CONTINUOUS SELECTION THEOREM, COINCIDENCE THEOREM AND INTERSECTION THEOREMS CONCERNING SETS WITH $H$-CONVEX SECTIONS

XIE-PING DING

(Received 31 January 1990; revised 30 June 1990)

Communicated by J. H. Rubinstein

Abstract

A continuous selection and a coincidence theorem are proved in $H$-spaces which generalize the corresponding results of Ben-El-Mechaiekh-Deguire-Granas, Browder, Ko-Tan, Lassonde, Park, Simon and Takahashi to noncompact and/or nonconvex settings. By applying the two theorems, some intersection theorems concerning sets with $H$-convex sections are obtained which generalize the corresponding results of Fan, Lassonde and Shih-Tan to $H$-spaces. Some applications to minimax principle are given.


Keywords and phrases: Continuous selection, coincidence, $H$-space, contractible, weakly $H$-convex, $H$-compact, $H-KKM$, compactly closed (open), upper semi-continuous, minimax principle.

1. Introduction

In our recent papers [7, 9], we have obtained some new matching theorems, fixed point theorems and minimax inequalities. By applying a minimax inequality in [7], some non-convex generalizations of well-known intersection theorems concerning sets with convex sections were proved in [8], but we would have to assume that the product space is a $H$-space.

In the present paper, we shall first show a continuous selection theorem, an $H-KKM$ theorem and a coincidence theorem which improve and generalize

This project was supported by the National Natural Science Foundation of China.
© 1992 Australian Mathematical Society 0263-6115/92 $A2.00 + 0.00
the corresponding results of Ben-El-Mechaiekh-Deguire-Granas [4], Browder [6], Ding-Tan [10], Ko-Tan [16], Lassonde [17], Park [19], Simon [20], and Takahashi [23] to noncompact and nonconvex settings. Next by applying our earlier results, some intersection theorems concerning sets with $H$-convex sections are obtained without the assumption that the product space is a $H$-space. These theorems generalize those of Fan [10, 12], Lassonde [17] and Shih-Tan [22] to noncompact and nonconvex settings. Some applications are given.

2. Preliminaries

Let $X$ be a nonempty set; we shall denote by $2^X$ the family of all subsets of $X$ and by $\mathcal{F}(X)$ the family of all nonempty finite subsets of $X$. Also $\Delta_n$ is the standard $n$ dimensional simplex with the vertices $e_0, e_1, \ldots, e_n$. If $J$ is a nonempty subset of $\{0, \ldots, n\}$, $\Delta_J$ will denote the convex hull of the vertices $\{e_j : j \in J\}$. Let $X$ and $Y$ be topological spaces and $D$ be a subset of $X$. $D$ is said to be compactly closed (open) in $X$ if $D \cap C$ is closed (open) in $C$ for each nonempty compact subset $C$ of $X$. A map $S: D \to 2^Y$ is said to be upper semi-continuous (u.s.c.) if for each $x \in D$ and for each open subset $U$ of $Y$ with $S(x) \subseteq U$, there exists an open neighborhood $V$ of $x$ in $X$ such that for each $z \in D \cap V$, $S(z) \subseteq U$. $S$ is said to be compactly valued if for each $x \in D$, $S(x)$ is compact in $Y$.

The following notions which were introduced by Bardaro-Ceppitelli in [2] were motivated by an earlier work of Horvath [15].

A pair $(X, \{F_A\})$ is called an $H$-space if $X$ is a topological space (which need not be Hausdorff) and $\{F_A\}$ is a family of nonempty contractible subsets of $X$ indexed by $A \in \mathcal{F}(X)$ such that $F_A \subseteq F_{A'}$ whenever $A \subseteq A'$. A subset $D$ of $X$ is said to be (i) $H$-convex if $F_A \subseteq D$ for each $A \in \mathcal{F}(D)$; (ii) weakly $H$-convex if $F_A \cap D$ is contractible for each $A \in \mathcal{F}(D)$ (this is equivalent to saying that $(D, \{F_A \cap D\})$ is an $H$-space); (iii) $H$-compact in $X$ if, for each $A \in \mathcal{F}(X)$, there exists a compact, weakly $H$-convex subset $D_A$ of $X$ such that $D \cup A \subseteq D_A$. A map $F: X \to 2^X$ is called $H-KKM$ if $F_A \subseteq \bigcup_{x \in A} F(x)$ for each $A \in \mathcal{F}(X)$.

3. Selection theorem, $H-KKM$ theorem and coincidence theorem

The proof of the following useful result is contained in the proof of [15, Theorem 1] (see also [9]).
**Lemma 3.1.** Let $X$ be a topological space. For each nonempty subset $J$ of $\{0, \ldots, n\}$, let $F_J$ be a nonempty contractible subset of $X$. If $J \subseteq J'$ imply $F_J \subseteq F_{J'}$, then there exists a continuous map $f: \Delta_n \to X$ such that $f(\Delta_J) \subseteq F_J$ for each nonempty subset $J$ of $\{0, \ldots, n\}$.

The following lemma is a slight improvement of [15, Corollary I.1] (also see [8]).

**Lemma 3.2.** Let $(Y, \{F_A\})$ be an $H$-space, $X$ be a nonempty subset of $Y$ and $G: X \to 2^Y$ be such that

(a) $G$ is an $H$-KKM map;
(b) for each $x \in X$, $G(x)$ is closed and for some $x_0 \in X$, $S(x_0)$ is compact.

Then $\bigcap_{x \in X} G(x) \neq \emptyset$.

