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VARIETIES OF ORTHOMODULAR LATTICES. II

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Introduction. In this paper we continue the study of equationally defined classes of orthomodular lattices started in [**1**].

The only atom in the lattice of varieties of orthomodular lattices is the variety of all Boolean algebras. Every nontrivial variety contains it. It follows from B. Jónsson [**4**, Corollary 3.2] that the variety [MO2] generated by the orthomodular lattice MO2 of Figure 1 covers the variety of all Boolean algebras. It was first shown by R. J. Greechie (oral communication) and is not difficult to see that every variety not consisting of Boolean algebras only contains [MO2]. Again it follows from the result of Jónsson's mentioned above that the varieties generated by one of the orthomodular lattices of Figures 2 to 5 cover [MO2]. The Figures 4 and 5 are to be understood in such a way that the orthocomplement of every element is on the vertical line through this element. Note that in Figure 5 the left and the right endpoints are "identified". It is the aim of this paper to prove the following

THEOREM. *Every finite orthomodular lattice which does not belong to* [MO2] *has one of the lattices of Figures 2 to 5 as the homomorpic image of a subalgebra.*

This theorem has the following

COROLLARY. *Every variety of orthomodular lattices which is not contained in* [MO2] *and is generated by its finite members contains a variety generated by one of the orthomodular lattices of Figures 2 to 5.*

Actually, the proof of our theorem goes through without modifications for all (possibly infinite) orthomodular lattices of finite height. We do not know, however, whether the theorem or its corollary is true without this restriction.

Our proof consists of a careful analysis of lattices of dimension 3 making extensive use of the results in Greechie [**2**] and [**3**]. The general result is then obtained by an inductive argument.

For general background information we refer to [**1**].

Our thanks go to M. Janowitz for comments which have influenced the set-up of this paper.

1. Blocks and their interaction. As in [**1**] an orthomodular lattice (abbreviated: OML) is considered as a universal algebra $(L; \vee, \wedge, ', 0, 1)$ with binary lattice operations $\vee$ and $\wedge$, the unary orthocomplementation operation $'$, and the two nullary operations (constants) 0 and 1, smallest and

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largest element of the lattice. All universal algebraic notions like subalgebra, homomorphic image etc. are to be understood in this sense.

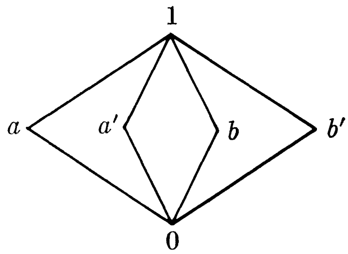
If d is an element of an OML we will frequently consider the interval $[0, d]$ as an OML. In this case the orthocomplement $a^{\#}$ of an element $a \in [0, d]$ is defined by $a^{\#} = a' \wedge d$. Equipped with an orthomodular structure in this way the interval $[0, d]$ becomes the homomorphic image of a subalgebra of L, namely the subalgebra S of all elements of L which commute with d. Since d is in the center of this subalgebra the interval $[0, d]$ is a homorphic image of S.

Throughout this paper L is a finite OML, $\mathscr{B}(L)$ is the set of all maximal Boolean subalgebras (blocks) of L, and $n(L)$ is the maximum of the cardinal numbers $|B|$ of the blocks $B \in \mathscr{B}(L)$. If C is a maximal chain in L then the subalgebra of L generated by C is a block. Conversely, every maximal chain in a block is also a maximal chain in L. In particular, every atom of a block is also an atom of L.

We frequently use the fact that a finite OML is simple if and only if it is subdirectly irreducible, if and only if it is directly irreducible, i.e. has a center consisting of 0 and 1 only.

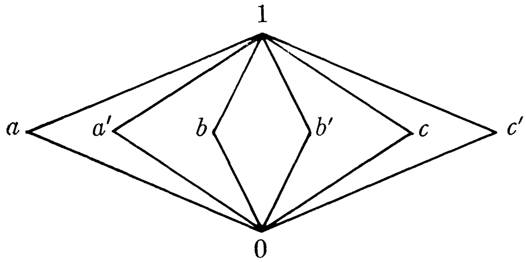
Given an element $e \in L$ we define $S(e) = [0, e'] \cup [e, 1]$. The sets $S(e)$ are called sections (Greechie [**2**]) of L.

