J. Aust. Math. Soc. **85** (2008), 305–322
doi:10.1017/S1446788708000839

INVOLUTION GRAPHS WHERE THE PRODUCT OF TWO ADJACENT VERTICES HAS ORDER THREE

ALICE DEVILLERS and MICHAEL GIUDICI ✉

(Received 27 November 2007; accepted 20 May 2008)

Communicated by Peter M. Neumann

Dedicated to Cheryl Praeger for her sixtieth birthday

Abstract

An S_3-involution graph for a group G is a graph with vertex set a union of conjugacy classes of involutions of G such that two involutions are adjacent if they generate an S_3-subgroup in a particular set of conjugacy classes. We investigate such graphs in general and also for the case where $G = \mathrm{PSL}(2, q)$.

2000 *Mathematics subject classification*: 20B25, 05C25.
Keywords and phrases: graphs, groups, involutions.

1. Introduction

In [11] an interesting tower of graphs was discovered and investigated that is associated with the subgroup chain

$$A_5 < \mathrm{PSL}(2, 11) < M_{11} < M_{12}.$$

The smallest graph in the tower is the line graph of the Petersen graph, while the largest graph is the Johnson graph $J(12, 4)$. (The *Johnson graph* $J(n, k)$ is the graph with vertices the k-subsets of an n-set such that two k-subsets are adjacent if their intersection has size $k - 1$.) The graphs associated with $\mathrm{PSL}(2, 11)$ and M_{11} are related to the Witt designs on 11 and 12 points. A uniform description of the graphs in the tower was achieved via involutions and S_3-subgroups of the groups in the subgroup chain. This leads to the following definition.

DEFINITION 1.1. Let G be a group with a nonempty set X of involutions closed under conjugation and a nonempty set $\mathcal{S}$ of S_3-subgroups also closed under conjugation. The S_3-*involution graph* $\Gamma(G, X, \mathcal{S})$ of G with respect to X and $\mathcal{S}$ is the graph with vertices the elements of X such that two vertices x, y are adjacent if and only if $\langle x, y\rangle \in \mathcal{S}$. In order to avoid degeneracies, we always require that X is the set of all involutions contained in elements of $\mathcal{S}$.

The tower of graphs is then given by a series of S_3-involution graphs for A_5, $\mathrm{PSL}(2, 11)$, M_{11} and M_{12}, where, for each group, both X and $\mathcal{S}$ are single conjugacy classes. The existence of this tower suggests the following natural problem.

PROBLEM 1.2. Investigate S_3-involution graphs for other families of groups.

This paper is intended as an initial investigation of S_3-involution graphs in general and we also investigate the S_3-involution graphs arising from the simple groups $\mathrm{PSL}(2, q)$, for $q \geq 4$.

In Section 2, after describing some examples of S_3-involution graphs, we investigate automorphisms, connectivity and triangles. Given an S_3-involution graph $\Gamma(G, X, \mathcal{S})$, the three involutions of each $S \in \mathcal{S}$ give rise to a triangle in the graph. The following gives a sufficient condition for these to be the only triangles. See Section 2.3 for a discussion about the converse.

THEOREM 1.3. *Let G be a finite group with conjugacy class X of involutions and union of conjugacy classes $\mathcal{S}$ of S_3-subgroups. If G has no subgroups of the form $C_3^2 \rtimes C_2$ or $C_p^2 \rtimes S_3$ for some prime p, then the only triangles of $\Gamma(G, X, \mathcal{S})$ are those given by subgroups in $\mathcal{S}$.*

In Section 3 we analyse the S_3-involution graphs for $\mathrm{PSL}(2, q)$. In particular we determine the full automorphism groups (Theorems 3.9 and 3.11) and show that there is a duality with the graph induced on S_3-triangles if and only if $q = 11$ and 13 (Theorem 3.8). We also give the following determination of the size of the largest cliques.

THEOREM 1.4. *Let $G = \mathrm{PSL}(2, q)$ for $q \geq 4$, let X be the unique conjugacy class of involutions in G and let $\mathcal{S}$ be a conjugacy class of S_3-subgroups. The size of the largest clique is 3^e if $q = 9^e$, 4 if $q = 25^e$ and 3 otherwise.*

The definition of an S_3-involution graph is reminiscent of Fischer's 3-*transposition groups*, that is, groups generated by a conjugacy class X of involutions such that any pair of noncommuting elements of X generates an S_3. The elements of X are called 3-*transpositions*. Fischer's investigation of such groups [13, 14] led to the discovery of three new sporadic simple groups. If G is a 3-transposition group with class X of 3-transpositions and $\mathcal{S}$ is the set of all S_3-subgroups generated by a pair of noncommuting 3-transpositions, the S_3-involution graph $\Gamma(G, X, \mathcal{S})$ is called the *diagram* of X and was used in [10, 13, 16] in the study of 3-transposition groups. In fact, a 3-transposition group with class X of 3-transpositions is a quotient of the Coxeter group with Coxeter diagram the diagram of X. We are interested in S_3-involution graphs for arbitrary groups. Indeed, the groups A_5, $\mathrm{PSL}(2, 11)$, M_{11} and M_{12} are not 3-transposition groups.

