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THE 2-PRIMITIVE IDEALS OF STRUCTURAL MATRIX NEAR-RINGS

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An isomorphism (as groups) is established between an arbitrary connected module over a structural matrix near-ring and a direct sum of appropriate modules over the base near-ring. This isomorphism leads to a characterization of the 2-primitive ideals of a structural matrix near-ring.

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1. Introduction

Until 1986 it had been unanimously decided in near-ring quarters that in most cases the notion of a matrix near-ring over an arbitrary near-ring does not make sense, since matrix multiplication is not associative as long as matrices are considered as arrays of entries for which addition and multiplication are defined in the familiar way (see Heatherly [1]). But then Meldrum and Van der Walt [3] used a functional approach in their definition of a matrix near-ring over an arbitrary near-ring R, by considering matrices as certain functions from R^n into R^n, where R^n denotes the direct sum of n copies of $(R,+)$.

Although there are many similarities to the ring case, there are also some striking differences, e.g., the correspondence between the two-sided ideals of the base near-ring R and the matrix near-ring $\mathbf{M}_n(R)$ is much more complex than in the ring case (see Van der Walt [9]). Since then matrix near-rings have been the object of study in various papers, e.g., [4, 6, 8–11].

Van der Walt [10] generalized the concept of a monogenic module to that of a connected module, and showed how G^n can be viewed as a (connected) $\mathbf{M}_n(R)$-module in case G is a connected R-module. Using the interplay between properties of G and G^n, he proved that R is 2-primitive if and only if $\mathbf{M}_n(R)$ is 2-primitive, which led to the following result:

$$\mathscr{T}_2(\mathbf{M}_n(R)) = (\mathscr{T}_2(R))^*, \tag{1}$$

where $(\mathscr{T}_2(R))^* := \{U \in \mathbf{M}_n(R) : Uu \in (\mathscr{T}_2(R))^n \text{ for every } u \in R^n\}$. (Obviously, $(\mathscr{T}_2(R))^* = \mathbf{M}_n(\mathscr{T}_2(R))$ in case R is a ring.) A crucial step in the building up to that proof was to establish an isomorphism (at least as groups) between an arbitrary connected $\mathbf{M}_n(R)$-module and a direct sum of appropriate R-modules. The generalization of this isomorphism to connected modules over structural matrix near-rings is the decisive starting point in the present paper.

A *structural matrix ring* $\mathbf{M}(B,R)$, R a ring and $B=[b_{ij}]$ a reflexive and transitive $n\times n$ Boolean matrix, is a subring of the (complete) matrix ring $\mathbf{M}_n(R)$, which is a ring only by virtue of the shape of its matrices, in the sense that substructures of R play no role:

$$\mathbf{M}(B,R):=\{U=[u_{ij}]\in\mathbf{M}_n(R):b_{ij}=0\Rightarrow u_{ij}=0\}.$$

In fact, $\mathbf{M}(B,R)$ is a generalized matrix ring, i.e. an S-graded ring $\bigoplus_{s\in S}R_s$, where $S:=\{E_{ij}:1\leqq i,j\leqq n \text{ and } b_{ij}=1\}\cup\{\#\}$, $R_s:=RE_{ij}$ if $s=E_{ij}$, and $R_{\#}:=0$. The E_{ij}'s denote the matrix units and

$$E_{ij}E_{kl}=\begin{cases}E_{il}, & \text{if } j=k\\ \#, & \text{otherwise.}\end{cases}$$

The author [12] showed that for a special radical $\mathscr{R}$ determined by a special class $\mathscr{M}$ of rings, such that $T\in\mathscr{M}$ if and only if $\mathbf{M}_n(T)\in\mathscr{M}$ whenever T has an identity, $\mathscr{R}(\mathbf{M}(B,R))$ is the sum of two two-sided ideals, namely, in the first place, the set of matrices with entries from $\mathscr{R}(R)$ in positions where B has ones, and zeroes elsewhere, and, secondly, the set of matrices with entries from R in the "antisymmetric part" of B, i.e. the positions (i,j) such that $b_{ij}=1$ and $b_{ji}=0$, and zeroes elsewhere. This result is, on the one hand, a generalization of the well known fact that $\mathscr{R}(\mathbf{M}_n(R))=\mathbf{M}_n(\mathscr{R}(R))$ and of the fact that, for a field F, the Jacobson radical of the lower triangular matrix ring

$$\begin{bmatrix}F&0&0\\F&F&0\\F&F&F\end{bmatrix}\text{ is }\begin{bmatrix}0&0&0\\F&0&0\\F&F&0\end{bmatrix},$$

and, on the other hand, a nice illustration of the Jacobson radical of a generalized matrix ring being a homogeneous ideal, as shown in [13]. (The description of the Jacobson radical of a generalized matrix ring is still lacking.)

