94.34 Regular polygons with integer coordinates

The results in this paper are all known, but I collect them as an extension topic for able sixth-formers studying advanced trigonometry. In our syllabus this appears in the Core 4 module and I have used this material with Double Maths students. Many of the propositions are worth proving individually, even if you do not have time to cover the whole series.

The question we will answer is this: What regular polygons can be drawn using only integer coordinates? With the right class, you could issue them with square dotty paper and see what they come up with. The chances are that they will come up with plenty of squares. They might also come up with equiangular octagons, equilateral octagons, equilateral hexagons but no truly regular polygons except squares.

**Proposition 1:** If $A$, $B$ and $C$ are points with integer coordinates in two dimensions, then $\tan \angle ABC$ is rational or undefined.

**Proof:** There are several configurations, but a typical one is shown in Figure 1.

Using the compound angle formula for tangent gives

\[
\tan \angle ABC = \tan (\angle ABX - \angle CBY) = \frac{\tan \angle ABX - \tan \angle CBY}{1 + \tan \angle ABX \tan \angle CBY}
\]

\[
= \frac{\frac{AX}{BX} - \frac{CY}{BY}}{1 + \frac{AX}{BX} \times \frac{CY}{BY}}
\]

which is rational. If $AB$ is perpendicular to $BC$ then $\tan \angle ABC$ is undefined.

**Remark:** The proposition above remains true if we replace integer coordinates by rational coordinates. The results below may also be generalised similarly.
Corollary 2: It is impossible to draw a regular polygon with $3m$ sides using only integer coordinates.

Proof: Joining up every $m$th vertex of the polygon forms an equilateral triangle $ABC$. But $\tan \angle BAC = \tan \frac{\pi}{3} = \sqrt{3}$, which is irrational.

Proposition 3: The exact value of $\tan \frac{\pi}{8}$ is $\sqrt{2} - 1$.

Proof: Let $t = \tan \frac{\pi}{8}$. Using the double angle formula for tangent

\[
1 = \tan \frac{\pi}{4} = \frac{2t}{1 - t^2}
\]

i.e. $t^2 + 2t - 1 = 0$

i.e. $t = -1 \pm \sqrt{2}$.

But $\tan \frac{\pi}{8}$ must be positive. (The negative root gives $\tan \frac{5\pi}{8}$.)

Corollary 4: It is impossible to draw a regular polygon with $8m$ sides using only integer coordinates.

Proof: Joining up every $m$th vertex of the polygon forms a regular octagon $ABCDEFGH$. But $\tan \angle BAC = \tan \frac{\pi}{8} = \sqrt{2} - 1$, which is irrational.

Proposition 5: The value of $\tan \frac{\pi}{5}$ is irrational.

Proof: Let $a = \frac{\pi}{5}$ and $t = \tan a$. Now $\tan 2a = -\tan 3a$, so using double and triple angle formulae for tangent gives

\[
\frac{2t}{1 - t^2} + \frac{3t - t^3}{1 - 3t^2} = 0
\]

i.e. $t(t^4 - 10t^2 + 5) = 0$

i.e. $t = 0$ or $t^2 = 5 - 2\sqrt{5}$ or $t^2 = 5 + 2\sqrt{5}$

Clearly $t \neq 0$. If $t$ were rational, then $t^2$ would be rational as well, which it is not.

Corollary 6: It is impossible to draw a regular polygon with $5m$ sides using only integer coordinates.

Proof: Joining up every $m$th vertex of the polygon forms a regular pentagon $ABCDE$. But $\tan \angle BAC = \tan \frac{\pi}{5}$, which is irrational.

It should now be clear that our main question reduces to: For what rational numbers $\frac{p}{q}$ is $\tan \frac{\pi}{4} \frac{p}{q}$ also rational? It turns out to be easier to deal first with cosines.
**Proposition 7**: If \( \frac{p}{q} \) is rational and \( \cos \frac{p\pi}{q} = \frac{x}{y} \) is also rational and in lowest terms with \( y > 0 \), then \( y = 1 \) or \( y = 2 \).

