THE WEAK COTYPE 2 AND THE ORLICZ PROPERTY OF THE LORENTZ SEQUENCE SPACE $d(a,1)$

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1. Preliminaries. The question "Does a Banach space with a symmetric basis and weak cotype 2 (or Orlicz) property have cotype 2?" is being seriously considered but is still open though the similar question for the r.i. function space on $[0,1]$ has an affirmative answer. (If X is a r.i. function space on $[0,1]$ and has weak cotype 2 (or Orlicz) property then it must have cotype 2.) In this note we prove that for Lorentz sequence spaces $d(a,1)$ they both hold.

Let $a_1 \geq \ldots \geq a_n \geq \ldots > 0$. The Lorentz sequence space $d(a,1)$ is the space of vectors in $\mathbf{R}^\infty$ with finite norm given by

$$\|x\| = \sum_i a_i x_i^*,$$

where (x_i^*) is the decreasing rearrangement of $(|x_i|)$.

A decreasing positive sequence $a=(a_n)$ is said to be *p-regular* $(p \geq 0)$ if there exists a constant $0 < C < \infty$ such that

$$\sum_{i \leq n} a_i^p \leq Cna_n^p \qquad (n \in \mathbf{N}).$$

Reisner proved in **[4]** that a is 1-regular if and only if $d(a,1)$ does not contain l_∞^n uniformly, which is also equivalent to the existence of $q<\infty$ such that $d(a,1)$ is q-concave (for definition of q-concave see below). Another result in **[4]** which plays an important role in our proof is that $d(a,1)$ is 2-concave if and only if a is 2-regular. It is well known that for Banach lattices 2-concave and cotype 2 are equivalent. See **[2]**.

The Banach space X is of *weak cotype* 2 if there is a constant $c \geq 0$, such that for any $n \in \mathbf{N}$ and n-dimensional subspace $E \subset X$, we have $\delta(E) \geq cn^{1/2}$ where

$$\delta(E) := \sup\Bigg\{\rho_2(x_1,\ldots,x_k) = \left(\frac{1}{2^k}\sum_{\epsilon_i=\pm 1}\left\|\sum_{i\leq k}\epsilon_i x_i\right\|^2\right)^{1/2} : x_1,\ldots,x_k \in E,\ \mu_2(x_1,\ldots,x_k) \leq 1,\ k \in \mathbf{N}\Bigg\}$$

$$\mu_p(x_1,\ldots,x_k) := \sup\left\{\left(\sum_{i\leq k}|f(x_i)|^p\right)^{1/p} : f \in E^*,\ \|f\| \leq 1\right\} \qquad (1 \leq p \leq \infty).$$

It is well known that weak cotype 2 implies cotype q for any $q>2$. See **[3]** for details of weak cotype 2.

The Banach space X has the *Orlicz property* if there exists a constant C such that for any $k \in \mathbf{N}$ and any $x_1,\ldots,x_k \in X$, we have

$$\left(\sum_{i\leq k}\|x_i\|^2\right)^{1/2} \leq C\mu_1(x_1,\ldots,x_k).$$

The Orlicz constant $\pi_{2,1}(X) := \inf C$. For more information about the Orlicz property see **[1]**.

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A Banach lattice X is said to be *q-concave* $(q \geq 2)$ if there exists a constant D such that

$$\left(\sum_{i\leq k} \|x_i\|^q\right)^{1/q} \leq D \left\|\left(\sum_{i\leq k} |x_i|^q\right)^{1/q}\right\|$$

for any $k \in \mathbf{N}$ and any $x_1, \ldots, x_k$ in X.

2. The weak cotype 2

THEOREM 1. *$d(a,1)$ is of weak cotype 2 if and only if it is of cotype 2.*

Proof. Assume $d(a,1)$ is of weak cotype 2, so $d(a,1)$ does not contain l_∞^n uniformly. By [4] $d(a,1)$ is q-concave for some $q<\infty$ and a is 1-regular. For any $n \in \mathbf{N}$, let $d(a,1;n)$ denote the n-dimensional subspace of $d(a,1)$ with all coordinates after the nth being 0. Then the q-concave constant of $d(a,1;n)$ is independent of n.