(1.1) *Definition*. The blocks B, C are said to meet in the section $S(e)$ if and only if $B \neq C, e \in (B \cap C) - \{1\}$, and $B \cap C = S(e) \cap (B \cup C)$.



(MO2)

Figure 1.



(MO3)

Figure 2.

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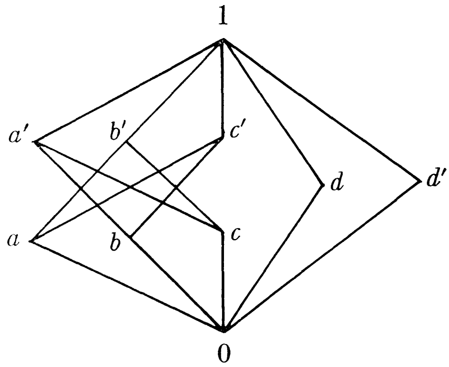

Figure 3.

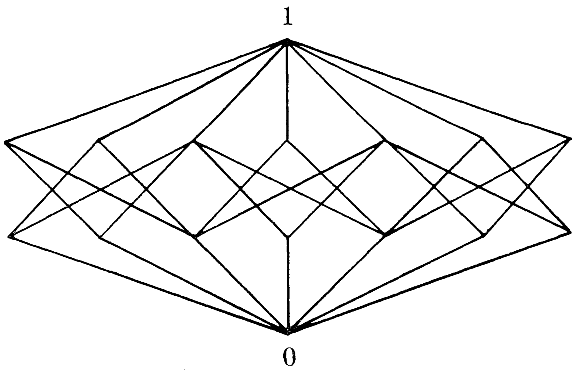

Figure 4.

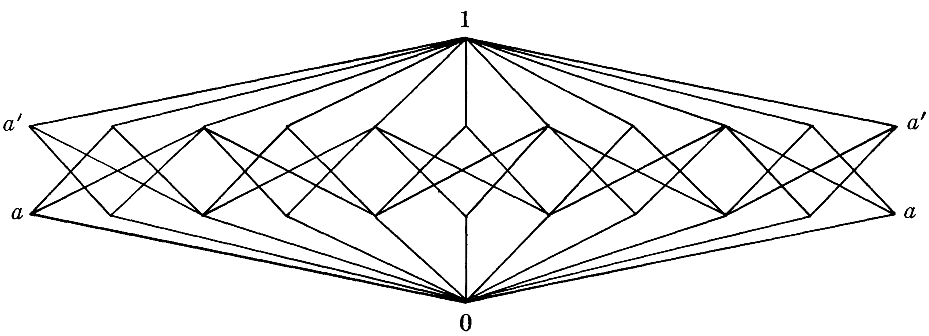

Figure 5.

(1.2) (Greechie [**2**, Lemma 4.3]). *Two (different) blocks B and C of L meet in the section $S(e)$ if and only if e is an atom of $B \cap C$, $e \neq 1$, and $[0, e'] \cap B = [0, e'] \cap C$.*

Note that if the blocks B and C meet in the section $S(e)$, then e is not an atom of L.

(1.3) *If B and C are blocks of L, then there exists at most one element $e \in B \cap C$ such that B and C meet in the section $S(e)$.*

Proof. Assume there were elements $e, f \in (B \cap C) - \{1\}$, $e \neq f$, such that $B \cap C = S(e) \cap (B \cup C) = s(f) \cap (B \cup C)$. Since e and f are both atoms of $B \cap C$ we would have $e \nleq f$ and, hence, $f \leqq e'$. Since e is not an atom of B there exists an atom $p \in B$, $p < e$. It would follow that $f \vee p < f \vee e$ and $f \vee p \in B$, and hence, $f \vee p \in S(f) \cap (B \cup C) \subseteq S(e)$. But $f \vee p \geqq e$ is impossible since $f \vee p < f \vee e$ and $f \vee p \leqq e'$ is impossible because of $p \leqq e$.