**Theorem 3.1.** Let $X$ be a compact topological space and $(Y, \{F_A\})$ be an $H$-space. Suppose that $S, T: X \to 2^Y$ are such that

(a) for each $x \in X$, $S(x) \neq \emptyset$ and $F_A \subseteq T(x)$ for each $A \in \mathcal{F}(S(x))$;
(b) for each $y \in Y$, $S^{-1}(y) = \{x \in X : y \in S(x)\}$ is open in $X$.

Then $T$ has a continuous selection $g: X \to Y$ and there exists a finite set $A \in \mathcal{F}(Y)$ such that $g(X) \subseteq F_A$.

**Proof.** By (a), we must have $X = \bigcup_{y \in Y} S^{-1}(y)$. From (b) and the compactness of $X$ it follows that there exists a finite set 

$$A = \{y_0, \ldots, y_n\} \in \mathcal{F}(Y)$$

such that $X = \bigcup_{i=0}^n S^{-1}(y_i)$. For each nonempty subset $J$ of $\{0, \ldots, n\}$, we define $F_J = F_{\{y_i\}_{i \in J}}$. Since $(Y, \{F_A\})$ is an $H$-space, $F_J$ is a contractible subset of $Y$ and $F_J \subseteq F_{J'}$ whenever $J \subseteq J'$. By Lemma 3.1, there is a continuous map $f: \Delta_n \to Y$ such that $f(\Delta_J) \subseteq F_J$ for each nonempty subset $J$ of $\{0, \ldots, n\}$. Let $\{\alpha_i\}_{i=0}^n$ be a continuous partition of unity subordinate to the open covering $\{S^{-1}(y_i)\}_{i=0}^n$. Define a map $\psi: X \to \Delta_n$ by 

$$\psi(x) = \sum_{i=0}^n \alpha_i(x)e_i.$$ 

For each $x \in X$, let $J(x) = \{i \in \{0, \ldots, n\} : \alpha_i(x) \neq 0\}$, then we have $\psi(x) \in \Delta_{J(x)}$ so that 

$$f \circ \psi(x) \in f(\Delta_{J(x)}) \subseteq F_{J(x)} \subseteq F_A.$$
Since \( x \in S^{-1}(y_j) \) for each \( j \in J(x) \), it follows that \( y_j \in S(x) \) for all \( j \in J(x) \). By (a), we obtain \( F_{J(x)} \subset T(x) \) so that \( f \circ \psi(x) \in T(x) \) for each \( x \in X \). Hence \( g = f \circ \psi \) is a continuous selection of \( T \) and there exists a finite set \( A \in \mathcal{F}(Y) \) such that \( g(X) \subset F_A \).

It would be of some interest to compare Theorem 3.1 with [15, Theorem 3].

Now we shall prove the following \( H - KKM \) theorem.

**Theorem 3.2.** Let \( X \) be a nonempty subset of an \( H \)-space \( (Y, \{F_A\}) \), \( Z \) be a topological space and \( G: X \rightarrow 2^Z \) be such that

(a) for each \( x \in X \), \( G(x) \) is compactly closed in \( Z \);

(b) there exists a compactly valued u.s.c. map \( S: Y \rightarrow 2^Z \) such that the map \( F: X \rightarrow 2^Y \) defined by \( F(x) = S^{-1}(G(x)) \) is \( H - KKM \);

(c) there exists an \( H \)-compact subset \( L \) of \( Y \) and a nonempty compact subset of \( Z \) such that for each \( B \in \mathcal{F}(X) \) and for each \( z \in S(L_B) \setminus K \), there is an \( x \in L_B \cap X \) such that \( x \notin G(x) \cap S(L_B) \). Then \( K \cap (\bigcap_{x \in X} G(x)) \neq \emptyset \).

**Proof.** For each \( x \in X \), let \( G_1(x) = G(x) \cap K \), then \( G_1(x) \) is closed in \( K \) by (a). We shall prove that the family \( \{G_1(x) : x \in X \} \) has the finite intersection property. Let \( B \in \mathcal{F}(X) \) be arbitrary fixed; then by (c), \( L_B \) is a compact, weakly \( H \)-convex subset of \( Y \) with \( L \cup B \subset L_B \) such that for each \( z \in S(L_B) \setminus K \), there is an \( x \in L_B \cap X \) satisfying \( z \notin G(x) \cap S(L_B) \). Now we define the map \( G_2: L_B \cap X \rightarrow 2^L_B \) by

\[
G_2(x) = F(x) \cap L_B = S^{-1}(G(x)) \cap L_B.
\]

Then we have the following properties.

1. By the weak \( H \)-convexity of \( L_B \), \( (L_B, \{F_A \cap L_B\}) \) is an \( H \)-space.

2. For each \( A \in \mathcal{F}(L_B \cap X) \subset \mathcal{F}(X) \), we have \( F_A \subset \bigcup_{x \in A} F(x) \) by (b) so that \( F_A \cap L_B \subset \bigcup_{x \in A} (F(x) \cap L_B) = \bigcup_{x \in A} G_2(x) \). Thus \( G_2 \) is also an \( H - KKM \) map.

3. Since \( S \) is compactly valued u.s.c. and \( L_B \) is compact in \( Y \), it follows that \( S(L_B) \) is compact in \( Z \) so that for each \( x \in X \), \( G(x) \cap S(L_B) \) is closed in \( Z \) by (a). By the upper semi-continuity of \( S \), \( S^{-1}(G(x) \cap S(L_B)) \) is a closed subset of \( X \). Hence, for each \( x \in L_B \cap X \),

\[
G_2(x) = S^{-1}(G(x)) \cap L_B = S^{-1}(G(x) \cap S(L_B)) \cap L_B
\]

is compact in \( L_B \).