Van der Walt and the author [11] showed that the two obvious definitions of a structural matrix near-ring $\mathbf{M}(B,R)$, R a (right) near-ring, somewhat unexpectedly yield the same near-ring. The main purpose of [11] was to describe $\mathscr{T}_2(\mathbf{M}(B,R))$. To this end, the characterization of the $\mathscr{T}_2$-radical of a near-ring as the intersection of its 2-primitive ideals, and the characterization of the $\mathscr{T}_2$-radical as the intersection of its strictly maximal left ideals, were considered. (Recall that a strictly maximal left ideal of R is a maximal left ideal which is also maximal as a left R-submodule of ${}_RR$.) Although Van der Walt [10] succeeded in characterizing the 2-primitive ideals of $\mathbf{M}_n(R)$ in terms of those of R, and then used them to arrive at (1), the authors of [11] did not manage to generalize the former's methods to structural matrix near-rings, the crucial problem being that they could not find an isomorphism (as groups) between an arbitrary connected $\mathbf{M}(B,R)$-module and a direct sum of "appropriate" R-modules (see the remark just after (1)). Consequently they opted in [**11**] for a characterization of the strictly maximal left ideals of $\mathbf{M}(B,R)$ in terms of those of R. However, quite a number of very technical results were needed to obtain that characterization, and the building up was rather slow.

In the present paper the sought after isomorphism (as groups) between an arbitrary connected $\mathbf{M}(B,R)$-module and a direct sum of appropriate R-modules (which need not be connected) is established. This leads, apart from additional results presented in Section 3 to provide a clear holistic picture of connected $\mathbf{M}(B,R)$-modules, relatively quickly to a number of elegant results (compared with those in [**11**]), terminating in Section 4 in a characterization of the 2-primitive ideals of $\mathbf{M}(B,R)$. As a result of this the same description of the $\mathcal{J}_2$-radical of a structural matrix near-ring as in [**11**] is obtained.

It is not known (see [**11**]) whether $\mathcal{J}_2(\mathbf{M}(B,R))$ can be expressed as the sum of two (two-sided) ideals, one of which is nilpotent, as in the ring case (see [**12**]). In the last part of Section 4 some progress is made in this direction, where it is shown that $\mathcal{J}_2(\mathbf{M}(B,R))$ contains the sum of two such ideals, which are precisely the two ideals in [**12**] in case R is a ring.

2. Preliminaries and notation

R will be a generic symbol for a zero-symmetric right near-ring with identity 1. The direct sum of n copies of a group $(G,+)$ is denoted by G^n, and the elements of G^n are thought of as column vectors, but written in transposed form with pointed brackets, e.g., $\langle g_1, g_2, \ldots, g_n \rangle$. The symbols ι_i and π_i denote the ith coordinate injection and projection functions respectively, $1 \leqq i \leqq n$.

For the ease of the reader we provide the pertinent definitions regarding matrix near-rings. Meldrum and Van der Walt [3] call the functions $f_{ij}^r\colon R^n \to R^n$, $1 \leqq i,j \leqq n$, defined by

$$f_{ij}^r := \iota_i \lambda(r) \pi_j,$$

the *elementary* $n \times n$ *matrices over* R, where $r \in R$ and $\lambda(r)\colon R \to R$ is the left multiplication $s \mapsto rs$, for all $s \in R$. They call the subnear-ring of $M(R^n)$, i.e. the near-ring of all mappings on R^n, generated by the elementary $n \times n$ matrices over R, the *near-ring of* $n \times n$ *matrices over* R, and denote it by $\mathbf{M}_n(R)$. The elements of $\mathbf{M}_n(R)$ are called *matrices.* Representations of matrices will be needed. Meldrum and Van der Walt defined the set $\mathscr{E}_n(R)$ of *matrix expressions*, i.e. the subset of the free semigroup over the alphabet of symbols

$$\{f_{ij}^r : r \in R, 1 \leqq i,j \leqq n\} \cup \{(,), +\},$$

recursively by the following rules:

(1) $f_{ij}^r \in \mathscr{E}_n(R)$ for $1 \leqq i, j \leqq n$ and all $r \in R$.

(2) If $E_1, E_2 \in \mathscr{E}_2(R)$, then $E_1 + E_2 \in \mathscr{E}_n(R)$.

(3) If $E_1, E_2 \in \mathscr{E}_n(R)$, then $(E_1)(E_2) \in \mathscr{E}_2(R)$.

The *weight* $w(U)$ of a matrix U is the length of an expression of minimal length representing U, where the *length* of an expression E is the number of f_{ij}^r's in E.

$B = [b_{ij}]$ will be a generic symbol for a reflexive and transitive $n \times n$ Boolean matrix. B determines and is determined by the binary relation $\leqq_B$ on $\{1, 2, \ldots, n\}$ defined by

$$i \leqq_B j :\Leftrightarrow b_{ij} = 1.$$

The quasi-order relation $\leqq_B$ gives rise in the usual way to an equivalence relation $\sim_B$ on $\{1, 2, \ldots, n\}$ defined by

$$i \sim_B j :\Leftrightarrow i \leqq_B j \quad \text{and} \quad j \leqq_B i.$$

The number of equivalence classes induced by $\sim_B$ is denoted by b, and $z_1, z_2, \ldots, z_b$ will be representatives of the equivalence classes, which we denote by $[z_a]$, $a = 1, 2, \ldots, b$. We denote the elements of $[z_a]$, $1 \leqq a \leqq b$, by $j_{1,a}, j_{2,a}, \ldots, j_{n_a,a}$, i.e. $|[z_a]| = n_a$.