By a hyperatom of L we mean an element $e \in L$ with the property that every maximal chain in the interval $[0, e]$ has exactly three elements.

(1.4) *Definition.* L is called homogeneous if and only if, whenever two blocks B and C meet in the section $S(e)$, e is a hyperatom of L.

(1.5) *If L is homogeneous and the blocks B and C of L meet in a section $S(e)$, then $|B| = |C|$.*

Proof. Let M be a maximal chain in $B \cap [e, 1] = C \cap [e, 1]$. Let $b < e$ be an atom of B and let $c < e$ be an atom of C. Then $M \cup \{0, b\}$ is a maximal chain in B and $M \cup \{0, c\}$ is a maximal chain in C. Since they have obviously the same number of elements, it follows that B and C have the same number of elements.

(1.6) *Definition.* A path in L is a finite sequence $(B_i)_{i=0, 1, \ldots n}$ $(n \geqq 0)$ of blocks in L such that B_i and B_{i+1} meet in a section $S(e_i)$ $(i = 0, 1, \ldots n - 1)$. The path $(B_i)_{i=0, 1, \ldots n}$ is said to connect the blocks B_0 and B_n. The natural number n is called the length of the path. For blocks $A, B \in \mathscr{B}(L)$, define $r(A, B)$ to be the smallest length of a path connecting A and B. If no such path exists we put $r(A, B) = \infty$. For a block A in L and an element b in L we define $r(A, b) = \min\{r(A, B) | \ b \in B \in \mathscr{B}(L)\}$. For elements a and b of L we define $r(a, b) = \min\{r(A, B) | \ a \in A \in \mathscr{B}(L),\ b \in B \in \mathscr{B}(L)\}$. The OML L is said to be connected if and only if $r(A, B) < \infty$ holds for any two blocks A and B of L.

The following statement (1.7) is a consequence of [**2**, Theorem 4.6].

(1.7) *If B and C are blocks in L and if $B \cap C \neq \{0, 1\}$, then there exists a path connecting B and C.*

(1.8) *If L is homogeneous and connected, then any two blocks of L have the same number of elements.*

Proof. This follows from (1.5) by induction.

(1.9) *Let $(B_0, B_1, \ldots B_n)$ be a path in L such that B_i and B_{i+1} meet in the section $S(e_i)$, and let $e \in L$ satisfy $e \leqq e_0' \wedge \ldots \wedge e'_{n-1}$. Then $B_0 \cap [0, e] = B_n \cap [0, e]$.*

Proof. This follows from (1.2) by induction.

(1.10) *Let L be subdirectly irreducible and connected and assume $n(L) \geqq 2^3$. Then there exists a path $(B_0, B_1, \ldots B_n)$ of length $n \geqq 2$ in L with the following properties: If $S(e_i)$ $(i = 0, 1, \ldots n - 1)$ is the section in which B_i and B_{i+1} meet, then $e_0 \neq e_{n-1}$, $e_0 \not\leqq e'_{n-1}$, $e_0 \leqq e_1' \wedge \ldots \wedge e'_{n-2}$, and $e_{n-1} \leqq e'_{n-2} \wedge \ldots \wedge e'_1$. (Note that the meet of the empty family is 1.)*

Proof. As the first step we prove the existence of a path satisfying all conditions of the theorem except possibly $e_{n-1} \leqq e'_{n-2} \wedge \ldots \wedge e_1'$.

Let D_1 be an arbitrary block in L. Since L is subdirectly irreducible and $n(L) \geqq 2^3$, L is not Boolean. Since L is also connected there exists a block $D_2 \neq D_1$ in L which meets D_1 in some section $S(d_1)$. Since L is subdirectly irreducible, d_1 is not in the center of L and there exists a block C not containing d_1. Since L is connected there exists a path $(D_2, \ldots D_t)$ with $D_t = C$, $t \geqq 3$. Let $S(d_i)$ $(i = 2, 3, \ldots t - 1)$ be the section in which D_i and D_{i+1} meet and let j be the largest index i for which $d_i = d_1$. Since $d_1 \notin D_t$ we have $j < t - 1$. Define $(B_0, B_1, \ldots B_s) = (D_j, D_{j+1}, \ldots D_t)$ and $e_i = d_{i+j}$ $(0 \leqq i < s)$. Then $s \geqq 2$, $e_0 \neq e_i$ $(1 \leqq i < s)$, and $e_0 \notin B_s$. By (1.9), $e_0 \leqq e_1' \wedge \ldots \wedge e'_{s-1}$ would imply $e_0 \in B_1 \cap [0, e_0] = B_s \cap [0, e_0]$, which would contradict $e_0 \notin B_s$. Hence $e_0 \not\leqq e_1' \wedge \ldots \wedge e'_{s-1}$. It follows that there exists a natural number n, $2 \leqq n \leqq s$, such that $e_0 \not\leqq e'_{n-1}$ and $e_0 \leqq e'_1 \wedge \ldots \wedge e'_{n-2}$. This establishes the existence of a path with all the desired properties except possibly the last.