Let $x_1, \ldots, x_k \in d(a,1;n)$. Then we have, using Hölder's inequality, the definition of q-concavity and Khintchine's inequality,

$$\begin{aligned}\rho_2(x_1,\ldots,x_k) &\leq \left(\frac{1}{2^k}\sum_{\epsilon_i=\pm 1}\left\|\sum_{i\leq k}\epsilon_i x_i\right\|^q\right)^{1/q}\\ &\leq C_1\left\|\left(\frac{1}{2^k}\sum_{\epsilon_i=\pm 1}\left|\sum_{i\leq k}\epsilon_i x_i\right|^q\right)^{1/q}\right\|\\ &\leq C_2\left\|\left\{\left(\sum_{i\leq k} x_i^2(j)\right)^{1/2}\right\}_{j=1}^n\right\|. \qquad (1)\end{aligned}$$

Without loss of generality we may assume that the numbers $\left(\sum_{i\leq k} x_i^2(j)\right)^{1/2}$, for $j=1,\ldots,n$, are decreasing. Then from (1) one has

$$\rho_2(x_1,\ldots,x_k) \leq C_2 \sum_{j\leq n}\left(\sum_{i\leq k}|x_i(j)|^2\right)^{1/2} a_j. \qquad (2)$$

Suppose that $\mu_2(x_1,\ldots,x_k)=1$. Since $y_\epsilon = \{\epsilon_j a_j\}_{j=1}^n \in d(a,1;n)^*$ has norm 1 for any $\epsilon = (\epsilon_1,\ldots,\epsilon_n) \in \{-1,1\}^n$, one has

$$\sum_{i\leq k}(y_\epsilon, x_i)^2 \leq 1.$$

Averaging over $\{-1,1\}^n$ we have

$$\sum_{j\leq n}\left(\sum_{j\leq k}|x_i(j)|^2\right)a_j^2 \leq 1. \qquad (3)$$

Combining (2), (3) and Lemma 1 below we have

$$\delta(d(a,1;n)) \leq C_3 \frac{\sum_{j\leq n} a_j}{\left(\sum_{j\leq n} a_j^2\right)^{1/2}}.$$

Meanwhile $\delta(d(a,1;n)) \geq cn^{1/2}$, so that

$$\left(n \sum_{j\leq n} a_j^2\right)^{1/2} \leq C_4 \sum_{j\leq n} a_j,$$

for $\forall n \in \mathbf{N}$. Hence a is 2-regular, since a is 1-regular, and so $d(a,1)$ is 2-concave by **[4]**, equivalently of cotype 2. ■

LEMMA 1. *Let* $a=(a_1,\ldots,a_n)$, $x=(x_1,\ldots,x_n)$, *with* $a_1 \geq \ldots \geq a_n \geq 0$ *and* $x_1 \geq \ldots \geq x_n \geq 0$. *Write* $ax=(a_1x_1,\ldots,a_nx_n)$. *Then*

$$\frac{\|ax\|_1}{\|ax\|_2} \leq \frac{\|a\|_1}{\|a\|_2}.$$

Proof. Define $R(x_1,\ldots,x_n)=\|ax\|_1/\|ax\|_2$ for any x as above. We show that for $1\leq k\leq n-1$, we have

$$R(x_1,\ldots,x_k,x_k,\ldots,x_k) \geq R(x_1,\ldots,x_k,x_{k+1},\ldots,x_{k+1}).$$

Then the conclusion follows because $R(x_1,\ldots,x_1)=\|a\|_1/\|a\|_2$. Define

$$F(t)=R(x_1,\ldots,x_k,t,\ldots,t)=\frac{A+Bt}{\sqrt{C+Dt^2}},$$

where

$$A=\sum_{i\leq k} a_ix_i, \qquad B=\sum_{j=k+1}^{n} a_j,$$

$$C=\sum_{i\leq k} a_i^2x_i^2, \qquad D=\sum_{j=k+1}^{n} a_j^2.$$

Then

$$\begin{aligned} F'(t) &= [B(C+Dt^2)-(A+Bt)Dt](C+Dt^2)^{-3/2} \\ &= (BC-ADt)(C+Dt^2)^{-3/2}. \end{aligned}$$

If $0\leq t\leq x_k$ (and hence $t\leq x_i$, for $i\leq k$), then

$$ADt=\sum_{1\leq i\leq k}\ \sum_{k+1\leq j\leq n} a_ia_j^2tx_i \leq \sum_i \sum_j a_i^2a_jx_i^2=BC.$$

Hence $F'(t)\geq 0$, and so $F(t)$ is increasing. The statement follows. ■

3. Some results on Lorentz sequence spaces.

LEMMA 2. (1) $\mu_2(e_1,\ldots,e_n)=\sqrt{\sum_{i\leq n} a_i^2}$ *in* $d(a,1)$.

(2) $\pi_{2,1}(d(a,1)) \geq \sqrt{n}\,\dfrac{a_1}{\sum_{i\leq n} a_i}$ *for each* n.