For the ease of the reader we state [**11**, Theorem 2.8], which will be invoked on several occasions in the sequel:

Theorem 2.1. ([**11**, Theorem 2.8]) *$\mathbf{M}(B, R)$ is the subnear-ring of $\mathbf{M}_n(R)$ generated by the set $\{f_{ij}^r : r \in R$ and $b_{ij} = 1\}$.*

Throughout the paper ideal will mean two-sided ideal. Notation and standard results not given here may be looked up in Meldrum [**2**] or Pilz [**7**].

3. Connected modules over structural matrix near-rings

Van der Walt [**10**] calls an R-module G *connected* if, for any $g_1, g_2 \in G$, there are $g \in G$ and $r, s \in R$ such that $g_1 = rg$ and $g_2 = sg$. This generalization of a monogenic module was needed in [**10**] to impose an $\mathbf{M}_n(R)$-module structure on G^n. It was shown in [**10**, Theorem 3.5] that if Γ is a connected $\mathbf{M}_n(R)$-module, then $(\Gamma, +) \cong (G^n, +)$ as groups, where G is an appropriate R-module.

Theorem 3.1. *Let Γ be a connected $\mathbf{M}(B, R)$-module. Then*

$$\Gamma \cong \sum_{a=1}^{b} \sum_{k=1}^{n_a} \iota_{j_{k,a}}(G_a)$$

as additive groups, for appropriate R-modules $G_1, G_2, \ldots, G_b$.

Proof. Let $1 \leqq a \leqq b$, and let $G_a = f_{z_a z_a}^1 \Gamma$. Then by [**10**, Lemma 3.2(2)] G_a is an

R-module by setting $r(f^1_{z_az_a}\gamma)=f^r_{z_az_a}\gamma$ for every $r\in R,\gamma\in\Gamma$, since by [3, Lemma 3.1(7)] $f^1_{z_az_a}\in(\mathbf{M}(B,R))_d$, the distributive part of $\mathbf{M}(B,R)$. Furthermore, as $f^1_{z_aj_{k,a}}\gamma=f^1_{z_az_a}(f^1_{z_aj_{k,a}}\gamma)\in G_a$, it follows that

$$\theta:\Gamma\to\sum_{a=1}^{b}\sum_{k=1}^{n_a}\iota_{j_{k,a}}(G_a),$$

defined by

$$\theta(\gamma)=\sum_{a=1}^{b}\sum_{k=1}^{n_a}\iota_{j_{k,a}}(f^1_{z_aj_{k,a}}\gamma),$$

is a group homomorphism. Next, if $\gamma_{j_{k,a}}$, for $a=1,2,\dots,b$ and $k=1,2,\dots,n_a$, are arbitrary elements of Γ, then

$$\begin{aligned}\theta\left(\sum_{a=1}^{b}\sum_{k=1}^{n_a}f^1_{j_{k,a}z_a}\gamma_{j_{k,a}}\right)&=\sum_{a=1}^{b}\sum_{k=1}^{n_a}\theta(f^1_{j_{k,a}z_a}\gamma_{j_{k,a}})\\&=\sum_{a=1}^{b}\sum_{k=1}^{n_a}\left(\sum_{c=1}^{b}\sum_{m=1}^{n_a}\iota_{j_{m,c}}(f^1_{z_cj_{m,c}}(f^1_{j_{k,a}z_a}\gamma_{j_{k,a}}))\right)\\&=\sum_{a=1}^{b}\sum_{k=1}^{n_a}\iota_{j_{k,a}}(f^1_{z_az_a}\gamma_{j_{k,a}}),\end{aligned}$$

and so θ is onto. Finally, if $\theta(\gamma)=0$, i.e. $f^1_{z_aj_{k,a}}\gamma=0$ for all k and a, then

$$\begin{aligned}\gamma&=(f^1_{11}+f^1_{22}+\cdots+f^1_{nn})\gamma\\&=\left(\sum_{a=1}^{b}\sum_{k=1}^{n_a}f^1_{j_{k,a}j_{k,a}}\right)\gamma\\&=\left(\sum_{a=1}^{b}\sum_{k=1}^{n_a}f^1_{j_{k,a}z_a}f^1_{z_aj_{k,a}}\right)\gamma\\&=\sum_{a=1}^{b}\sum_{k=1}^{n_a}f^1_{j_{k,a}z_a}(f^1_{z_aj_{k,a}}\gamma)\\&=0,\end{aligned}$$

which establishes the result.