Among the paths satisfying these conditions now take a path $(B_0, B_1, \ldots B_n)$ of smallest length $n \geqq 2$ and assume $e_{n-1} \not\leqq e'_{n-2} \wedge \ldots \wedge e'_1$. Let j be the largest index i, $1 \leqq i \leqq n - 2$, for which $e_{n-1} \not\leqq e_i'$. We then have $e_{n-1} \not\leqq e_j'$, $e_{n-1} \leqq e'_{n-2} \wedge \ldots \wedge e'_{j+1}$. Since $e_0 \not\leqq e'_{n-1}$ and $e_0 \leqq e_1' \wedge \ldots \wedge e'_{n-2}$ we also have $e_{n-1} \neq e_j$. This means that the path $(B_n, B_{n-1}, \ldots B_j)$ satisfies all the conditions of our theorem except possibly the last one and has a length strictly less than n. This contradicts the choice of n and proves that our path $(B_0, B_1, \ldots B_n)$ has all the desired properties, proving (1.10).

2. The case $n(L) \leqslant 2^3$. In this section we prove our theorem for an OML L satisfying $n(L) \leqq 2^3$. If $n(L) \leqq 2^2$, then L either belongs to [MO2] or it contains the OML MO3 of Figure 2 as a subalgebra. Hence we may and will assume throughout this section that L is an OML satisfying $n(L) = 2^3$. In this case two blocks A and B meet in a section if and only if $A \cap B$ consists of 0, 1, an atom and a co-atom of L. Moreover, the blocks A and B meet in a section if and only if $A \neq B$ and $A \cap B \neq \{0, 1\}$.

Note that a four-element block does not meet any block in a section and, consequently, that for every four-element block A and any other block B, $r(A, B) = \infty$ holds.

(2.1) *If the blocks A, B satisfy $r(A, B) = 1$, then $a \vee b \neq 1$ holds for all atoms $a \in A$, $b \in B$.*

Proof. If $a \in B$ or $b \in A$, then clearly $a \vee b \neq 1$. If not, let c be the co-atom in $A \cap B$. Then $a \leqq c$ and $b \leqq c$; hence $a \vee b = c \neq 1$.

(2.2) *If a, b are atoms of L and $a \neq b'$, then $a \vee b = 1$ if and only if $r(a, b) \geqq 2$.*

Proof. If $a \vee b = 1$, then no block contains both a and b and it follows that $r(a, b) \geqq 1$. But $r(a, b) = 1$ is impossible by (2.1). Assume now $a \vee b < 1$. Then a and $a \vee b$ are contained in a block A and b and $a \vee b$ are contained in a block B. Since $0, 1 \neq a \vee b \in A \cap B$, this gives $r(a, b) \leqq 1$.

(2.3) *Let A be a block and d an atom of L. If $a \leqq d'$ for some $a \in A - \{0\}$, then $r(a, d) \leqq 1$. If $a \vee d \neq 1$ for some $a \in A - \{0\}$, then $r(A, d) \leqq 2$.*

Proof. If $a \leqq d'$ there exists a block B containing a and d' (and hence d) such that $0, 1 \neq a \in A \cap B$, implying $r(a, d) \leqq 1$. Assume $a \vee d \neq 1$ for some $a \in A - \{0\}$. Then there exist blocks B, C with $a, a \vee d \in B$ and $d, a \vee d \in C$. Since $0, 1 \neq a \in A \cap B$ and $0, 1 \neq a \vee d \in B \cap C$, it follows that $r(A, d) \leqq 2$.

