CERTAIN INTEGRALS FOR CLASSES OF $p$-VALENT MEROMORPHIC FUNCTIONS

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In this paper we introduce two classes, namely $T_p(m, M)$ and $E_p(m, M)$, of functions

$$f(z) = \frac{1}{z^p} + \frac{a_0}{z^{p-1}} + \ldots + \frac{a_{n+p-1}}{z^n} + \ldots,$$

regular and $p$-valent in $D = \{z : |z| < 1\}$ where $D = \{z : |z| < 1\}$. We show that, for suitable choices of real constants $\alpha$, $\beta$, and $\gamma$, the integral operators of the form

$$F(z) = \left[\frac{\gamma-p(\alpha+\beta)+2}{z^{\gamma-p\beta+2}} \int_0^z t^{\gamma+1} (f(t))^{\alpha} g(t)^{\beta} dt\right]^{-1/\alpha}$$

map $\Gamma_p^*(\rho) \times \Gamma_p(m, M)$ into $\Gamma_p^*(\rho)$, where $\Gamma_p^*(\rho)$ is the class of $p$-valent meromorphically starlike functions of order $\rho$, $0 \leq \rho < 1$. For the classes $T_p(m, M)$ and $E_p(m, M)$, we obtain class preserving integral operators of the form

$$F(z) = \left[\frac{\gamma-p\alpha+1}{z^{\gamma+1}} \int_0^z t^\gamma f(t)^\alpha dt\right]^{-1/\alpha},$$

with suitable restrictions on real constants $\alpha$ and $\gamma$.

Our results generalize almost all known results obtained so far in this direction.

Received 13 August 1981.
1. Introduction

Let $I^+$ denote the set of positive integers. We denote by $\Gamma_p$, $p \in I^+$, the set of the functions

$$f(z) = \frac{1}{z^p} + \frac{a_0}{z^{p-1}} + \cdots + \frac{a_{n+p-1}}{z^n} + \cdots,$$

regular and $p$-valent in $D - \{0\}$, where $D = \{z : |z| < 1\}$ and $f'(z) \neq 0$ there. A function $f$ of $\Gamma_p$ is said to belong to $\Gamma_p^*$, the class of $p$-valent meromorphically starlike functions, if and only if

$$\text{Re}\{z[f'(z)/f(z)]\} < 0,$$

$z \in D$. A function $f$ of $\Gamma_p$ is said to belong to $\Gamma_p^*(\rho)$, the class of $p$-valent meromorphically starlike functions of order $\rho$, $0 \leq \rho < 1$, if and only if

$$\text{Re}\{z[f'(z)/f(z)]\} < -\rho,$$

$z \in D$. A function $f$ of $\Gamma_p$ is said to belong to $\Sigma_p^*(\rho)$, the class of $p$-valent meromorphically convex functions of order $\rho$, $0 \leq \rho < 1$, if and only if

$$\text{Re}\{1+z[f''(z)/f'(z)]\} < -\rho,$$

$z \in D$. The class $\Sigma_p^*$ of $p$-valent meromorphically convex functions is identified by $\Sigma_p^* = \Sigma_p(0)$.

Now we define two subclasses, namely $\Gamma_p(m, M)$ and $\Sigma_p(m, M)$, of $\Gamma_p^*$ and $\Sigma_p^*$ respectively.

A function $f$ of $\Gamma_p$ belongs to the class $\Gamma_p(m, M)$ if and only if

$$|z[f'(z)/f(z)] + m| < M,$$

$z \in D$ where

$$(m, M) \in E_p = \{(m, M) : |m-p| < M \leq m\}.$$

A function $f$ of $\Gamma_p$ belongs to the class $\Sigma_p(m, M)$ if and only if

$$|1+z[f''(z)/f'(z)] + m| < M,$$

$z \in D$, where

$$(m, M) \in E_p = \{(m, M) : |m-p| < M \leq m\}.$$

It is clear that $\Gamma_p(m, M) \subset \Gamma_p^*[(m-M)/p] \subset \Gamma_p^* \subset \Gamma_p$ and $\Sigma_p(m, M) \subset \Sigma_p[(m-M)/p] \subset \Sigma_p \subset \Gamma_p$. Also, a function $f$ belongs to $\Sigma_p(m, M)$ if and only if $zf'(z)/p$ belongs to $\Gamma_p(m, M)$.