Henceforth G_a, $a=1,2,\ldots,b$, will denote the R-modules defined in the proof of Theorem 3.1. In the complete matrix near-ring case, i.e. where B is the universal $n\times n$ Boolean matrix (and $\sim_B$ induces a single equivalence class), G_1 need not be connected, even if R is a ring (see [**5**, page 70]). However, Van der Walt showed in [**10**, Theorem 3.10] that G_1 is of type 2 (and hence connected) in case Γ is of type 2. We shall show in Theorem 3.6 that every G_a, $1\leqq a\leqq b$, is of type 2 (and hence connected) in case Γ is of type 2, but we first prove that without such a restriction on Γ, the shape of B in the structural matrix near-ring case may force some of the G_a's to be connected.

Proposition 3.2. *Let Γ be a connected $\mathbf{M}(B,R)$-module. If $|[z_a]|=1$ and z_a is a maximal element of $\{z_1,z_2,\ldots,z_b\}$ with respect to $\leqq_B$, then G_a is a connected R-module; moreover, if Γ is monogenic (by α), then G_a is monogenic (by $f^1_{z_az_a}\alpha$).*

Proof. Let $f^1_{z_az_a}\gamma$, $f^1_{z_az_a}\delta\in G_a$. Since Γ is connected, there are $U,V\in\mathbf{M}(B,R)$ and $\alpha\in\Gamma$ such that $U\alpha=\gamma$ and $V\alpha=\delta$. Hence, $f^1_{z_az_a}\gamma=(f^1_{z_az_a}U)\alpha$ and $f^1_{z_az_a}\delta=(f^1_{z_az_a}V)\alpha$. We use induction on the weight $w(U)$ of U to show that $f^1_{z_az_a}U=f^r_{z_az_a}$ for some $r\in R$. If $w(U)=1$, then by Theorem 2.1 we have $U=f^r_{ij}$ for some $r\in R$, where $b_{ij}=1$. Hence, by [**3**, Lemma 3.1(3)] we have

$$f^1_{z_az_a}U=\begin{cases} f^r_{z_aj}, & \text{if}\quad i=z_a\\ f^0_{z_aj}=f^0_{z_az_a}, & \text{otherwise.}\end{cases}$$

If $i=z_a$, then the conditions on z_a ensure that $j=z_a$, and so the assertion is true if $w(U)=1$. The rest of the induction process is straightforward if one keeps in mind that $f^1_{z_az_a}\in(\mathbf{M}(B,R))_d$ and that $f^r_{z_az_a}=f^r_{z_az_a}f^1_{z_az_a}$. We conclude that $f^1_{z_az_a}\gamma=f^r_{z_az_a}\alpha=r(f^1_{z_az_a}\alpha)$. Similarly, $f^1_{z_az_a}\delta=s(f^1_{z_az_a}\alpha)$ for some $s\in R$. Therefore G_a is a connected R-module. The monogenic case is treated in the same way.

Example 3.3. If

$$B=\begin{bmatrix} 1 & 1 & 1\\ 1 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}$$

and Γ is a connected $\mathbf{M}(B,R)$-module, then $\Gamma\cong\langle G_1,G_1,G_2\rangle$ as groups, where $G_1:=f^1_{11}\Gamma$ and $G_2:=f^1_{33}\Gamma$ are R-modules, the latter being connected.

Following [**5**], and supported by Theorem 2.1, we call a matrix of the form $\sum_{i(i\leqq_B j)}f^{r_i}_{ij}$, with the r_i's elements of R and $1\leqq j\leqq n$, a *jth column matrix* in $\mathbf{M}(B,R)$. The set of all jth column matrices in $\mathbf{M}(B,R)$ will be denoted by $\mathscr{C}_j$. The proofs of [**5**, Proposition 1.24] and [**5**, Lemma 2.4] serve to a great extent as the proofs of the following two results:

Lemma 3.4. *$\mathscr{C}_j$ is an $\mathbf{M}(B,R)$-submodule of $\mathbf{M}(B,R)$ for $j=1,2,\dots,n$.*

Proof. It follows immediately that $(\mathscr{C}_j,+)$ is a subgroup of $(\mathbf{M}(B,R),+)$ if one keeps in mind that $f^{r_i}_{ij}+f^{r_k}_{kj}=f^{r_k}_{kj}+f^{r_i}_{ij}$ in case $i\neq k$. We now use induction on the weight $w(U)$ of U to show that $U(\sum_{i(i\leqq_B j)}f^{r_i}_{ij})\in\mathscr{C}_j$ for every $U\in\mathbf{M}(B,R)$. If $w(U)=1$, then by Theorem 2.1 we have $U=f^r_{kl}$ for some $r\in R$, where $b_{kl}=1$. Then $f^r_{kl}(\sum_{i(i\leqq_B j)}f^{r_i}_{ij})=f^0_{jj}$ if $l\nleqq_B j$, and $f^r_{kl}(\sum_{i(i\leqq_B j)}f^{r_i}_{ij})=f^{rr_l}_{kj}$ if $l\leqq_B j$. But $f^{rr_l}_{kj}\in\mathscr{C}_j$, since $\leqq_B$ is transitive. The rest of the induction process is straightforward.

