heliocentrical orbit similar to the Earth's and was later captured by the Earth. The capture of the Moon defines also a severe heating event where the Earth's kinetic energy of rotation is largely dissipated by the mechanism of tidal friction.

A corrected tidal theory (Singer, 1977) leads to a dramatically different result of MacDonald's theory (MacDonald, 1964) suggesting lunar capture from a prograde orbit.

A conventional way to model the tidal perturbation is by means of a tidal "bulge" on the solid and/or ocean-covered Earth that lags in time. The angle δ of the bulge depends on the elastic constants of the Earth; its current value is taken to be $2^\circ\!.16$ (MacDonald, 1964), corresponding to Earth angular velocity ω and Moon mean motion n. The angle has been allowed to remain constant in some orbital evolution. More realistically, δ is taken to be proportional to the relative angular velocity $(\omega - n)$. Singer (1977) describes a theoretical extension of the tidal theory that departs from the existing theory in one important aspect: the phase angle is made dependent on instantaneous frequency $\delta = b(\omega - df/dt)$.

The major result of the push-pull tidal theory is the fact that under certain conditions an orbit can be "reflected" (Singer, 1977; Goldreich *et al.*, 1975), i.e. approach to a minimum distance and then expand again, even if the *inclination* of the orbit is zero. This results from the fact that δ

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may reverse its sign over a sufficient portion of the orbit to yield an effect due to the force which reverses the sign.

A detailed orbit calculation (Goldreich *et al.*, 1975) shows that – going backward in time – the Moon approaches the Earth until it reaches the Earth's synchronous orbit, i.e. until its angular velocity around the orbit, n , matches the spin angular velocity of the Earth, ω. In other words, the length of the month approaches the length of the day. Thus, the major effect of the Moon on the Earth is the despinning of the Earth as the lunar orbit recedes from the Earth after capture. The Earth's kinetic energy of rotation is dissipated into heat by means of tidal friction.

In General Relativity and Gravitation the expressions for the observable depend on the reference frame of the observer while the non-observable, like positions and velocities, depend on the coordinate system (Soffel, 1989).

Even after having chosen the reference frame and the coordinates to answer the question: "how large are the relativistic effects?" one faces in addition the following difficulty: if we simply add the corresponding "post-Newtonian correction terms" to a Newtonian theory then the solution will very rapidly show large deviations as a function of time. This is because each theory possesses its own set of initial conditions and dynamical variables that have to be fitted to observational data ("the Newtonian mass of the Earth $\neq$ post-Newtonian mass of the Earth"). The magnitude of such "relativistic effects" depends not only on the reference frame of the observer, but also on the arbitrary choice of the coordinates. Since the Schwarzschild radius of the Earth $2GM_\oplus/c^2$ is about 1 cm one would expect this length scale to determine the "magnitude of relativistic effects" in the vicinity of the Earth and in a geocentrical reference frame.

According to Brumberg (1958), a solution of the restricted post-Newtonian 3-body problem is presented based on the Hill-Brown method with circular motion for the Earth-Moon system.

More recently, post-Newtonian theories of great accuracy have been formulated by Brumberg *et al.* (1982) and by Lestrade *et al.* (1982). The motion of the Moon in a Fermi-frame moving with the Earth-Moon barycenter has been studied by Mashhoon (1985) and Soffel *et al.* (1986). Brumberg's (1958) post-Newtonian Hill-Brown theory for the lunar motion was expressed in barycentric PN coordinates. The mean motion of the Earth-Moon system is given by

$$n' = \left(\frac{GM_\odot}{a'^3}\right)^{1/2} [1 - 1/2(2\beta + \gamma)\sigma] \tag{1}$$

and

$$\sigma = \frac{GM_\odot}{c^2 a'} = 10^{-8} \tag{2}$$

characterizes the relativistic terms, a' is the "solar parallax"; β, γ are the PPN parameters. The dominant relativistic oscillation of the coordinate distance has an amplitude of $\sim$ 100 cm and a period of half a synodic month. The dominant contribution to the secular advances of the perigee $\dot{\omega}$ and $\dot{\Omega}$ results from the geodetic precession. The Einstein-Infeld-Hoffmann equations of motion for the lunar orbit due to relativistic kinematics yield a perturbing acceleration in the Earth-Moon system that is *roughly constant*. In the co-moving proper frame the amplitude of the dominant relativistic range oscillation amounts to $\sim$ 2 cm (Mashhoon, 1985; Soffel *et al.*, 1980). The general relativistic contributions to $< \dot{\omega} >$ and $< \dot{\Omega} >$ amount to (using $\beta = \gamma = 1$; Goldreich *et al.*, 1975; Brumberg *et al.*, 1982)