Downloaded from https://www.cambridge.org/core. IP address: 54.70.40.11, on 21 May 2019 at 13:35:53, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S1446788700032833
Continuous selection theorem

By Lemma 3.2, \( \bigcap_{x \in L_B \cap X} G_2(x) \neq \emptyset \). Take any \( y \in \bigcap_{x \in L_B \cap X} G_2(x) \), then we have

\[
S(y) \cap \left( \bigcap_{x \in L_B \cap X} (G(x) \cap S(L_B)) \right) \neq \emptyset.
\]

By (c), we must have

\[
S(y) \cap \left( \bigcap_{x \in L_B \cap X} (G(x) \cap S(L_B)) \right) \subset S(y) \cap \left( \bigcap_{x \in L_B \cap X} (G(x) \cap K) \right)
\]

\[
\subset S(y) \cap \left( \bigcap_{x \in B} (G(x) \cap K) \right) = S(y) \cap \left( \bigcap_{x \in B} G_1(x) \right) \subset \bigcap_{x \in B} G_1(x).
\]

It follows that \( \bigcap_{x \in B} G_1(x) \neq \emptyset \). By the compactness of \( K \), \( \bigcap_{x \in X} G_1(x) \neq \emptyset \), that is, \( K \cap (\bigcap_{x \in X} G(x)) \neq \emptyset \).

**Remark 3.1.** If \( S \) is a single-valued continuous map, Theorem 3.2 reduces to [10, Theorem 1] and in turn generalizes [1, Theorem 1]. It is easy to see that condition (c) of Theorem 3.2 is equivalent to the condition:

(c\text{\_}) there exists an \( H \)-compact subset \( L \) of \( Y \) and a nonempty compact subset \( K \) of \( Z \) such that for each \( B \in \mathcal{F}(X) \),

\[
\bigcap_{x \in B \cap X} (G(x) \cap S(L_B)) \subset K.
\]

We also note that under hypothesis (a) of Theorem 3.2, condition (c\text{\_}) is implied by the condition: there exists an \( H \)-compact subset \( L \) of \( Y \) such that \( \bigcap_{x \in X} G(x) \) is compact in \( Z \). Since every convex space is an \( H \)-space [17], Theorem 3.2 generalizes [17, Theorem 1] (which is equivalent to [19, Theorem 6]) to an \( H \)-space with a weaker assumption.

In the following we shall prove a coincidence theorem.

**Theorem 3.3.** Let \( X \) be a nonempty subset of an \( H \)-space \( (Y, \{F_A\}) \), \( Z \) be a topological space and \( A, B: X \to 2^Z \) be such that

(a) for each \( z \in Z \), \( B^{-1}(z) \neq \emptyset \) and \( F_D \subset A^{-1}(z) \) for each \( D \in \mathcal{F}(B^{-1}(z)) \);

(b) for each \( x \in X \), \( B(x) \) is compactly open in \( Z \);

(c) there exists an \( H \)-compact subset \( L \) of \( Y \) and a nonempty compact subset \( K \) of \( Z \) such that for each \( B \in \mathcal{F}(X) \) and for each \( z \in Z \backslash K \), there is an \( x \in L_B \cap X \) such that \( z \in B(x) \).

Then for any compactly valued u.s.c. map \( S: Y \to 2^Z \), there exists an \( x_0 \in X \) such that \( S(x_0) \subset A(x_0) \).
PROOF. Define a map $G : X \to 2^X$ by

$$G(x) = Z \setminus B(x) \quad \text{for each } x \in X.$$ 

Then we have the following properties.

(1) For each $x \in X$, $G(x)$ is compactly closed by (b).

(2) By (c), there exist an $H$-compact subset $L$ of $Y$ and a nonempty compact subset $K$ of $Z$ such that for each $B \in \mathcal{F}(X)$ and for each $z \in Z \setminus K$, there is an $x \in L_B \cap X$ such that $z \notin G(x)$ so that $z \notin G(x) \cap S(L_B)$ for any compactly valued u.s.c. map $S : Y \to 2^Z$.

Now for any given compactly valued u.s.c. map $S : Y \to 2^Z$, define a map $F : X \to 2^Y$ by

$$F(x) = S^{-1}(G(x)) \quad \text{for each } x \in X.$$ 

If $F$ is an $H - KKM$ map, it follows from Theorem 3.2 that

$$\bigcap_{x \in X} G(x) = \bigcap_{x \in X} (Z \setminus B(x)) = Z \setminus \bigcup_{x \in X} B(x) \neq \emptyset.$$ 

But condition (a) implies $Z = \bigcup_{x \in X} B(x)$, we obtain a contradiction so that $F$ is not an $H - KKM$ map. Therefore there exists $D \in \mathcal{F}(X)$ and $x_0 \in F_D$ such that $x_0 \notin \bigcup_{x \in D} F(x) = \bigcup_{x \in D} S^{-1}(G(x))$. It follows that $S(x_0) \cap (\bigcup_{x \in D} G(x)) = S(x_0) \cap (\bigcup_{x \in D} (Z \setminus B(x))) = \emptyset$. Thus, $S(x_0) \subset B(x)$ for all $x \in D$ so that for any given $z \in S(x_0)$, we have $D \in \mathcal{F}(B^{-1}(z))$. By (a), $F_D \subset A^{-1}(z)$. It follows that $x_0 \in A^{-1}(z)$ and so $z \in A(x_0)$. From the arbitrariness of $z \in S(x_0)$ it follows that $S(x_0) \subset A(x_0)$.