(2.4) *Let A, B be blocks in L with $A \cap B = \{0, 1\}$. Then $r(a, b) = 0$ holds for at most one pair of atoms $a \in A$, $b \in B$.*

Proof. Assume that (a, b), $(x, y) \in A \times B$ are different pairs of atoms with $r(a, b) = r(x, y) = 0$. We may assume w.l.o.g. that $b \neq y$. Let $a = x$. Since $r(a, b) = r(x, y) = 0$ there exist a block C containing a and b and a block D containing x and y. It follows that $b \leqq a'$ and $y \leqq x' = a'$, which implies $b \vee y = a' \in A \cap B$, contradicting $A \cap B = \{0, 1\}$. If $a \neq x$, we obtain by a similar argument: $x' = a \vee y = b' \in A \cap B$, again contradicting $A \cap B = \{0, 1\}$.

(2.5) *For an* OML *L with $n(L) = 2^3$, the following are equivalent:*

1. *The* OML *of Figure 3 is a subalgebra of L.*
2. *There exist $A \in \mathscr{B}(L)$ and $d \in L$ with $r(A, d) \geqq 3$.*

Proof. $1 \Rightarrow 2$. Assume that the OML of Figure 3 is a subalgebra of L. With the notation of Figure 3 put $A = \{0, 1, a, a', b, b', c, c'\}$. Then A is a block of L. We may assume w.l.o.g. that d is an atom of L. Clearly $r(A, d) \geqq 1$. By (2.1), $r(A, d) \geqq 2$. Assume $r(A, d) = 2$. Then there exists a path (A, B, C) with $d \in C$ and by symmetry we may assume that $a \in B$, i.e. $r(a, d) = 1$. By (2.2) it would follow that $a \vee d \neq 1$, contradicting the assumption that the OML of Figure 3 is a subalgebra of L. This proves $r(A, d) \geqq 3$.

$2 \Rightarrow 1$. Assume the block A in condition 2 is a four-element Boolean algebra. Since $n(L) = 2^3$ there exists an eight-element block B. If a is an atom in A we have $r(B, a) = \infty \geqq 3$. Hence we may assume in condition 2 w.l.o.g. that A is an eight-element Boolean algebra. Furthermore, we may assume that d is an atom of L. We claim that $A \cup \{d, d'\}$ is a subalgebra of L isomorphic with the OML of Figure 3. Let a be an atom of A. By symmetry and duality it is enough to show that $a \vee d = a \vee d' = 1$. This is a consequence of (2.3) and $r(A, d) \geqq 3$ since d' is a co-atom and hence $a \vee d' = 1$ if and only if $a \nleq d'$.

(2.6) *For an* OML L *with* $n(L) = 2^3$, *the following are equivalent*:

1. *The* OML MO3 *is a subalgebra of* L.

2. *There exist elements* $a, b, c \in L$ *satisfying* $r(a, b) \geqq 2$, $r(a, c) \geqq 2$ *and* $r(b, c) \geqq 2$.

Proof. $1 \Rightarrow 2$. With the notation of Figure 2 we may assume w.l.o.g. that a, b, c are atoms of L. Application of (2.2) gives the desired result.

$2 \Rightarrow 1$. We may assume w.l.o.g. that a, b, c are atoms of L. By (2.2), $a \vee b = a \vee c = b \vee c = 1$. Since $a \leqq b'$ would imply $r(a, b) = 0$ and since b' is a co-atom of L, it follows that $a \vee b' = 1$. Using symmetry and duality we obtain that $\{0, 1, a, a', b, b', c, c'\}$ is a subalgebra of L isomorphic with MO3.

(2.7) *Definition* (Greechie [**3**]). A loop of order n ($n \geqq 3$) in L is a sequence $(B_i)_{i=0,1,\ldots n-1}$ of blocks in L satisfying:

1. B_i and B_{i+1} meet in a section ($i = 0, 1, \ldots n-1 \pmod n$),
2. $B_i \cap B_j = \{0, 1\}$ if $|i - j| \geqq 2$ and $\{i, j\} \neq \{0, n-1\}$,
3. $B_i \cap B_j \cap B_k = \{0, 1\}$ if $0 \leqq i < j < k \leqq n - 1$.