$$< \dot{\omega} >= 1.728''/century; < \dot{\Omega} >= 1.901''/century \tag{3}$$

In this report for the plane motion we are completely analyzing the system of differential equations for gravitational capture of the early Moon at the resonance rotation under the action of post-Newtonian gravitational and tidal torques by qualitative analysis and bifurcation theory of dynamical systems. The separation of a 3-dimensional parameter space of the dynamical system by bifurcation surfaces is obtained. The gallery of more than seventy phase portraits of gravitational capture extends the known scenario of cosmogonic evolution of the Moon on the early time.

2. Basic Equations

In accordance with vector angular momentum theory we can write (Beletsky, 1975):

$$\frac{d\vec{L}}{dt} = \vec{M}, \tag{4}$$

where $\vec{M}$ is the momentum of perturbed forces, $\vec{L}$ the kinematic angular momentum.

The important case of perturbed motion has equations of motion of a body with dynamical symmetry and inertia tensor component $A = B \neq C$.

Let us consider the system of equations describing the motion nearby the resonance rotation (Beletsky, 1975):

$$\frac{d^2\kappa}{dt^2} + \frac{3}{2}\Phi_k \frac{A-C}{B} \sin 2\kappa = -\alpha(f_0 \frac{d\kappa}{\mathrm{dt}} + \beta), \tag{5}$$

where κ is a new variable describing a deviation from pure resonance rotation. The second term in the left part of the equation is connected with the post-Newtonian gravitational angular momentum. The term in the right part describes a dissipation factor.

The gravitational angular momentum can itself provide stable resonance rotation, which is described by generalized Cassini laws. However, the initial data of motion can lie outside of a resonance zone. So, a mechanism

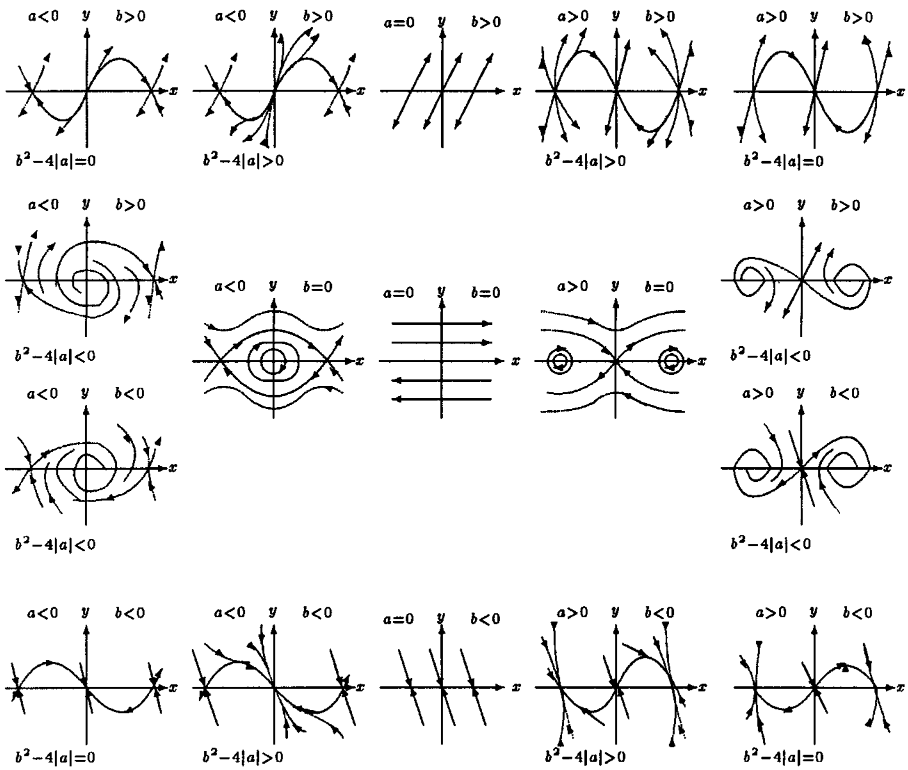

Figure 1. Phase portraits for any values of $a, b; c \simeq 0$.

which provides sufficiently high probability of capture in the resonance zone should be proposed. The tidal friction theory (Singer, 1968, 1970, 1977; MacDonald, 1964) is a very interesting model for this case.