REMARK 3.2. We note that condition (c) of Theorem 3.3 is equivalent to the following condition:

(c') there exists an $H$-compact subset $L$ of $Y$ and a compact subset $K$ of $Z$ such that

$$Z \setminus \bigcup_{x \in L_B \cap X} B(x) \subset K.$$ 

Theorem 3.3 improves and generalizes [4, Theorem 1, 6, Theorem 1, 16, Theorem 3.1, 17, Theorem 1.1, 19, Theorem 6, 20, Theorem 4.3 and 23, Theorem 2 and 5].

4. Intersection theorems concerning sets with $H$-convex sections

In this section, we always assume that every $H$-space $(X, \{F_A\})$ has the following property: for each $A \in \mathcal{F}(X)$, $F_A$ is $H$-compact in $X$. Clearly,
each convex space \( X \) is an \( H \)-space [17] with the property that \( F_A = \text{co}(A) \) for each \( A \in \mathcal{F}(X) \).

The following notations are used throughout this section. Let \((X_i, \{F_{A_i}\}), i = 1, \ldots, n\), be \( n \) \((\geq 2)\) \( H \)-spaces and \( X = \prod_{i=1}^n X_i \). For each \( i \in \{1, \ldots, n\} \), let \( \hat{X}_i = \prod_{j \neq i} X_j \). Also \( \hat{x}_i \) denotes an element of \( \hat{X}_i \). For each \( i = 1, \ldots, n \), \( X_i \times \hat{X}_i = X \) and \((x_i, \hat{x}_i)\) denotes an element of \( X \) (with the appropriate ordering).

We shall prove the following intersection theorems.

**Theorem 4.1.** Let \((X_i, \{F_{A_i}\}), i = 1, \ldots, n\), be \( n \) \((\geq 2)\) \( H \)-spaces and \( X = \prod_{i=1}^n X_i \). If \( M_1, \ldots, M_n, N_1, \ldots, N_n \) are \( 2n \) subsets of \( X \) such that

(a) for each \( i \in \{1, \ldots, n\} \) and for each \( x_i \in X_i \), the section \( M_i(x_i) = \{\hat{y}_i \in \hat{X}_i : (x_i, \hat{y}_i) \in M_i\} \) is compactly open in \( \hat{X}_i \);

(b) for each \( i \in \{1, \ldots, n\} \) and for each \( \hat{y}_i \in \hat{X}_i \), the section \( M_i(\hat{y}_i) = \{x_i \in X_i : (x_i, \hat{y}_i) \in M_i\} \neq \emptyset \) and \( F_{D_i} \subset N_i(\hat{y}_i) = \{x_i \in X_i : (x_i, \hat{y}_i) \in N_i\} \) for each \( D_i \in \mathcal{F}(M_i(\hat{y}_i)) \);

(c) for at least \( (n - 1) \) indices \( i \), there exists an \( H \)-compact subset \( L_i \) of \( X_i \) such that \( \hat{X}_i \setminus \bigcup_{x_i \in L_i} M_i(x_i) \) is compact in \( \hat{X}_i \). Then \( \bigcap_{i=1}^n N_i \neq \emptyset \).

**Proof.** We may assume without loss of generality that condition (c) holds for \( i = 2, \ldots, n \). By (b), we have

\[
\hat{X}_i = \bigcup_{x_i \in X_i} M(x_i) \quad \text{for each } i = 1, \ldots, n.
\]

From (a), (c) and (4.1) it follows that for each \( i = 2, \ldots, n \), there exists a finite set \( B_i = \{x_i^1, \ldots, x_i^{k_i}\} \in \mathcal{F}(X_i) \) such that

\[
\hat{X}_i \setminus \bigcup_{x_i \in L_i} M_i(x_i) \subset \bigcup_{j=1}^{k_i} M_i(x_i^j).
\]

Thus, we have

\[
\hat{X}_i \subset \bigcup_{x_i \in L_i \cup \{x_i^1, \ldots, x_i^{k_i}\}} M_i(x_i).
\]

Since \( L_i \) is \( H \)-compact in \( X_i \), there exists a compact, weakly \( H \)-convex subset \( C_i \) of \( X_i \) with \( L_i \cup B_i \subset C_i \) and (4.2) imply

\[
\hat{X}_i \subset \bigcup_{x_i \in C_i} M_i(x_i).
\]
Now we define the maps $M_1, N_1: \prod_{i=2}^n C_i \to 2^{X_1}$ as follows: for each $\hat{y}_1 \in \prod_{i=2}^n C_i$,

$$M_1(\hat{y}_1) = \{x_1 \in X_1: (x_1, \hat{y}_1) \in M_1\}$$

and

$$N_1(\hat{y}_1) = \{x_1 \in X_1: (x_1, \hat{y}_1) \in N_1\}.$$  

By (b), for each $\hat{y}_1 \in \prod_{i=2}^n C_i$, $M_1(\hat{y}_1) \neq \emptyset$ and $F_{D_i} \subset N_1(\hat{y}_1)$ for each $D_i \in \mathcal{F}(M_1(\hat{y}_1))$. For each $x_1 \in X_1$,

$$M_1^{-1}(x_1) = \left\{\hat{y}_1 \in \prod_{i=2}^n C_i: (x_1, \hat{y}_1) \in M_1\right\} = \prod_{i=2}^n C_i \cap M_1(x_1)$$

is open in $\prod_{i=2}^n C_i$ by (a). It follows from Theorem 3.1 that there is a continuous map $g: \prod_{i=2}^n C_i \to X_1$ and $A_1 \in \mathcal{F}(X_1)$ such that $g(\hat{y}_1) \in N_1(\hat{y}_1)$ for each $y_1 \in \prod_{i=2}^n C_i$ and $g(\prod_{i=2}^n C_i) \subset F_{A_1}$. By the assumption that $F_{A_i}$ is $H$-compact, there exists a compact weakly $H$-convex subset $C_1$ of $X_1$ with $F_{A_i} \subset C_1$. Hence, we have $g(\prod_{i=2}^n C_i) \subset C_1$ and $(g(\hat{y}_1), \hat{y}_1) \in N_1$ for each $\hat{y}_1 \in \prod_{i=2}^n C_i$.