In [1], [2], [3], the integral operators of the forms
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\[ F(z) = \left[ \frac{\beta-2\alpha+2}{2^{\beta-2\alpha+2}} \int_0^{\beta} t^\alpha\left( f(t)g(t)^\alpha dt \right) \right]^{1/\beta} \]

and

\[ F(z) = \left[ \frac{\alpha+1}{2^{\alpha+1}} \int_0^{\alpha} t^\beta f(t)^{\alpha} dt \right]^{1/\alpha}, \]

with suitable restrictions on the real constants $\alpha$, $\beta$ and $\gamma$, and for $f$ and $g$ belonging to some favoured classes of meromorphic functions have been studied. The purpose of this paper is to obtain the integral operators that are more general and transform $\Gamma_p^*(\rho) \times \Gamma_p^*(m, M)$ into $\Gamma_p^*(\rho)$, $\Gamma_p(m, M)$ into $\Gamma_p(m, M)$, and $\Sigma_p(m, M)$ into $\Sigma_p(m, M)$. Our results generalize almost all known results obtained so far in this direction [1], [2], [3].

2. Preliminary lemmas

Let $S(m, M)$ be the class of functions $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ regular and satisfying

\[ z \frac{h'(z)}{h(z)} - m < M \text{ in } D, \]

where

\[ (m, M) \in E = \{(m, M) : m > \frac{1}{2}, |m-1| < M \leq m\}. \]

This class $S(m, M)$ was introduced by Jakubowski [5]. It is worth mentioning here that the requirement $m > \frac{1}{2}$ in (2.2) is superfluous and may be dropped since it follows directly from the inequality $|m-1| < m$ in (2.2).

The proof of the following lemma is based on the lines of a result of Silverman [7, Theorem 1], the only difference is that in the definition of $S(m, M)$ Silverman [7] has taken equality also in (2.1) and restricted $m, M$ by the inequalities

\[ m + M \geq 1, \quad M \leq m \leq M + 1 \]

which are equivalent to the inequalities
\[|m-1| \leq M \leq m.\]

However we follow the definition of \(S(m, M)\) given by Jakubowski [5].

**Lemma 2.1.** The function \(h(z) = z + \sum_{n=2}^{\infty} a_n z^n\) belongs to \(S(m, M)\) if and only if there exists a function \(w(z)\) regular in \(D\) and satisfying \(w(0) = 0\), \(|w(z)| < 1\) for \(z\) in \(D\), such that

\[
Z \frac{h'(z)}{h(z)} = \frac{1+a'w(z)}{1-b'w(z)}, \quad z \in D,
\]

where

\[
a' = \frac{M^2-m^2+m}{M} \quad \text{and} \quad b' = \frac{m-1}{M}.
\]

**Proof.** First suppose that \(h \in S(m, M)\). Then

\[
|z \frac{h'(z)}{h(z)} - \frac{m}{M}| < 1.
\]

Let \(g(z) = z\left[\frac{h'(z)}{h(z)} - \frac{m}{M}\right]\) and

\[
w(z) = \frac{g(z) - g(0)}{1 - g(0)g(z)} = \frac{z\left[h'(z)/h(z)\right] - 1}{M + \left[(m-1)/M\right] \left[z\left[h'(z)/h(z)\right] - m\right]}.
\]

Then \(w(0) = 0\), \(|w(z)| < 1\). Rearranging (2.6) and using (2.5) we get (2.4).