The dynamical system (2) can be written (Salangina, 1994):

$$\left\{ \begin{array}{lcl} \dot{x} & = & y = P(x,y), \\ \dot{y} & = & a \sin x + by + c = Q(x,y,a,b,c), \end{array} \right. \tag{6}$$

where $a = -3\Phi_k \frac{A-C}{B}$, $b = -\alpha f_0$, $c = -2\alpha\beta$ are parameters of the model.

Let us find an equilibrium state of dynamical systems which are determined by conditions (Bautin, 1990; Arnol'd, 1978):

$$P(x_0, y_0) = Q(x_0, y_0, a, b, c) = 0,$$

$$\Delta = \left(\begin{array}{cc} P'_x(x_0,y_0) & P'_y(x_0,y_0) \\ Q'_x(x_0,y_0) & Q'_y(x_0,y_0) \end{array} \right) = \left(\begin{array}{cc} 0 & 1 \\ a \cos x(x_0,y_0) & b \end{array} \right), \tag{7}$$

$$\text{Trace } \sigma = P'_x(x_0,y_0) + Q'_y(x_0,y_0) = b. \tag{8}$$

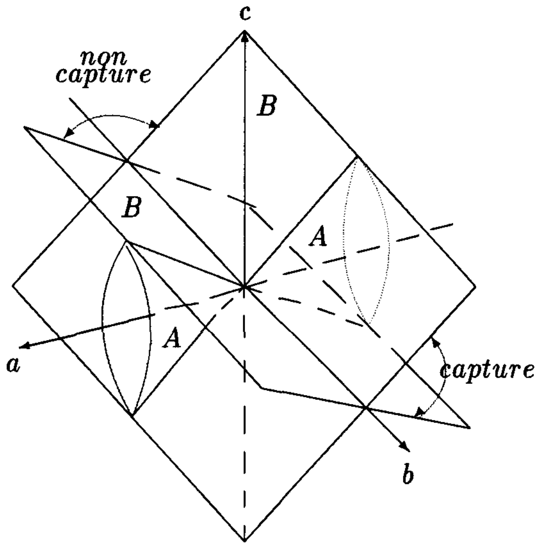

Figure 2. **3-dimensional bifurcation spaces of parameters.**

As shown in Figure 1 there are simple equilibrium states by the saddle, node, focus, center types and/or complicated equilibrium states. Moving along the phase portrait gallery we can meet stable and unstable equilibrium states.

The separation of 3-dimensional parameter spaces of dynamical systems by bifurcational surfaces and lines can be seen in Figure 2. The conditions $| a |>| c |$ are necessary conditions for the existence of an equilibrium state, connecting to resonance rotations of the Moon. The additional condition $b < 0$ is also necessary for gravitational capture at the resonance rotation of the Moon.

In this connection, the attraction of phase trajectory at the resonance zone occurs in qualitatively various ways for $c = 0$ and $c \neq 0$. For $c = 0$ the conditions $b < 0$ $(a \neq 0)$ are necessary and sufficient for gravitational capture at the resonance zone. For $c \neq 0, b < 0$ there are initial data, which do not lead to capture. We can say that for $c = 0$ the probability of capture is equal to one, but for $c \neq 0$ it is less than or equal to one. For $b = 0$, the probability is equal to zero.

3. Conclusions

1. The seventeen phase portraits from the gallery of more than seventy phase portraits of gravitational capture extend the known scenario of global evolution of the Moon in the frame of the Moon's push-pull capture.

2. The results obtained for representing the initial data permit one to estimate the probability of gravitational capture at resonance rotation of the early Moon.

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