Let $C = \prod_{i=1}^n C_i$ and $\hat{C}_i = \prod_{j \neq i} C_j$. For each $i \in \{2, \ldots, n\}$, we define the maps $M_i, N_i: C_i \to 2^{\hat{C}_i}$ by

$$M_i(x_i) = \{\hat{y}_i \in \hat{C}_i: (x_i, \hat{y}_i) \in M_i\}$$

and

$$N_i(x_i) = \{\hat{y}_i \in \hat{C}_i: (x_i, \hat{y}_i) \in N_i\}$$

for each $x_i \in C_i$. Then, for each $x_i \in C_i$, $M_i(x_i)$ is open in $\hat{C}_i$ by (a) and for each $\hat{y}_i \in \hat{C}_i$, $M_i^{-1}(\hat{y}_i) = \{x_i \in C_i: (x_i, \hat{y}_i) \in M_i\} = C_i \cap M_i(\hat{y}_i) \neq \emptyset$ and $F_{D_i} \subset N_i^{-1}(\hat{y}_i)$ for each $D_i \in \mathcal{F}(M_i^{-1}(\hat{y}_i))$ by (b) and (4.3). From Theorem 3.3 with $X = Y = C_i$ and $Z = \hat{C}_i = K$ it follows that for any compactly valued u.s.c. map $S: C_i \to 2^{\hat{C}_i}$ there is an $x_i \in C_i$ such that $S(x_i) \subset N_i(x_i)$.

Now, let $p_i: \hat{C}_1 \to C_i, i = 2, \ldots, n$ and $q_i: \hat{C}_i \to C_1, i = 1, \ldots, n$ be the projective maps, then $p_i, q_i$ are continuous open maps. We consider the following map

$$q_i^{-1} \circ g \circ p_i^{-1}: C_i \to 2^{\hat{C}_i}, \quad i = 2, \ldots, n.$$  

Since $p_i$ and $q_i$ are continuous open maps and $g$ is continuous, it is easy to see that $q_i^{-1} \circ g \circ p_i^{-1}$ is compactly valued and u.s.c. on $C_i$. Thus for $i = 2, \ldots, n$, there exists $x_i \in C_i$ such that

$$q_i^{-1} \circ g \circ p_i^{-1}(x_i) \subset N_i(x_i).$$

(4.4)
Let \( \hat{x}_1 = (x_2, \ldots, x_n) \) and \( g(\hat{x}_1) = x_1 \), then

\[
x = (x_1, \ldots, x_n) \in N_1.
\]

Since, for \( i = 2, \ldots, n \),

\[
x_1 = g(\hat{x}_1) \in g(C_2 \times \cdots \times C_{i-1} \times \{x_i\} \times C_i \times \cdots \times C_n)
\]

and

\[
q_i^{-1} \circ g \circ p_i^{-1} = g(C_2 \times \cdots \times C_{i-1} \times \{x_i\} \times C_i \times \cdots \times C_n)
\]

\[
\times C_2 \times \cdots \times C_{i-1} \times C_i \times \cdots \times C_n,
\]

we must have

\[
\hat{x}_i = \prod_{j \neq i} x_j \in q_i^{-1} \circ g \circ p_i^{-1}(x_i) \subset N_i(x_i) \text{ for } i = 2, \ldots, n.
\]

Hence \( x = (x_1, \ldots, x_n) \in N_i \) for all \( i = 1, \ldots, n \) so that \( \prod_{i=1}^n N_i \neq \emptyset \).

**Remark 4.1.** Theorem 4.1 generalizes [17, Theorem 1.9] to \( 2n \) sets and \( H \)-spaces with weaker assumptions. We observe that condition (c) of Theorem 4.1 is implied by the following condition:

(c) at least \( (n - 1) \) of the \( X_i \)'s (say \( X_2, \ldots, X_n \)) are compact. Indeed, in the case, (c) is satisfied by \( L_i = X_i \) for \( i = 2, \ldots, n \), because by (b) the set \( \hat{X}_i \setminus \bigcup_{x_i \in X_i} M_i(x_i) = \emptyset \). Thus Theorem 4.1 also generalizes [11, Theorem 1] to \( H \)-spaces. It would be of some interest to compare Theorem 4.1 with [3, Theorem 2].

**Theorem 4.2.** Let \( (X_i, \{F_{A_i}\}), i = 1, \ldots, n \), be \((\geq 2)\) \( H \)-spaces and \( X = \prod_{i=1}^n X_i \). If \( M_1, \ldots, M_n, N_1, \ldots, N_n \) are \( 2n \) subsets of \( X \) such that

(a) for each \( i \in \{1, \ldots, n\} \) and for each \( x_i \in X_i \), the section \( M_i(x_i) \) is compactly open in \( \hat{X}_i \);

(b) for each \( i \in \{1, \ldots, n\} \) and for each \( \hat{y}_i \in \hat{X}_i \), the section \( M_i(\hat{y}_i) \neq \emptyset \) and \( F_{D_i} \subset N_i(\hat{y}_i) \) for each \( D_i \in \mathcal{F}(M_i(\hat{y}_i)) \);  

(c) for at least \( (n - 1) \) indices \( i \), there exists an \( H \)-compact subset \( L_i \) of \( X_i \) and a compact subset \( \hat{K}_i \) of \( \hat{X}_i \) such that \( L_i \cap M_i(\hat{y}_i) \neq \emptyset \) for each \( \hat{y}_i \in \hat{X}_i \setminus \hat{K}_i \).