Conversely, suppose that \(h(z)\) satisfies (2.4). Then

\[
z \frac{h'(z)}{h(z)} - m = M\frac{(1-m)/M + w(z)}{1 + ((1-m)/M)w(z)} = MG(z), \quad \text{say}.
\]

In view of (2.2), \(|(1-m)/M| < 1\). Thus the function

\[G(z) = \frac{(1-m)/M + w(z)}{1 + ((1-m)/M)w(z)}\]

satisfies \(|G(z)| < 1\). From (2.7), it follows now that \(h \in S(m, M)\).

This completes the proof of the lemma.

**Lemma 2.2.** The function \(f\) is in \(\Gamma_p(m, M)\) if and only if there exists a function \(w(z)\) regular and satisfying \(w(0) = 0\), \(|w(z)| < 1\), for \(z\) in \(D\) such that
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where $a = \frac{(M^2 - m^2 + mp)}{Mp}$ and $b = \frac{(m-p)}{M}$.

Proof. Since $f \in \Gamma_p(m, M)$, $z\left(\frac{f'(z)}{f(z)}\right) = -p + \ldots$. Let $\left(z\frac{f'(z)}{f(z)}\right) / -p = z\left(h'(z)/h(z)\right) = 1 + \ldots$, then from Lemma 2.1, for $|(m/p) - 1| < M/p \leq m/p$,

$$\left|\frac{z(f'(z)/f(z))}{-p} - \frac{m}{p}\right| < \frac{M}{p}$$

if and only if

$$z\left(\frac{f'(z)}{f(z)}\right) = \frac{1 + aw(z)}{1 - bw(z)}$$

where

$$a = \frac{(M/p)^2 - (m/p)^2 + (m/p)}{M/p} \quad \text{and} \quad b = \frac{(m/p) - 1}{M/p};$$

or, for $|m-p| < M \leq m$,

$$\left|z\frac{f'(z)}{f(z)} + m\right| < M$$

if and only if

$$z\frac{f'(z)}{f(z)} = -p \frac{1 + aw(z)}{1 - bw(z)},$$

where

$$a = \frac{m^2 - m^2 + mp}{Mp} \quad \text{and} \quad b = \frac{m-p}{M}.$$

**Lemma 2.3.** If the function $w(z)$ is regular for $|z| \leq r < 1$, $w(0) = 0$ and $|w(z_1)| = \max_{|z|=r} |w(z)|$, then

$$w(z_1)^{k} = k\omega(z_1), \quad k \geq 1.$$ 

A proof of Lemma 2.3, which is due to Jack, may be found in [4].

**Lemma 2.4.** A function $f$ belongs to $\Gamma_p^*(\rho)$, $0 \leq \rho < 1$, if and only if there exists a function $w(z)$ regular and satisfying $w(0) = 0$, $|w(z)| < 1$ in $D$ such that
Proof. Let \( P(z) = \left( \frac{z(f'(z)/f(z))}{-p} - \rho \right)/(1-P) = 1 + \ldots \); then \( \text{Re}\{P(z)\} > 0 \) and hence by a well known result [6], \( P(z) \) can be written as

\[
P(z) = \frac{1-w(z)}{1+w(z)},
\]

where \( w(z) \) is regular and \( w(0) = 0, |w(z)| < 1 \), for \( z \) in \( D \). Thus

\[
\frac{z(f'(z)/f(z))}{-p} - \rho = \frac{1-w(z)}{1+w(z)}
\]

and hence the result follows.

3. Integral operators that map \( \Gamma^*(\rho) \times \Gamma_p(m, M) \) into \( \Gamma^*(\rho) \)

**THEOREM 3.1.** Let \( \alpha, \beta \) and \( \gamma \) be real constants such that

\[
\alpha > 0, \beta \geq 0 \quad \text{and} \quad \gamma - p(\alpha + \beta) + 1 > -1.
\]

If \( f \in \Gamma^*_p(\rho) \) and \( g \in \Gamma_p(m, M) \), \( (m, M) \in E^*_p = \{(m, M) : |m-p| < M \leq m^*\} \)

where \( m^* = \min\{m, (m+p)+\{\alpha p(1-p)/2\gamma(\gamma-pp-p\alpha+2)\}\} \), then

\[
F(z) = \left[ \frac{\gamma-p(\alpha+\beta)+2}{z^\gamma-p^2+2} \int_0^z t^{\gamma+1} f(t)^\alpha g(t)^\beta dt \right]^{1/\alpha} = \frac{1}{z^p} + \ldots
\]

also belongs to \( \Gamma^*_p(\rho) \). In (3.1) all powers are principal ones.