Note that the condition 2 is vacuous if $n = 3$ and that condition 2 implies condition 3 if $n \geqq 4$.

As a special case of Greechie [**3**] we have:

(2.8) L *does not contain a loop of order* 3 *or* 4.

(2.9) *Let* B_i ($i = 0, 1, 2, 3$) *be pairwise different blocks of* L *and let* a_i ($i = 0, 1, 2$) *be pairwise different atoms of* L *such that* $a_i \in B_i \cap B_{i+1}$ ($i = 0, 1, 2$). *Then* $B_0 \cap B_2 = B_0 \cap B_3 = B_1 \cap B_3 = \{0, 1\}$.

Proof. If $B_0 \cap B_2 \neq \{0, 1\}$, then (B_0, B_1, B_2) would be a loop of order 3 contradicting (2.8). Hence $B_0 \cap B_2 = \{0, 1\}$ and, by symmetry, $B_1 \cap B_3 = \{0, 1\}$. If $B_0 \cap B_3 \neq \{0, 1\}$, then (B_0, B_1, B_2, B_3) would be a loop of order 4, again contradicting (2.8). Hence $B_0 \cap B_3 = \{0, 1\}$.

We consider the following condition:

(α) No element $a \neq 0, 1$ in L is contained in at least three blocks.

(2.10) *Assume that* L *satisfies* (α), *let* $(C_i)_{0 \leqq i \leqq 4}$ *be a loop in* L, *and let* $a \in C_3 \cap C_4$ *and* $b \in C_1 - (C_0 \cup C_2)$ *be atoms of* L. *Then* $r(a, b) = 2$.

(2.11) *Assume that* L *satisfies* (α), *let* $(C_i)_{0 \leqq i \leqq 5}$ *be a loop in* L, *and let* $a \in C_3 \cap C_4$ *and* $b \in C_0 \cap C_1$ *be atoms of* L. *Then* $r(a, b) = 2$.

Proofs of (2.10) *and* (2.11). Clearly $r(a, b) \leqq 2$. Since $C_1 \cap C_3 = \{0, 1\}$ it follows from (2.4) that $r(a, b) \geqq 1$. Assume $r(a, b) = 1$. Then there exists a path (C, D) with $a \in C$ and $b \in D$. By (α) one has $C = C_3$ or $C = C_4$. By symmetry we may assume $C = C_3$. Since $C \cap D \neq \{0, 1\}$ there exists an atom $e \in C_3$ with $r(e, b) = 0$. For the atoms $c \in C_1 \cap C_2$ and $d \in C_2 \cap C_3$ one also has $r(c, d) = 0$. Since $b \neq c$ this contradicts (2.4). It follows that $r(a, b) = 2$.

(2.12) *Let* L *satisfy* (α) *and assume that there are no elements* e, f, g *in* L *satisfying* $r(e, f) \geqq 2$, $r(e, g) \geqq 2$, *and* $r(f, g) \geqq 2$. *Let* $(C_i)_{0 \leqq i \leqq 4}$ *be a loop in a* L. *Then* $\bigcup_{i=0}^{4} C_i$ *is a subalgebra of* L.

Proof. Assume that $\bigcup_{i=0}^{4} C_i$ is not a subalgebra of L. Then, by symmetry and duality we may assume that there exist atoms $b \in C_0 - C_1$ and $c \in C_2 - (C_1 \cup C_3)$ such that $b \vee c \notin \bigcup_{i=0}^{4} C_i$. But $b \in C_4$, by (2.10), would imply $r(b, c) = 2$ which, by (2.2), would give $b \vee c = 1 \in \bigcup_{i=0}^{4} C_i$. Hence we may also assume $b \in C_0 - (C_1 \cup C_4)$. From (2.2) and (2.4) we obtain $r(b, c) = 1$. Let C be the block containing b and $b \vee c$ and let D be the block containing c and $b \vee c$. Using (2.9) it is easy to check that (C_0, C, D, C_2, C_1) and $(C_0, C, D, C_2, C_3, C_4)$ are loops in L. Pick atoms e, f, g such that $e \in C_3 \cap C_4, f \in C_1 - (C_0 \cup C_2)$, and $g \in C \cap D$. Then (2.10) and (2.11) give $r(e, f) = r(e, g) = r(f, g) = 2$, contradicting the assumptions of (2.12). Hence $\bigcup_{i=0}^{4} C_i$ is a subalgebra of L.