Lemma 3.5. *Let $U\in\mathbf{M}(B,R)$ and let $1\leqq a\leqq b$. Then $U(\sum_{i(i\leqq_B z_a)}f^{r_i}_{iz_a})=\sum_{i(i\leqq_B z_a)}f^{s_i}_{iz_a}$ (in $\mathbf{M}(B,R)$) if and only if $U(\sum_{i(i\leqq_B z_a)}\iota_i(r_i))=\sum_{i(i\leqq_B z_a)}\iota_i(s_i)$ (in R^n), where the r_i's and s_i's are arbitrary elements of R.*

Proof. Consideration of the action of $U(\sum_{i(i\leqq_B z_a)}f^{r_i}_{iz_a})=\sum_{i(i\leqq_B z_a)}f^{s_i}_{iz_a}$ on $\iota_{z_a}(1)$ gives the desired result.

Theorem 3.6. *Let Γ be an $\mathbf{M}(B,R)$-module of type 2. Then G_a is an R-module of type 2 for $a=1,2,\dots,b$.*

Proof. G_a is faithful, since $r\in Ann_R G_a$ if and only if $f^r_{z_az_a}\in Ann_{\mathbf{M}(B,R)}\Gamma$. Next, let $f^1_{z_az_a}\gamma$ be any nonzero element of G_a. If $f^1_{z_az_a}\delta\in G_a$, then $U(f^1_{z_az_a}\gamma)=f^1_{z_az_a}\delta$ for some $U\in\mathbf{M}(B,R)$, since Γ is of type 2. It follows from Lemma 3.5 that there are elements r_i of R such that $U(f^1_{z_az_a})=\sum_{i(i\leqq_B z_a)}f^{r_i}_{iz_a}$, and so

$$f^1_{z_az_a}\delta=f^1_{z_az_a}(f^1_{z_az_a}\delta)=f^1_{z_az_a}\left(\sum_{i(i\leqq_B z_a)}f^{r_i}_{iz_a}\right)\gamma$$

$$=f^{r_{z_a}}_{z_az_a}\gamma=r_{z_a}(f^1_{z_az_a}\gamma).$$

Hence G_a is of type 2.

4. The 2-primitive ideals and the $\mathscr{T}_2$-radical of $\mathbf{M}(B,R)$

Let L be a left ideal of R, and let $1\leqq a\leqq b$. The following $\mathbf{M}(B,R)$-ideal of the *structural* $\mathbf{M}(B,R)$*-module* $R^n(a,R)$ was introduced in [11]:

$$R^n(a,L):=\left\{u\in R^n\colon \pi_i u=\begin{cases} l & (\text{for some } l\in L),\ \text{ if } \ z_a\sim_B i\\ 0, & \text{if } \ z_a\leqq_B i \ \text{ and } \ z_a\nsim_B i\end{cases}\right\}.$$

Proposition 4.1. *If Γ is a (nonzero) $\mathbf{M}(B,R)$-module of type 2, then $Ann_{\mathbf{M}(B,R)}\Gamma=(R^n(a,I)\colon R^n(a,R))$ for some ideal I of R and some a, $1\leqq a\leqq b$.*

Proof. By Theorem 3.1 there is a minimal element z_a (say) of $\{z_1, z_2, \ldots, z_b\}$ (with respect to $\leqq_B$) such that $G_a \neq 0$. Set $I := Ann_R G_a$. In order to arrive at the desired result, firstly, let $U \in Ann_{\mathbf{M}(B,R)}\Gamma$. Take any $\sum_{a=1}^{b} \sum_{k=1}^{n_a} \iota_{j_{k,a}}(r_{k,a}) \in R^n(a,R)$, and let $U(\sum_{a=1}^{b} \sum_{k=1}^{n_a} \iota_{j_{k,a}}(r_{k,a})) = \sum_{a=1}^{b} \sum_{k=1}^{n_a} \iota_{j_{k,a}}(s_{k,a})$. Then $s_{m,c} = 0$ for $m = 1, 2, \ldots, n_a$, in case $z_a \leqq_B z_c$ and $z_a \not\geq_B z_c$, because $R^n(a,R)$ is an $\mathbf{M}(B,R)$-module. Furthermore, it follows from Lemma 3.5 and the minimality of z_a that

$$U\left(\sum_{k=1}^{n_a} f^{r_{k,a}}_{j_{k,a}z_a}\right) = \sum_{k=1}^{n_a} f^{s_{k,a}}_{j_{k,a}z_a}, \quad \text{and so} \quad \left(\sum_{k=1}^{n_a} f^{s_{k,a}}_{j_{k,a}z_a}\right) f^{1}_{z_a z_a}\gamma = 0$$