Proof. Let us choose a function \( w(z) \) such that

\[
z \frac{f'(z)}{f(z)} = -p \frac{1+(2p-1)w(z)}{1+w(z)}, \quad w(0) = 0,
\]

and \( w(z) \) is either regular or meromorphic in \( D \).

From (3.1) and (3.2), we have

\[
z^{pB} \frac{f(z)^\alpha}{F(z)^\alpha} g(z)^\beta = \frac{1+(\xi/\delta)w(z)}{1+w(z)},
\]

where \( \xi = \gamma + p(\alpha-\beta) + 2(1-\alpha p) \) and \( \delta = \gamma - p(\alpha+\beta) + 2 \).

Logarithmic differentiation of (3.3) yields
Let $r^*$ be the distance from the origin of the pole of $w(z)$ nearest the origin. Then $w(z)$ is regular in $|z| < r_0 = \min\{r^*, 1\}$. By Lemma 2.3, for $|z| \leq r$ $(r < r_0)$ there is a point $z_0$ such that

$$z_0w'(z_0) = kw(z_0), \quad k \geq 1.$$  

From (3.4) and (3.5) we have

$$\Re\left\{\frac{f'(z_0)}{f(z_0)}\right\} > \frac{\beta}{\alpha} (m-p) - \frac{\beta}{\alpha} \left| \frac{g'(z_0)}{g(z_0)} + m \right| - p \frac{\Re\left[1 + (2p-1)\omega(z_0)\right]}{1 + \omega(z_0)}$$

$$+ \frac{2p(1-\rho)k\Re\left[\omega(z_0)\right]}{1 + \omega(z_0)} + \frac{2p(1-\rho)k\Re\left[\omega(z_0)\right]}{1 + \omega(z_0)}.$$  

Now suppose that it were possible to have $M(r, \omega) = \max_{|z|=r} |\omega(z)| = 1$ for some $r$ $(r < r_0 \leq 1)$. At the point $z_0$ where this occurred, we would have $|\omega(z)| = 1$. Then, from (3.6),

$$\Re\left\{\frac{f'(z_0)}{f(z_0)}\right\} > -pp + \frac{2(\gamma-p\beta-\alpha p+2)[(1-p)p+(2\beta/\alpha)(m-p)(\gamma-p\beta-\alpha p+2)]}{\delta^2 + 2\delta \xi \Re\left(z_0^{a} + \xi}^{2}}\right.$$  

$$\geq -pp, \quad \text{provided } M \leq (m-p) + \frac{\alpha p(1-p)}{2\beta(\gamma-p\beta-\alpha p+2)}.$$  

But this is contrary to the fact that $f \in \Gamma^*(\rho)$. So we cannot have $M(r, \omega) = 1$. Thus $|\omega(z)| \neq 1$ in $|z| < r_0$. Since $\omega(0) = 0$, $|\omega(z)|$
is continuous in \(|z| < r_0\) and \(|\omega(z)| \neq 1\) there, \(\omega(z)\) cannot have a pole at \(|z| = r_0\). Therefore \(|\omega(z)| < 1\) and \(\omega(z)\) is regular in \(D\).

Hence from (3.2) and Lemma 2.4, \(F \in \Gamma^*_p(p)\).

REMARK. Taking \(p = 1\), the undermentioned results follow from Theorem 3.1:

(i) for \(\alpha = 1\), \(\beta = 0\), \(\gamma = 0\) and \(\rho = 0\), a result of Bajpai [1];

(ii) for \(\alpha = \beta = 1\), a result of Dwivedi [2];

(iii) for \(\alpha = \beta\), a result of Dwivedi, Bhargava and Shukla [3].