(2.13) *Let* L *satisfy* (α) *and let* (C_1, C_2, C_3) *be a path in* L *such that* $\bigcup_{i=1}^{3} C_i$ *is not a subalgebra of* L. *Then there exists a loop* $(C_0, C_1, C_2, C_3, C_4)$ *in* L.

Proof. Because of (α) and the fact that L does not contain a loop of order 3, it is $C_1 \cap C_3 = \{0, 1\}$. Since $C_1 \cup C_2$ and $C_2 \cup C_3$ are subalgebras of L there exist atoms $c \in C_1 - C_2$ and $d \in C_3 - C_2$ with $c \vee d \notin C_1 \cup C_2 \cup C_3$. From (2.2) and (2.4) it follows that $r(c, d) = 1$. Let C_0 be the block containing c and $c \vee d$ and let C_4 be the block containing d and $c \vee d$. It is obvious that $(C_0, C_1, C_2, C_3, C_4)$ is a loop.

(2.14) *Let* L *be an* OML *which is not contained in* [MO2] *and satisfies* $n(L) \leqq 2^3$. *Then one of the* OML*s of Figures* 3, 4, 5 *is a subalgebra of* L *or* MO3 *is the homomorphic image of a subalgebra of* L.

Proof. We may assume w.l.o.g. that L is subdirectly irreducible and that $n(L) = 2^3$. If condition (α) is not satisfied then there exists a co-atom e in L which is contained in at least three blocks. Then MO3 is a subalgebra of $[0, e]$, hence the homomorphic image of a subalgebra of L. We assume now that MO3 is not a homomorphic image of a subalgebra of L. Then, as we have just seen, (α) is satisfied and by (2.6) there are no elements a, b, c in L satisfying

$r(a, b) \geqq 2$, $r(a, c) \geqq 2$, and $r(b, c) \geqq 2$. If L is not connected there exist a block A and an atom $d \in L$ with $r(A, d) \geqq 3$ and by (2.5) the OML of Figure 3 is a subalgebra of L. If L is connected, then, by (1.10), there exists a path (C_1, C_2, C_3) in L. Since L satisfies (α) we have $C_1 \cap C_3 = \{0, 1\}$. If $C_1 \cup C_2 \cup C_3$ is a subalgebra of L then this subalgebra is isomorphic with the OML of Figure 4. If it is not a subalgebra of L, then, by (2.13), there exists a loop $(C_i)_{0 \leqq i \leqq 4}$ in L. By (2.12), $\bigcup_{4=0}^{i} C_i$ is a subalgebra of L and this subalgebra is obviously isomorphic with the OML of Figure 5. This proves (2.14).

3. The general case.

(3.1) *Let L be subdirectly irreducible, homogeneous, connected and assume $n(L) \geqq 2^4$. Then there exists an element $d \in L$ such that $n([0, d]) = 2^3$ and such that $[0, d]$ is subdirectly irreducible.*