Then \( \bigcap_{i=1}^n N_i \neq \emptyset \).

**Proof.** We shall show that condition (c) is equivalent to condition (c) of Theorem 4.1 and hence Theorem 4.2 follows from Theorem 4.1. Suppose that condition (c) of Theorem 4.1 holds. Let \( \hat{X}_i \setminus \bigcup_{x_i \in L_i} M(x_i) = \hat{K}_i \), then \( \hat{K}_i \) is a compact subset of \( \hat{X}_i \) and for each \( \hat{y}_i \in \hat{X}_i \setminus \hat{K}_i \), \( \hat{y}_i \in \bigcup_{x_i \in L_i} M(x_i) \).
Thus, there exists $x_i \in L_i$ such that $(x_i, \hat{y}_i) \in M_i$, that is $x_i \in L_i \cap M_i(\hat{y}_i)$ and hence $L_i \cap M_i(\hat{y}_i) \neq \emptyset$. Therefore condition (c) of Theorem 4.2 holds. If condition (c) of Theorem 4.2 holds, then for each $\hat{y}_i \in \hat{X}_i \setminus \bigcup_{x_i \in L_i} M_i(x_i)$, $\hat{y}_i \not\in M_i(x_i)$ for all $x_i \in L_i$ so that $x_i \not\in M_i(\hat{y}_i)$ for all $x_i \in L_i$. Thus $L_i \cap M_i(\hat{y}_i) = \emptyset$. It follows that $\hat{y}_i \in \hat{K}_i$ and

$$\hat{X}_i \setminus \bigcup_{x_i \in L_i} M_i(x_i) \subset \hat{K}_i.$$ 

By (a), $\hat{X}_i \setminus \bigcup_{x_i \in L_i} M_i(x_i)$ is closed in $\hat{K}_i$ so that it is compact in $\hat{X}_i$. This proves that condition (c) of Theorem 4.1 holds.

**THEOREM 4.3.** Let $(X_i, \{F_{A_i}\}), i = 1, \ldots, n$, be $n$ $(\geq 2)$ $H$-spaces and $X = \prod_{i=1}^n X_i$. If $M_1, \ldots, M_n, N_1, \ldots, N_n$ are $2n$ subsets of $X$ such that

(a) for each $i \in \{1, \ldots, n\}$ and for each $x_i \in X_i$, the section $M_i(x_i)$ is compactly open in $\hat{X}_i$;

(b) for each $i \in \{1, \ldots, n\}$ and for each $\hat{y}_i \in \hat{X}_i$, the section $M_i(\hat{y}_i) \neq \emptyset$ and $F_{D_i} \subset N_i(\hat{y}_i)$ for each $D_i \in \mathcal{F}(M_i(\hat{y}_i))$;

(c) there exists a compact subset $K$ of $X$ such that for each $i = 1, \ldots, n$, the projection $L_i$ of $K$ on $X_i$ is $H$-compact in $X_i$ and such that $K \cap (\prod_{i=1}^n M_i(y_i)) \neq \emptyset$ for each $y \in X \setminus K$.

Then $\bigcap_{i=1}^n N_i \neq \emptyset$.

**PROOF.** For each $i = 1, \ldots, n$, let $L_i$ and $\hat{K}_i$ be the projections of $K$ on $X_i$ and $\hat{X}_i$, respectively, then $L_i$ is $H$-compact in $X_i$ by the assumption and $\hat{K}_i$ is a compact subset of $\hat{X}_i$. The condition (c) of Theorem 4.3 imply that for each $i = 1, \ldots, n$, $L_i \cap M_i(\hat{y}_i) \neq \emptyset$ for each $\hat{y}_i \in \hat{X}_i \setminus \hat{K}_i$. By Theorem 4.2, $\bigcap_{i=1}^n N_i \neq \emptyset$.

**REMARK 4.3.** Theorem 4.3 generalizes [12, Theorem 11] in several ways. We note that if condition (b) of Theorem 4.3 is replaced by the following condition:

(b) for each $i \in \{1, \ldots, n\}$ and for each $y_i \in X_i$, the section $M_i(y_i) \neq \emptyset$ and for at least $q$ $(\geq 2)$ indices $i$, $F_{D_i} \subset N_i(\hat{y}_i)$ for each $D_i \in \mathcal{F}(M_i(\hat{y}_i))$ and for each $\hat{y}_i \in \hat{X}_i$.

Then at least $q$ of the sets $N_1, \ldots, N_n$ have a nonempty intersection by applying Theorem 4.3 for the $q$ $H$-spaces satisfying condition (b). Thus Theorem 4.3 also generalizes [13, Theorem 15].
5. Some applications to the von Neumann Minimax Theorem

For convenience, we state the special case $n = 2$ of Theorem 4.1.