THEOREM 3.2. Let \(\beta\) and \(\gamma\) be real constants such that \(\beta \geq 0\) and \(\gamma > 0\). If \(f \in \Gamma^*_p(p)\) and \(g \in \Gamma_p(m, M)\),

\[(m, M) \in E^{**}_p = \{(m, M) : |m-p| < M < m^*\} \]

where

\[m^* = \min\{m, (m-p) + \frac{p(1-p)}{2\beta(\gamma+p(1-p))}\}.\]

Then

\[(3.7) \quad F(z) = \frac{\gamma}{z^{\gamma+p}} \int_0^z t^{\gamma+p(1+\beta)-1} f(t) \cdot g(t)^{\beta} dt = \frac{1}{z^p} + \ldots \]

also belongs to \(\Gamma^*_p(p)\). In (3.7) all powers are principal ones.

Proof. Let us choose a function \(\omega(z)\) such that

\[(3.8) \quad z \frac{F'(z)}{F(z)} = -p \frac{1+(2p-1)\omega(z)}{1+\omega(z)}, \quad \omega(0) = 0,\]

\(\omega(z)\) is either regular or meromorphic in \(D\).

From (3.7) and (3.8), we have

\[(3.9) \quad z^p \frac{F_f(z) g(z)^{\beta}}{F(z)} = 1 + \left(\frac{\gamma+2p(1-p)}{\gamma}\right) \frac{1+\omega(z)}{1+\omega(z)}.\]

Logarithmic differentiation of (3.9) yields
(3.10) $z \frac{f'(z)}{f(z)} = -p\beta - \beta \frac{g'(z)}{g(z)} - p \frac{1+(2p-1)\omega(z)}{1+\omega(z)} + \frac{2p(1-p)\omega'(z)}{(1+\omega(z))^2}\frac{1+\omega(z)}{1+\omega(z)^2}. $

Let $r^*$ be the distance from the origin of the pole of $\omega(z)$ nearest the origin. Then $\omega(z)$ is regular in $|z| < r_0 = \min\{r^*, 1\}$. By Lemma 2.3, for $|s| \leq r \ (r < r_0)$ there is a point $z_0$ such that

(3.11) $z_0\omega'(z_0) = k\omega(z_0), \ k \geq 1.$

From (3.10) and (3.11), we have

$$
\Re\left\{z_0 \frac{f'(z_0)}{f(z_0)}\right\} \geq \beta(m-p) - \beta \left|z_0 \frac{g'(z_0)}{g(z_0)} + m\right| - p \frac{\Re\left\{1+(2p-1)\omega(z_0)\right\}}{1+\omega(z_0)^2} + \frac{2p(1-p)k\Re\left\{\omega(z_0)\right\}}{|1+\omega(z_0)|^2}\left[1+\omega(z_0)^2\right].
$$

or

(3.12)

$$
\Re\left\{z_0 \frac{f'(z_0)}{f(z_0)}\right\} > -\beta(M-(m-p)) - p \frac{1+2p\Re\omega(z_0)+(2p-1)|\omega(z_0)|^2}{1+2\Re\omega(z_0)+|\omega(z_0)|^2} + \frac{2p(1-p)k\Re\left\{\omega(z_0)\right\}}{|1+2\Re\omega(z_0)+|\omega(z_0)|^2|^2}\left[\omega(z_0)^2\right].
$$

Now suppose that it were possible to have $M(r, \omega) = \max_{|z|=r} |\omega(z)| = 1$ for some $r \ (r < r_0)$. At the point $z_0$ where this occurred we would have $|\omega(z_0)| = 1$. Then, from (3.12),

$$
\Re\left\{z_0 \frac{f'(z_0)}{f(z_0)}\right\} > -p\rho + \frac{2\left(\gamma+p(1-p)\right)\left[\gamma+p(1-p)\right]}{\left[\gamma^2+2\gamma\left(\gamma+p(1-p)\right)\Re\omega(z_0)\right]} \left[\gamma^2+2\gamma\left(\gamma+p(1-p)\right)\Re\omega(z_0)\right] \geq -p\rho, \ \text{provided} \ M \leq (m-p) + \frac{p(1-p)}{2\beta\left(\gamma+p(1-p)\right)}.
$$