for all $\gamma \in \Gamma$, since $(\sum_{k=1}^{n_a} f^{r_{k,a}}_{j_{k,a}z_a}) f^1_{z_a z_a}\gamma \in \Gamma$. Hence, for every $p, 1 \leqq p \leqq n_a$, we have $0 = f^1_{z_a j_{p,a}}(\sum_{k=1}^{n_a} f^{s_{k,a}}_{j_{k,a}z_a}) f^1_{z_a z_a}\gamma = f^{s_{p,a}}_{z_a z_a}\gamma$, which implies that $s_{p,a} \in I$, since $I = \{r \in R : f^r_{z_a z_a}\Gamma = 0\}$. Therefore, $\sum_{a=1}^{b} \sum_{k=1}^{n_a} \iota_{j_{k,a}}(s_{k,a}) \in R^n(a,I)$, from which we conclude that $U \in (R^n(a,I) : R^n(a,R))$. Secondly, let $U \in (R^n(a,I) : R^n(a,R))$, and let $\gamma \in \Gamma$. There is a $\delta \in \Gamma$ such that $f^1_{z_a z_a}\delta \neq 0$, otherwise $G_a = 0$. Since Γ is of type 2 over $\mathbf{M}(B,R)$, there is a $V \in \mathbf{M}(B,R)$ such that $V f^1_{z_a z_a}\delta = \gamma$, and so by Lemma 3.5 and the minimality of z_a we have $U\gamma = U(\sum_{k=1}^{n_a} f^{t_k}_{j_{k,a}z_a})\delta = (\sum_{k=1}^{n_a} f^{u_k}_{j_{k,a}z_a})\delta$ for some $t_k \in R$, $u_k \in I$, $k = 1, 2, \ldots, n_a$. Also, $f^{u_k}_{j_{k,a}z_a}\delta = (f^1_{j_{k,a}z_a} f^{u_k}_{z_a z_a})\delta = f^1_{j_{k,a}z_a}(f^{u_k}_{z_a z_a}\delta) = f^1_{j_{k,a}z_a}0$, since $u_k \in I$, and so it follows from [**3**, Lemma 3.1(6)] that $U\gamma = 0$, since R is zero-symmetric. Therefore $U \in Ann_{\mathbf{M}(B,R)}\Gamma$, which completes the proof.

Eventually we want to show that the ideals $(R^n(a,I) : R^n(a,R))$, for $a = 1, 2, \ldots, b$ and I a 2-primitive ideal of R, compose the set of all the 2-primitive ideals of $\mathbf{M}(B,R)$. The following near-ring isomorphism is essential in this regard:

Proposition 4.2. *$\mathbf{M}(B,R)/(R^n(a,I) : R^n(a,R)) \cong \mathbf{M}_{n_a}(R/I)$ for $a = 1, 2, \ldots, b$ and every ideal I of R.*

Proof. Let $1 \leqq a \leqq b$, and consider the near-ring epimorphisms $\Phi : \mathbf{M}(B,R) \to \mathbf{M}_{n_a}(R)$ in [**11**, Theorem 3.10] and $\phi : \mathbf{M}_{n_a}(R) \to \mathbf{M}_{n_a}(R/I)$ in [**10**, Lemma 4.2], defined respectively by

$$\Phi(U) = \mu(\theta_2(E_U)), \quad \text{where } E_U \in \mu^{-1}(U),$$

and

$$\phi(V) = \mu(\theta_1(E_V)), \quad \text{where} \quad E_V \in \mu^{-1}(V).$$

Here $\theta_1(E_V)$ is the expression derived from E_V by changing every f^r_{ij} into f^{r+I}_{ij}, and $\theta_2(E_U)$ is the expression derived from E_U by replacing every $f^s_{j_k j_m}$ in E_U by f^s_{km}, $1 \leqq k, m \leqq n_a$, and everything else by $f^0_{z_a z_a}$, whereas $\mu(E)$ denotes the matrix represented by an expression E. (Recall from [3] that every matrix expression represents a matrix, but the same matrix may be represented by many different expressions, since every matrix is merely a function from R^n into R^n.) It is easily seen that the composite function $\phi \circ \Phi : \mathbf{M}(B,R) \to \mathbf{M}_{n_a}(R/I)$ maps $U \in \mathbf{M}(B,R)$ onto $\mu(\theta_1(\theta_2(E_U)))$. We show that $\mathrm{Ker}(\phi\circ\Phi)=(R^n(a,I):R^n(a,R))$. Let $U\in(R^n(a,I):R^n(a,R))$, and let $r_1,r_2,\ldots,r_{n_a}\in R$. Then $u:=\sum_{k=1}^{n_a}\iota_{j_k}(r_k)\in R^n(a,R)$, and so $\pi_{j_k}(Uu)\in I$ for $k=1,2,\ldots,n_a$, i.e. $\Phi(U)\langle r_1,r_2,\ldots,r_{n_a}\rangle\in I^{n_a}$, since by [**11**, Lemma 3.9] we have

$$\pi_m(\Phi(U)\langle r_1,r_2,\ldots,r_{n_a}\rangle)=\pi_{j_m}\left(U\left(\sum_{k=1}^{n_a}\iota_{j_k}(r_k)\right)\right)\quad\text{for}\quad m=1,2,\ldots,n_a.$$