Proof. Let $(B_0, B_1, \ldots B_n)$ be a path in L with the properties stated in (1.10). By (1.9) we have $B_1 \cap [0, e_0] = B_{n-1} \cap [0, e_0]$ which implies $e_0 \in B_{n-1}$. Since $e_{n-1} \in B_{n-1}$, $e_0 \neq e_{n-1}$ and $e_0 \nleq e'_{n-1}$, $a = e_0 \wedge e_{n-1}$ is an atom in B_{n-1}. Let b, c be the atoms of B_{n-1} for which $a \vee b = e_0$ and $a \vee c = e_{n-1}$. Put $d = e_0 \vee e_{n-1} = a \vee b \vee c$. Since a, b, c are pairwise different and belong to B_{n-1} we obtain from (1.8) that $n([0, d]) = 2^3$. From $B_1 \cap [0, e_0] = B_{n-1} \cap [0, e_0]$ it follows that $a, b \in B_1$. But (1.9) also implies that $B_{n-1} \cap [0, e_{n-1}] = B_1 \cap [0, e_{n-1}]$, which gives $c \in B_1$ and hence $d \in B_1$. Since by the dual of (1.2) we have $B_0 \cap [e_0, 1] = B_1 \cap [e_0, 1]$, it follows that $d \in B_0$. By the same argument we obtain $d \in B_n$. It follows that $B_0 \cap [0, d]$ and $B_n \cap [0, d]$ are blocks of $[0, d]$. We show that $B_0 \cap B_n \cap [0, d] = \{0, d\}$ which proves that $[0, d]$ is directly and hence subdirectly irreducible.

Since $a < e_0$, $a \in B_1$, it is $a \notin B_0$. Since $c \nleq e_0$, hence $c \leqq e_0'$, it is $c \in B_0$. Since $e_{n-1} \in B_0$ would imply $a = (a \vee c) \wedge c' = e_{n-1} \wedge c' \in B_0$, it is $e_{n-1} \notin B_0$. Let a_1, a_2 be the atoms of B_n which satisfy $a_1 \vee a_2 = e_{n-1}$. Then $a_1 \vee c = a_2 \vee c = e_{n-1}$, which gives $a_1, a_2 \notin B_0$. But b is an atom of $B_n \cap [0, d]$ different from a_1, a_2. Consequently, a_1, a_2, b are the three atoms of $B_n \cap [0, d]$. Since $b \in B_1$ and $b < e_0$ we have $b \notin B_0$, i.e. none of the atoms of $B_n \cap [0, d]$ belongs to $B_0 \cap [0, d]$. This means that $B_0 \cap B_n \cap [0, d] = \{0, d\}$, which was to be proved.

(3.2) *Let L be subdirectly irreducible and $n(L) \geqq 2^4$. Then the* OML *of Figure* 3 *is a subalgebra of L or there exists $e \in L - \{1\}$ such that $[0, e]$ is subdirectly irreducible and $n([0, e]) \geqq 2^3$.*

Proof. Assume first that L is not connected. Then there exist blocks A, B in L with $r(A, B) = \infty$. If one of the blocks A, B, say B, consists of four elements only, then $r(C, B) = \infty$ holds for every block $C \neq B$ in L. Since $n(L) \geqq 2^4$ it follows that we may assume that A has at least eight elements. Let S be an eight-element subalgebra of A. We claim that $a \vee b = 1$ and (dually) $a \wedge b = 0$

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holds for all $a \in S - \{0, 1\}$, $b \in B - \{0, 1\}$. Assume $a \vee b \neq 1$ holds for at least one such pair of elements a, b. Then there exist blocks D, E such that $a, a \vee b \in D$ and $b, a \vee b \in E$. Since $A \cap D \neq \{0, 1\}$, $D \cap E \neq \{0, 1\}$, and $E \cap B \neq \{0, 1\}$, repeated application of (1.7) yields the existence of a path connecting A and B, which contradicts $r(A, B) = \infty$. If d is an arbitrary element of $B - \{0, 1\}$ it is now easily seen that $S \cup \{d, d'\}$ is a subalgebra of L isomorphic with the OML of Figure 3.

Assume next that L is not homogeneous. Then there exist blocks C, D in L which meet in a section $\mathrm{S}(e)$ where $e \neq 0, 1$ is neither an atom nor a hyperatom of L. It follows $n([0, e]) \geqq 2^3$. Since $C \cap [0, e]$ and $D \cap [0, e]$ are blocks in $[0, e]$ and $C \cap D \cap [0, e] = \{0, e\}$, the OML $[0, e]$ is subdirectly irreducible.

If, finally, L is homogeneous and connected the claim follows from (3.1).

We are now in the position to give the

Proof of the theorem. It is enough to prove the theorem for subdirectly irreducible L. Statement (2.14) gives the theorem if $n(L) \leqq 2^3$ and (3.2) allows an obvious induction on $n(L)$.

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