But this is contrary to the fact that $f \in \Gamma^+(\rho)$. So we can not have $M(r, \omega) = 1$. Thus $|\omega(z)| \neq 1$ in $|z| < r_0$. Since $|\omega(0)| = 0$,
|w(z)| is continuous in |z| < \rho_0 and |w(z)| \neq 1 there, w(z) can not have a pole at |z| = \rho_0. Therefore \ |w(z)| < 1 and w(z) is regular in D.

Hence from (3.8) and Lemma 2.4, \ F \in \Gamma_p^*(\rho).

4. Class preserving integral operators for \ \Gamma_p(m, M) and \ \Sigma_p(m, M)

**THEOREM 4.1.** If \ f \in \Gamma_p(m, M) and \ F(z) is defined by

\[
(4.1) \quad F(z) = \left[ \frac{\gamma - \rho_a + 1}{\alpha + 1} \right] \int_0^z \frac{t^\gamma f(t)^\alpha}{dt} dt = \frac{1}{z^\rho} + \ldots ,
\]

\(\alpha > 0\). Then \ F \in \Gamma_p(m, M) \ if \ \gamma > \max\{pa(1+\alpha)+b-1\}/(1-b), p\alpha-1\}.

In (4.1) all powers are principal ones.

**Proof.** From (4.1) we have

\[
(4.2) \quad z \frac{F'(z)}{F(z)} + (\gamma + 1) = (\gamma - p\alpha + 1) \left\{ \frac{f(z)}{F(z)} \right\}^\alpha .
\]

Let us choose a function \ w(z) such that

\[
(4.3) \quad z \frac{F'(z)}{F(z)} = -p \frac{1 + aw(z)}{1 - bw(z)} , \ w(0) = 0 ,
\]

and \ w(z) is either regular or meromorphic in D. From (4.2) and (4.3), we have

\[
(4.4) \quad (\gamma - p\alpha + 1) \left\{ \frac{f(z)}{F(z)} \right\}^\alpha = \frac{\gamma - p\alpha + 1 - (a\alpha + b + b\gamma)w(z)}{1 - bw(z)} .
\]

Logarithmic differentiation of (4.4) yields

\[
(4.5) \quad z \frac{F'(z)}{F(z)} + m = \frac{[(m-p)k - (m-p)\theta + (ap + bm)k]w(z) + (ap + bm)\theta w^2(z) - (a + b)p\omega w'(z)}{[k - (a\alpha + b(2-p\alpha) + 2b\gamma)\omega(z) + b\theta w^2(z)]}
\]

where \(k = \gamma - p\alpha + 1\) and \(\theta = a\alpha + b + b\gamma\).

Let \(r^*\) be the distance of the pole of \(w(z)\) nearest the origin. Then \(w(z)\) is regular in \(|z| < r_0 = \min\{r^*, 1\}\). By Lemma 2.3, for
$|z| \leq r$ $\left( r < r_0 \right)$, there is a point $z_0$ such that

\begin{equation}
\tag{4.6}
z_0w'(z_0) = kw(z_0), \quad k \geq 1.
\end{equation}

From (4.5) and (4.6), we have

\begin{equation}
\tag{4.7}
z_0 \frac{f'(z_0)}{f(z_0)} + m = \frac{N(z_0)}{D(z_0)},
\end{equation}

where

$N(z_0) = (m-p)(\gamma-p\alpha+1) - \{(m-p)(a\alpha+p+b\gamma)+(ap+bm)(\gamma-p\alpha+1)+(a+b)pm\}w(z_0)$

$+ \{(ap+bm)(a\alpha+p+b\gamma)\}w^2(z_0)$

and

$D(z_0) = (\gamma-p\alpha+1) - \{a\alpha+p(2-p\alpha)+2b\gamma\}w(z_0) + b(a\alpha+p+b\gamma)w^2(z_0)$.