It follows from [**10**, Lemma 4.2] that $\phi(\Phi(U))\langle r_1+I,r_2+I,\ldots,r_{n_a}+I\rangle=0$ in $(R/I)^{n_a}$, since by [**4**, Proposition 6] $(R/I)^{n_a}\cong R^{n_a}/I^{n_a}$ as $\mathbf{M}_{n_a}(R)$-modules. Therefore $U\in\mathrm{Ker}(\phi\circ\Phi)$. Conversely, let $U\in\mathrm{Ker}(\phi\circ\Phi)$, and let $u:=\langle u_1,u_2,\ldots,u_n\rangle\in R^n(a,R)$. Then by the definition of $R^n(a,R)$ we have $u=\sum_{k=1}^{n_a}\iota_{j_k}(u_{j_k})+\sum_{i(z_a\not\leq_B i)}\iota_i(u_i)$. We must show that $\pi_{j_m}(Uu)\in I$ for $m=1,2,\ldots,n_a$. Again by [**11**, Lemma 3.9] we have $\pi_{j_m}(Uu)=\pi_m(\Phi(U)\langle u_{j_1},u_{j_2},\ldots,u_{j_{n_a}}\rangle)$, and so it follows from [**10**, Lemma 4.2] that $\pi_{j_m}(Uu)=\pi_m(\phi(\Phi(U))\langle u_{j_1}+I,u_{j_2}+I,\ldots,u_{j_{n_a}}+I\rangle)=0$ in R/I, from which we conclude that $\pi_{j_m}(Uu)\in I$. Therefore $Uu\in R^n(a,I)$, and so $U\in(R^n(a,I):R^n(a,R))$. Since $\phi\circ\Phi$ is onto, the desired near-ring isomorphism follows.

Proposition 4.1, Proposition 4.2 and [**10**, Theorem 3.10] together lead to

Theorem 4.3. *The set of* $(R^n(a,I):R^n(a,R))$*'s, for* $a=1,2,\ldots,b$ *and* I *a* 2*-primitive ideal of* R*, is the set of all the* 2*-primitive ideals of* $\mathbf{M}(B,R)$.

The characterization of the $\mathcal{T}_2$-radical of a near-ring as the intersection of its 2-primitive ideals is now used to describe $\mathcal{T}_2(\mathbf{M}(B,R))$ in terms of $\mathcal{T}_2(R)$:

Theorem 4.4. ([**11**, Theorem 3.17])

$$\mathcal{T}_2(\mathbf{M}(B,R))=\bigcap_{a=1}^{b}(R^n(a,\mathcal{T}_2(R)):R^n(a,R)).$$

Proof. It follows directly from Theorem 4.3 and [7, Proposition 1.44].

Whether $\mathcal{T}_2(\mathbf{M}(B,R))$ can be written as the sum of two ideals, one of which is nilpotent (as in the ring case), was raised as an open problem in [**11**]. We conclude by making some progress in this direction.

Theorem 4.5. $(\mathcal{T}_2(R))^*+\bigcap_{a=1}^{b}(R^n(a,0):R^n(a,R))\subseteq\mathcal{T}_2(\mathbf{M}(B,R))$.

Proof. Let $U\in(\mathcal{T}_2(R))^*$ and let $V\in\bigcap_{a=1}^{b}(R^n(a,0):R^n(a,R))$. It follows from the definition of $R^n(a,\mathcal{T}_2(R))$, $1\leqq a\leqq b$, and from $R^n(a,R)$ being an $\mathbf{M}(B,R)$-module that $U(R^n(a,R))\subseteq R^n(a,\mathcal{T}_2(R))$ if we can show that $\pi_{j_{k,a}}(U(R^n(a,R)))\subseteq\mathcal{T}_2(R)$ for $k=1,2,\ldots,n_a$. But this is certainly the case, since $U(R^n)\subseteq(\mathcal{T}_2(R))^n$. Furthermore, $V(R^n(a,R))\subseteq R^n(a,\mathcal{T}_2(R))$, because $R^n(a,0)\subseteq R^n(a,\mathcal{T}_2(R))$. The desired result follows now immediately from Theorem 4.4.

Note that $(\mathcal{T}_2(R))^* = \mathbf{M}(B, \mathcal{T}_2(R))$, and that $\bigcap_{a=1}^{b} (R^n(a,0): R^n(a,R))$ is the antisymmetric radical of $\mathbf{M}(B,R)$, in case R is a ring. (See [12], where it is shown that $\mathcal{T}_2(\mathbf{M}(B,R))$, R a ring, is the sum of $\mathbf{M}(B,\mathcal{T}_2(R))$ and the antisymmetric radical of $\mathbf{M}(B,R)$.)

The next result shows that the elementary matrices in $\mathcal{T}_2(\mathbf{M}(B,R))$ are contained in the sum of the two ideals in Theorem 4.5.