**Theorem 5.1.** Let $(X, \{F_A\})$ and $(Y, \{F_A\})$ be two $H$-spaces and let $M_1, M_2, N_1, N_2$ be subsets of $X \times Y$. Suppose that

(a) for each $x \in X$, the section $M_1(x) = \{y \in Y : (x, y) \in M_1\}$ is compactly open in $Y$, the section $M_2(x) = \{y \in Y : (x, y) \in M_2\} \neq \emptyset$ and $F_A \subset N_2(x)$ for each $A \in \mathcal{F}(M_2(x))$;

(b) for each $y \in Y$, the section $M_2(y) = \{x \in X : (x, y) \in M_2\}$ is compactly open in $X$, the section $M_1(y) = \{x \in X : (x, y) \in M_1\} \neq \emptyset$ and $F_A \subset N_1(y)$ for each $A \in \mathcal{F}(M_1(y))$;

(c) there exists an $H$-compact subset $X_0$ of $X$ such that the intersection $\bigcap_{x \in X_0} (Y \setminus M_1(x))$ is compact in $Y$.

Then the intersection $N_1 \cap N_2$ is nonempty.

**Remark 5.1.** If the coercive condition (c) is replaced by the following condition:

\begin{itemize}
  \item[(c$_1$)] there exists an $H$-compact subset $Y_0$ of $Y$ such that the intersection $\bigcap_{y \in Y_0} (X \setminus M_2(y))$ is compact in $X$, then the inclusion of Theorem 5.1 still holds. We also note that if at least one of $X$ or $Y$ is compact, then condition (c) of Theorem 5.1 holds. Theorem 5.1 improves and generalizes [22, Theorem 2] and Ha's result [14] in several ways.
\end{itemize}

**Theorem 5.2.** Let $(X, \{F_A\})$ and $(Y, \{F_A\})$ be two $H$-spaces and $f, s, t, g : X \times Y \to \mathbb{R}$ and $\lambda \in \mathbb{R}$ be such that

(a) $s \leq t$ on $X \times Y$;

(b) for each $x \in X$, $y \mapsto f(x, y)$ is lower semi-continuous on each compact subset of $Y$ and for each $y \in Y$, $x \mapsto g(x, y)$ is upper semi-continuous on each compact subset of $X$;

(c) for each $x \in X$, $A \in \mathcal{F}(\{y \in Y : g(x, y) < \lambda\})$ imply $F_A \subset \{y \in Y : t(x, y) < \lambda\}$ and for each $y \in Y$, $A \in \mathcal{F}(\{x \in X : f(x, y) > \lambda\})$ imply $F_A \subset \{x \in X : s(x, y) > \lambda\}$;

(d) there exists an $H$-compact subset $X_0$ of $X$ such that the intersection $\bigcap_{x \in X_0} (Y \setminus \{y \in Y : f(x, y) > \lambda\})$ is compact in $Y$.

Then either there exists $\hat{y} \in Y$ such that $f(x, \hat{y}) \leq \lambda$ for all $x \in X$ or there exists $\hat{x} \in X$ such that $g(\hat{x}, y) \geq \lambda$ for all $y \in Y$. 

PROOF. Suppose that the conclusion does not hold. Let

\[ M_1 = \{(x, y) \in X \times Y : f(x, y) > \lambda\}, \]
\[ M_2 = \{(x, y) \in X \times Y : g(x, y) < \lambda\}, \]
\[ N_1 = \{(x, y) \in X \times Y : s(x, y) > \lambda\}, \]
\[ N_2 = \{(x, y) \in X \times Y : t(x, y) < \lambda\}. \]

Then for each \( x \in X \),

\[ M_2(x) = \{y \in Y : g(x, y) < \lambda\} \neq \emptyset \]

and for each \( y \in Y \),

\[ M_1(y) = \{x \in X : f(x, y) > \lambda\} \neq \emptyset. \]

Moreover,

(i) for each \( x \in X \), \( M_1(x) = \{y \in Y : f(x, y) > \lambda\} \) is compactly open in \( Y \) and for each \( y \in Y \), \( M_2(y) = \{x \in X : g(x, y) < \lambda\} \) is compactly open in \( X \) by (a);

(ii) for each \( x \in X \), \( F_A \subset N_2(x) \) whenever \( A \in \mathcal{F}(M_2(x)) \) and for each \( y \in Y \), \( F_A \subset N_1(y) \) whenever \( A \in \mathcal{F}(M_1(y)) \) by (c);

(iii) condition (c) of Theorem 5.1 holds by (d).

Thus all hypotheses of Theorem 5.1 are satisfied so that \( N_1 \cap N_2 \neq \emptyset \). Take any \((\hat{x}, \hat{y}) \in N_1 \cap N_2\), then \( s(\hat{x}, \hat{y}) > \lambda \) which contradicts (a). Therefore the conclusion must hold.

Recall that a real-valued function \( \varphi \) defined on an \( H \)-space \((X, \{F_A\})\) is said to be \( H \)-quasi-concave if for each real number \( t \), the set \( \{x \in X : \varphi(x) > t\} \) is \( H \)-convex; \( \varphi \) is said to be \( H \)-quasi-convex if \( -\varphi \) is \( H \)-quasi-concave.

**Corollary 5.1.** Let \((X, \{F_A\})\) and \((Y, \{F_A\})\) be two \( H \)-spaces and \( f, s, t, g : X \times Y \to \mathbb{R} \) be such that

(a) \( f \leq s \leq t \leq g \) on \( X \times Y \);

(b) for each \( x \in X \), \( y \mapsto f(x, y) \) is lower semi-continuous on each compact subset of \( Y \) and for each \( y \in Y \), \( x \mapsto g(x, y) \) is upper semi-continuous on each compact subset of \( X \);

(c) for each \( x \in X \), \( t(x, y) \) is an \( H \)-quasi-convex function of \( y \) on \( Y \) and for each \( y \in Y \), \( s(x, y) \) is an \( H \)-quasi-concave function of \( x \) on \( X \);

(d) there exists an \( H \)-compact subset \( X_0 \) of \( X \) such that for each \( t \in \mathbb{R} \), the intersection \( \bigcap_{x \in X_0} (Y \setminus \{y \in Y : f(x, y) > t\}) \) is compact in \( Y \).