Now suppose that it were possible to have $M(r, w) = \max |\omega(z)|$ for some $r$ $\left( r < r_0 \right)$. At the point $z_0$ where this occurred we would have $\omega(z_0) = 1$. Then we have

\begin{equation}
\tag{4.8}
|N(z_0)|^2 - M^2|D(z_0)|^2 = A - 2B \Re\{\omega(z_0)\},
\end{equation}

where

$A = kp(a+b)\left[kp(a+b)+2m(\gamma-p\alpha+1)+2M(ba\alpha+p+b\gamma)\right]$ 

and

$B = kp(a+b)M[a\alpha+p(2-p\alpha)+2b\gamma]$.

From (4.8), we have

\begin{equation}
\tag{4.9}
|N(z_0)|^2 - M^2|D(z_0)|^2 > 0,
\end{equation}

provided $A+2B > 0$.

Now

$A + 2B = kp(a+b)[kp(a+b)+2m(1+b)\{(1+b)+(1-p\alpha+p\alpha)\}]$

$> 0$

provided $\gamma \geq (-p\alpha-b+p\alpha)/(1+b)$ and
\[ A - 2B = kp(a+b)(kp(a+b)+2M(1-b))(\gamma(1-b)+(1-pa-b-ap\alpha)) \]
\[ > 0 , \]
provided \( \gamma \geq \frac{(ap\alpha+b+pa-1)/(1-b)}{1-b} \).

Thus from (4.7) and (4.9), it follows that
\[ \left| \frac{f'(z_0)}{f(z_0)} + m \right| > M , \text{ if } \gamma \geq \frac{ap(1+\alpha)+b-1}{1-b} . \]

But this is contrary to the fact that \( f \in \Gamma_p(m, M) \). So we can not have \( M(r, \omega) = 1 \). Thus \( |w(z)| \neq 1 \) in \( |z| < r_0 \). Since \( |w(0)| = 0 \), \( |w(z)| \) is continuous and \( |w(z)| \neq 1 \) in \( |z| < r_0 \), \( w(z) \) can not have a pole at \( |z| = r_0 \). Therefore \( w(z) \) is regular and \( |w(z)| < 1 \), for \( z \) in \( D \).

Hence from (4.3) and Lemma 2.2, \( F \in \Gamma_p(m, M) \).

REMARK. Taking \( p = 1 \), the undermentioned results follow from Theorem 4.1:

(i) for \( \alpha = \gamma = 1 \), \( m = M \) and \( m \to \infty \), a result of Bajpai [1];

(ii) for \( \alpha = 1 \), a result of Dwivedi [2];

(iii) a result of Dwivedi, Bhargava and Shukla [3].

**THEOREM 4.2.** If \( f \in \Sigma_p(m, M) \) and \( F(z) \) is defined by

\[ F(z) = \frac{Y-p+1}{z^{Y+1}} \int_0^z t^Y f(t) dt = \frac{1}{z^p} + \ldots . \]

Then \( F \in \Sigma_p(m, M) \) if \( \gamma > \max\{[p(1+\alpha)+b-1]/(1-b), p-1]\} .

In (4.10) all powers are principal ones.

Proof. We can write (4.10) as

\[ zF''(z) = \frac{Y-p+1}{z^{Y+1}} \int_0^z t^Y f'(t) dt . \]

Since \( f \in \Sigma_p(m, M) \) if and only if \( zf'(z)/p \in \Gamma_p(m, M) \) and hence from
Theorem 4.1, and (4.11), we get \( zF'(z)/p \) belongs to \( \Gamma_p(m, M) \). So
\( F(z) \) belongs to \( \Sigma_p(m, M) \).

**REMARK.** Taking \( p = 1 \), the undermentioned results follow from Theorem 4.2:

(i) for \( m = M \) and \( m + \infty \), a result of Bajpai [1].
(ii) a result of Dwivedi [2].

**References**