(3) *If* a *is* 1*-regular, then for* $k\geq l$,

$$l(a_{k+1}^2+\ldots+a_{k+l}^2) \leq 4C^2(a_{k+1}+\ldots+a_{k+l})^2,$$

where C *is a constant independent of* k *and* l.

Proof. (1) and (2) are fairly easy. We only prove (3). By 1-regularity of a, there is a constant C such that

$$(k+1)a_{k+1} \le a_1 + \ldots + a_{k+l} \le (k+l)Ca_{k+l}$$

and so $a_{k+1} \le \dfrac{k+l}{k+1} Ca_{k+l} \le 2Ca_{k+l}$. Hence

$$l(a_{k+1}^2 + \ldots + a_{k+l}^2) \le l^2 a_{k+1}^2 \le 4C^2 l^2 a_{k+l}^2 \le 4C^2(a_{k+1} + \ldots + a_{k+l})^2,$$

which completes the proof. ■

We now prove a result using Talagrand's "isoperimetric inequality" (see [6] for details). Let $x_1, \ldots, x_n$ be elements of any linear normed space X, $\mu_2(x_1, \ldots, x_n) = \sigma$. Let P_n be the usual probability measure on $D_n = \{-1, 1\}^n$. For $\epsilon \in D_n$, write $s(\epsilon) = \sum_i \epsilon_i x_i$. For $t > 0$, let $A_t = \{\epsilon \in D_n : \|s(\epsilon)\| \le t\}$, $B_t = D_n \backslash A_t$. The inequality states that for $M > 0$, $P_n(A_M)P_n(B_{M+t}) \le \exp(-t^2/8\sigma^2)$. If $M = \int \|s(\epsilon)\|$, then $P_n(B_{2M}) \le \frac{1}{2}$ by since Chebyshev's inequality $M \ge \int_{B_{2M}} \|s(\epsilon)\| \ge 2MP_n(B_{2M})$. Hence $P_n(B_{2M+t}) \le 2\exp(-t^2/8\sigma^2)$.

PROPOSITION. *Let $y_1, \ldots, y_N$ be positive elements of $d(a, 1; n)$ with $\sum_{i \le N} y_i^2 \le 1$. Let the q-concave constant of $d(a, 1; n)$ be C, for some $q < \infty$. Then there exist $z_1, \ldots, z_N \in d(a, 1; n)$ and $C' = C'(C)$ such that $|z_i(j)| = y_i(j)(\forall i, j)$ and*

$$\mu_1(z_1, \ldots, z_N) \le C'\left(\sum a_i + \sqrt{N \sum a_i^2}\right).$$

This generalizes Lemma 6.4 of [5]. The proof is similar but more transparent.

Proof. Let D_{Nn} denote the set of $\epsilon = \{\epsilon_{i,j} : i \le N, j \le n\}$ with $\epsilon_{i,j} \in \{-1, 1\}$. Assuming equal probability 2^{-Nn} to each such ϵ, define $z_i^\epsilon(j) = \epsilon_{i,j} y_i(j)$ for $1 \le i \le N$, $1 \le j \le n$. Clearly $|z_i^\epsilon(j)| = y_i(j)$.

Let $\eta_i \in \{-1, 1\}$ for $1 \le i \le N$. Then

$$\sum_i \eta_i z_i^\epsilon(j) = \sum_i \epsilon_{i,j}^* y_i(j) = \sum_i z_i^{\epsilon^*}(j),$$

where $\epsilon_{i,j}^* = \eta_i \epsilon_{i,j}$. The map $\epsilon \mapsto \epsilon^*$ maps D_{Nn} onto itself.

Write $y_{i,j} = y_i(j)e_j$, so that

$$z_i^\epsilon = \sum_j \epsilon_{i,j} y_{i,j}.$$

If f is a linear functional on $d(a, 1; n)$ with $\|f\| \le 1$, then

$$\sum_i \sum_j f(y_{i,j})^2 = \sum_j \sum_i y_i(j)^2 f(e_j)^2$$
$$\le \sum_j f(e_i)^2 \le \sum_i a_i^2 \tag{4}$$

by (1) of Lemma 2 and $\sum_i y_i(j)^2 \le 1$. Hence $\mu_2\{y_{i,j} : i \le N, j \le n\} \le \sqrt{k\, a_i^2}$.