Then for each \( \lambda \in \mathbb{R} \), either there exists \( \hat{y} \in Y \) such that \( f(x, \hat{y}) \leq \lambda \) for all \( x \in X \) or there exists \( \hat{x} \in X \) such that \( g(\hat{x}, y) \geq \lambda \) for all \( y \in Y \).
Remark 5.2. Theorem 5.2 and Corollary 5.1 improve and generalize [5, Theorem 5.4]. It would be of some interest to compare Theorem 5.2 and Corollary 5.1 with [8, Theorem 4 and Corollary 4].

Theorem 5.3. Let \((X, \{F_A\})\) and \((Y, \{F_A\})\) be two H-spaces and \(f, s, t, g : Y \times Y \to \mathbb{R}\) be such that

(a) \(s \leq t\) on \(X \times Y\);

(b) for each \(x \in X\), \(y \mapsto f(x, y)\) is lower semi-continuous on each compact subset of \(Y\) and for each \(y \in Y\), \(x \mapsto g(x, y)\) is upper semi-continuous on each compact subset of \(X\);

(c) for each \(y \in \mathbb{R}\) and for each \(x \in X\), \(F_A \subseteq \{y \in Y : t(x, y) < y\}\) whenever \(A \in \mathcal{S}\) and for each \(y \in Y\), \(F_A \subseteq \{x \in X : s(x, y) > y\}\) whenever \(A \in \mathcal{S}\);

(d) there exists an H-compact subset \(L\) of \(X\) and a compact subset \(K\) of \(Y\) such that

\[
\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \inf_{y \in Y \setminus K} \sup_{x \in L} f(x, y).
\]

Then the following minimax inequality holds,

\[
\alpha = \inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} \inf_{y \in Y} g(x, y) = \beta.
\]

Proof. Without loss of generality, we may assume that \(\alpha \neq -\infty\) and \(\beta \neq +\infty\). Assume to the contrary that \(\alpha > \beta\). Choose a real number \(\lambda\) such that \(\alpha > \lambda > \beta\). Let

\[
M_1 = \{(x, y) \in X \times Y : f(x, y) > \lambda\},
\]

\[
M_2 = \{(x, y) \in X \times Y : g(x, y) < \lambda\},
\]

\[
N_1 = \{(x, y) \in X \times Y : s(x, y) > \lambda\},
\]

\[
N_2 = \{(x, y) \in X \times Y : t(x, y) < \lambda\}.
\]

Then \(\alpha > \lambda\) implies that for each \(y \in Y\), \(M_1(y) \neq \emptyset\); and \(\lambda > \beta\) implies that for each \(x \in X\), \(M_2(x) \neq \emptyset\). The condition (d) implies that \(\bigcap_{x \in L} (Y \setminus M_1(x)) \subseteq K\) and each \(M_1(x)\) is compactly open in \(Y\) by (b), thus \(\bigcap_{x \in L} (Y \setminus M_1(x))\) is compact in \(Y\). The other conditions of Theorem 5.1 are easily verified. By Theorem 5.1, \(N_1 \cap N_2 \neq \emptyset\) so that there exists \((\hat{x}, \hat{y}) \in X \times Y\) such that \(s(\hat{x}, \hat{y}) > \lambda\) and \(t(\hat{x}, \hat{y}) < \lambda\) which contradicts (a). This completes the proof.

Corollary 5.2. Let \((X, \{F_A\})\) and \((Y, \{F_A\})\) be two H-spaces and \(f, s, t, g : X \times Y \to \mathbb{R}\) be such that

(a) \(f \leq s \leq t \leq g\) on \(X \times Y\);
(b) for each \( x \in X \), \( y \mapsto f(x, y) \) is lower semi-continuous on each compact subset of \( Y \) and for each \( y \in Y \), \( x \mapsto g(x, y) \) is upper semi-continuous on each compact subset of \( X \);

(c) for each \( x \in X \), \( t(x, y) \) is an H-quasi-convex function of \( y \) on \( Y \) for each \( y \in Y \), \( s(x, y) \) is an H-quasi-concave function of \( x \) on \( X \);

(d) there exists an H-compact subset \( L \) of \( X \) and a compact subset \( K \) of \( Y \) such that

\[
\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \inf_{y \in Y \setminus Y} \sup_{x \in L} f(x, y).
\]

Then the following minimax inequality holds,

\[
\alpha = \inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} \inf_{y \in Y} g(x, y) = \beta.
\]

Remak 5.3. Theorem 5.3 and Corollary 5.2 generalize [22, Theorem 4(2), 3, Corollary 5.5] and Liu’s result [18] in several ways. When \( f = s = t = g \), the conclusion of Corollary 5.2 (respectively Theorem 5.3) implies the following minimax equality, which generalizes the minimax principle of the von Neumann type due to Sion [21]:

\[
\inf_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \inf_{y \in Y} f(x, y).
\]

It would be of some interest to compare the minimax equality with the corresponding result of Barbaro-Ceppitelli in [3].

References

Continuous selection theorem


Sichuan Normal University Chengdu,
Sichuan People’s Republic of China