Proposition 4.6.

$$\{f_{ij}^r : f_{ij}^r \in \mathcal{T}_2(\mathbf{M}(B,R))\} \subseteq (\mathcal{T}_2(R))^* + \bigcap_{a=1}^{b} (R^n(a,0): R^n(a,R)).$$

Proof. Let $f_{ij}^r \in \mathcal{T}_2(\mathbf{M}(B,R))$. It follows from Theorem 2.1 that $i \leqq_B j$. Let $i \sim_B z_{a'}$ (say), and let $j \sim_B z_{a''}$ (say), $1 \leqq a', a'' \leqq b$. We consider the following two possibilities, viz. $z_{a''} \leqq_B z_{a'}$ or $z_{a''} \nleqq_B z_{a'}$. In the first case $a' = a''$. For every $\langle u_1, u_2, \ldots, u_n \rangle \in R^n$ we have $f_{ij}^r \langle u_1, u_2, \ldots, u_n \rangle = \iota_i(ru_j) = f_{ij}^r(\iota_j(u_j))$. Since $\iota_j(u_j) \in R^n(a',R)$ and $f_{ij}^r \in (R^n(a', \mathcal{T}_2(R)): R^n(a',R))$, it follows that $\iota_i(ru_j) \in R^n(a', \mathcal{T}_2(R))$, and so $ru_j \in \mathcal{T}_2(R)$. Therefore $f_{ij}^r \in (\mathcal{T}_2(R))^*$. In the second case, let $1 \leqq a \leqq b$ and let $u := \langle u_1, u_2, \ldots, u_n \rangle \in R^n(a,R)$. If $z_a \leqq_B z_{a'}$, then it follows from the transitivity of $\leqq_B$ that $z_a \leqq z_{a''}$ and $z_{a''} \nleqq_B z_a$ (otherwise $z_{a''} \leqq_B z_{a'}$), and so the definition of $R^n(a,R)$ ensures that $u_j = 0$, because $j \sim_B z_{a''}$. Hence $f_{ij}^r u = 0 \in R^n(a,0)$. If $z_a \nleqq_B z_{a'}$, then $\pi_i(R^n(a,0)) = R$, because $i \sim_B z_{a'}$. Hence we have $f_{ij}^r u \in R^n(a,0)$ in this case, too. Consequently, $f_{ij}^r \in \bigcap_{a=1}^{b} (R^n(a,0): R^n(a,R))$. This completes the proof.

Set $\mathcal{S} := \{f_{ij}^r : f_{ij}^r \in \mathcal{T}_2(\mathbf{M}(B,R))\}$. It follows directly from Proposition 4.6 that every sum of f_{ij}^r's in $\mathcal{S}$ is an element of $(\mathcal{T}_2(R))^* + \bigcap_{a=1}^{b} (R^n(a,0): R^n(a,R))$. Our final result shows that $\mathcal{S}$ resembles the union of the sets $\{rE_{ij} : r \in \mathcal{T}_2(R)$, and $i,j \sim_B z_{a'}$ for some a', $1 \leqq a' \leqq b\}$ and $\{rE_{ij} : r \in R, i \sim_B z_{a'}$ and $j \sim_B z_{a''}$ for some $a' \neq a''$, $1 \leqq a'$, $a'' \leqq b\}$ in the ring case.

Proposition 4.7. *$\mathcal{S} = \{f_{ij}^r : r \in \mathcal{T}_2(R)$, and $i,j \sim_B z_{a'}$ for some a', $1 \leqq a' \leqq b\} \cup \{f_{ij}^r : r \in R, i \sim_B z_{a'}$ and $j \sim_B z_{a''}$ for some $a' \neq a''$, $1 \leqq a', a'' \leqq b\}$.*

Proof. Let $f_{ij}^r \in \mathcal{S}$, and suppose that $i,j \sim_B z_{a'}$ for some a', $1 \leqq a' \leqq b$. Then by the first part of the proof of Proposition 4.6 we have $f_{ij}^r(\iota_j(1)) \in (\mathcal{T}_2(R))^n$, which implies that $r \in \mathcal{T}_2(R)$. Next, suppose that $r \in \mathcal{T}_2(R)$ and $i,j \sim_B z_{a'}$ for some a', $1 \leqq a' \leqq b$. Since $\mathcal{T}_2(R)$ is a right ideal of R, it follows that $\iota_i(ru_j) \in R^n(a, \mathcal{T}_2(R))$ for every $\langle u_1, u_2, \ldots, u_n \rangle \in R^n(a,R)$ and every $a, 1 \leqq a \leqq b$, and so $f_{ij}^r \in \mathcal{S}$. Lastly, suppose that $r \in R$, $i \sim_B z_{a'}$ and $j \sim_B z_{a''}$ for some $a' \neq a''$, $1 \leqq a'$, $a'' \leqq b$. Then an argument almost precisely the one pursued in the last part of the proof of Proposition 4.6 shows that $f_{ij}^r \in \mathcal{S}$, and so the desired equality has been established.

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