We will show that

$$\int_{D_{Nn}} \left\| \sum_{i,j} \epsilon_{i,j} y_{i,j} \right\| dP(\epsilon) \le C' \sum a_i. \tag{5}$$

By the isoperimetric inequality it will then follow that

$$P_{Nn}\left\{\epsilon : \left\| \sum_i z_i^\epsilon \right\| \ge C' \sum a_i + t\right\} \le 2\exp\left(-t^2/8 \sum a_i^2\right).$$

If we then choose t with $t^2 = 8N \sum a_i^2$, then for each $(\eta_i) \in D_N$, we have

$$P_{Nn}\left\{\epsilon : \left\| \sum \eta_i z_i^\epsilon \right\| \ge C' \sum a_i + t\right\} \le 2e^{-N}.$$

Hence the probability of the union of such sets for all (η_i) is not greater than $2^{N+1}e^{-N}$(<1, if $N > 3$). So there is an ϵ belonging to none of these sets, so that $\|\sum \eta_i z_i^\epsilon\| \le C' \sum a_i + t$ for all (η_i). To prove (5), note first that for each s, $\sum_j y_{i,j}(s)^2 = y_i(s)^2$, and so $\sum_{i,j} y_{i,j}(s)^2 = \sum_i y_i(s)^2 \le 1$. By the q-concavity and Khintchine's inequality, it is easy to obtain the statement. ■

4. The Orlicz property.

THEOREM 2. *If $d(a,1)$ has the Orlicz property then a is 2-regular; equivalently $d(a,1)$ is of cotype 2.*

Proof. Throughout the proof we use $M, M_1, M_2, \ldots$ for the constants which are independent of n. Since $\pi_{2,1}(d(a,1)) < \infty$, $d(a,1)$ does not contain l_∞^n uniformly, so by [**4**], a is 1-regular. Then it is enough to prove that there exists M such that $n \sum_{i \le n} a^2 \le M\left(\sum_{i \le n} a_i\right)^2$, $\forall n \in \mathbf{N}$.

Without loss of generality we may assume that $n = 2^m$. Let

$$k' = \frac{\left(\sum_{i \le n} a_i\right)^2}{\sum_{i \le n} a_i^2}.$$

Then $1 \le k' \le n$. Let k by the largest integer of the form 2^s with $k \le k'$, so that $2k \ge k'$. Note that

$$\left(k \sum_{i \le n} a_i^2\right)^{1/2} \le \sum_{i \le n} a_i \le \left(2 \sum_{i \le n} a_i^2\right)^{1/2}.$$

Let $l = n/k$ (an integer). We must show that $l \le M_1$. Let

$$y(m) = a_{(m-1)l+1} + \ldots + a_{ml}, \qquad m = 1, \ldots, k,$$

so that $y(1) \ge \ldots \ge y(k)$. Let $y_1 = y \in \mathbf{R}^k$ and let $y_2, \ldots, y_k$ be the elements of $\mathbf{R}^k$

obtained from y by cyclic permutation. Let x_i be the element $(y_i, \ldots, y_i)$ of $(\mathbf{R}^k)^l = \mathbf{R}^n$ for $1 \le i \le k$. Then

$$\|x_i\| = y(1)(a_1 + \ldots + a_l) + y(2)(a_{l+1} + \ldots + a_{2l}) + \ldots + y(k)[a_{(k-1)l+1} + \ldots + a_{kl}] = \sum_{j \le k} y(j)^2 = \|y\|_2^2. \tag{6}$$

Also $\left(\sum_{i \le k} x_i^2\right)^{1/2} = \|y\|_2(1, \ldots, 1)$.

Since $d(a, 1)$ is q-concave for some $q < \infty$, by the Proposition there exist $z_1, \ldots, z_k$ such that $|z_i| = x_i$ (hence $\|z_i\| = \|y\|_2^2$) and

$$\mu_1(z_1, \ldots, z_k) \le \|y\|_2 M_2 \sum_{i \le n} a_i.$$

Since $(\sum \|z_i\|^2)^{1/2} \le \pi_{2,1}(d(a, 1))\mu_1(z_1, \ldots, z_k)$, we have

$$\|y\|_2 \le M_3 \left(\sum_{i \le n} a_i^2\right)^{1/2}. \tag{7}$$

By (2) of Lemma 2, $\sqrt{l}(a_1^2 + \ldots + a_l^2) \le M_4 y(1)^2$, and by (3) of Lemma 2

$$l(a_{(m-1)l+1}^2 + \ldots + a_{ml}^2) \le M_5 y(m)^2 \quad (m = 2, \ldots, k).$$

Hence

$$\sqrt{l} \sum_{i \le n} a_i^2 \le M_6 \|y\|_2^2 \le M_7 \sum_{i \le n} a_i^2,$$

and so $l \le M_1$. ■

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