**Proof**: Define a rational sequence by \( a_0 = \cos \frac{p\pi}{q} \), \( a_{n+1} = 2a_n^2 - 1 \). Repeated application of a double angle formula for cosine shows that \( a_n = \cos \left( 2^n \frac{p\pi}{q} \right) \). Because cosine has period \( 2\pi \), the sequence can only take up to \( 2q \) different values. Let \( a_N = \frac{x}{y} \) be the term with largest denominator when written in lowest terms with \( y > 0 \).

If \( y \) is odd, then

\[
a_{N+1} = \frac{2x^2 - y^2}{y^2}
\]

is in lowest terms. If \( y > 1 \) then \( y^2 > y \), which is impossible.

If \( y \) is even, then

\[
a_{N+1} = \frac{x^2 - \frac{1}{2}y^2}{\frac{1}{2}y^2}
\]

is in lowest terms. If \( y > 2 \) then \( \frac{1}{2}y^2 > y \), which is impossible.

**Remark**: The above proof comes from [1], which Nick Lord drew to my attention. It also gave rise to question 5 of a recent Olympiad paper [2, p. 133].

**Proposition 8**: If \( \frac{p}{q} \) is rational and \( \tan \frac{p\pi}{q} = \frac{x}{y} \) is also rational and in lowest terms with \( y > 0 \), then \( y = 1 \).

**Proof**: Let \( t = \tan \frac{p\pi}{q} \). Then using the half-tangent formula for cosine gives

\[
\frac{1 - t^2}{1 + t^2} = \cos \frac{2p\pi}{q} = c, \text{ say.}
\]

The left-hand side is rational, so Proposition 7 shows that the right-hand side must be \(-1, -\frac{1}{2}, 0, \frac{1}{2} \) or 1. Now

\[
1 - t^2 = c + ct^2
\]

i.e. \( t = \pm \sqrt{\frac{1 - c}{1 + c}} \).

Checking the five possible values of \( c \) shows that the only rational values are \( t = \pm 1 \) when \( c = 0 \) and \( t = 0 \) when \( c = 1 \).

**Corollary 9**: It is impossible to draw a regular polygon with \( m \) sides using only integer coordinates unless \( m = 4 \).
Proof: Let the regular polygon be labelled \( ABC \ldots \). Then 
\[
\tan \angle BAC = \tan \frac{\pi}{m},
\]
which is irrational unless \( m = 4 \).

We may generalise the question to three dimensions by asking: What regular polyhedra can be constructed using only integer coordinates? It is not too difficult to see that the octahedron can be constructed as well as the cube. More surprisingly, the tetrahedron can also be constructed.

Exercise 10: Find points \( A, B, C \) and \( D \) in three dimensions with integer coordinates such that \( ABCD \) is a regular tetrahedron.

These turn out to be the only possibilities. This is a consequence of the following result.

Proposition 11: It is impossible to draw a regular plane polygon with \( m \) sides using only integer coordinates in three dimensions unless \( m = 3, 4 \) or 6.

Proof: Let the regular polygon be labelled \( ABC \ldots \). Then
\[
\cos \angle ABC = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}| \cdot |\overrightarrow{BC}|}.
\]
The numerator is an integer and the denominator is an integer since \( \overrightarrow{BA} = \overrightarrow{BC} = \sqrt{BC \cdot BC} \).

Proposition 7 ensures that \( \angle ABC \) is \( \frac{\pi}{3}, \frac{\pi}{2} \) or \( \frac{2\pi}{3} \). It is easy to show that all these possibilities can occur.

Remark: Proposition 11 remains true in more than three dimensions.

Exercise 12: Use Proposition 11 to prove that it is impossible to construct a regular icosahedron in three dimensions using only integer coordinates.

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References

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