https://doi.org/10.1017/S1446788708000839 Published online by Cambridge University Press

Given a 3-transposition group G with a class X of 3-transpositions and $\mathcal{S}$ the set of all S_3-subgroups generated by pairs of elements of X, one can construct a partial linear space known as a *Fischer space*, whose points are the elements of X and lines are the sets of three involutions contained in an S_3 in $\mathcal{S}$. Moreover, Fischer spaces are precisely the partial linear spaces such that each plane is either an affine plane over GF(3) or a dual affine plane over GF(2) (see [9, 10]). The S_3-involution graph $\Gamma(G, X, \mathcal{S})$ is the collinearity graph of the Fischer space.

Another graph with vertex set a conjugacy class of 3-transpositions was also used in the investigation of 3-tranposition groups. Given a group G and conjugacy class X of involutions, the *commuting involution graph* $\mathcal{C}(G, X)$ is the graph with vertices the elements of X such that two vertices are adjacent if they commute. If G is a 3-transposition group such that X is a class of 3-transpositions and $\mathcal{S}$ is the set of all S_3-subgroups generated by pairs of elements of X, then $\mathcal{C}(G, X)$ is the complement of $\Gamma(G, X, \mathcal{S})$. Commuting involution graphs for groups other than 3-transposition groups have recently been studied in [2–4].

2. General theory

We begin this section with a few simple examples.

EXAMPLE 2.1. Let $G = S_n$, X the conjugacy class of transpositions and $\mathcal{S}$ the conjugacy class of S_3-subgroups generated by two transpositions. Note that X is a class of 3-transpositions. The map from X to the set of 2-subsets of an n-set that maps each transposition x to the set of two points moved by x yields a one-to-one correspondence between X and the vertex set of $J(n, 2)$. Moreover, two transpositions generate an S_3 if and only if their 2-cycles have a unique point in common. Thus $\Gamma(G, X, \mathcal{S}) \cong J(n, 2)$.

EXAMPLE 2.2. Let V be a $(2n)$-dimensional vector space over GF(2) equipped with an alternating form $(.,.)$. Let $G = \mathrm{Sp}(2n, 2)$ be the group of all linear transformations of V that preserve $(.,.)$ and let X be the set of all transvections contained in G, that is all maps

$$\begin{array}{rccl} t_v : & V & \to & V \\ & x & \mapsto & x + (x, v)v, \end{array}$$

where v is a nonzero vector of V. Calculations show that if $(v, u) = 0$ then $t_v t_u$ has order two, otherwise $t_u t_v$ has order three and $t_u^{t_v} = t_{u+v}$. Thus X is a class of 3-transpositions for G. Letting $\mathcal{S}$ be the set of all S_3-subgroups of G generated by pairs of noncommuting elements of X, it follows that $\Gamma(G, X, \mathcal{S})$ is isomorphic to the graph with vertices the nonzero vectors of V such that two vertices u, v are adjacent if and only if $(v, u) = 1$, that is if and only if they lie in a hyperbolic line. Moreover, the third vector of that line corresponds to the third involution of $\langle t_u, t_v \rangle$. Hence the Fischer space of G is the partial linear space whose points are the nonzero vectors of V and whose lines are the hyperbolic lines of V. This space is called a symplectic copolar space over GF(2) and is denoted by $\overline{W(2n-1, 2)}$ or $\overline{\mathrm{Sp}(2n, 2)}$.

https://doi.org/10.1017/S1446788708000839 Published online by Cambridge University Press

Similar graphs can be constructed from the other classical groups which are also 3-transposition groups.

EXAMPLE 2.3. Let $G = M_{11}$. Then G has a unique conjugacy class X of involutions and two classes of S_3-subgroups. Now $|X| = 165$ and there exists a bijection from X to the set of 3-subsets of an 11-set where each involution is mapped to its set of fixed points. Also G has two conjugacy classes $\mathcal{S}_1$, $\mathcal{S}_2$ of S_3-subgroups with $|\mathcal{S}_1| = 220$ and $|\mathcal{S}_2| = 660$.

The graph $\Gamma(G, X, \mathcal{S}_1)$ is the graph associated with M_{11} in the tower investigated in [11]. By [11, Theorem 1.1] it has valency 8 and M_{11} as its full automorphism group. Moreover, $\Gamma(G, X, \mathcal{S}_1)$ is isomorphic to the graph with vertex set the set of 3-subsets of an 11-set such that two vertices are adjacent if they are disjoint and the complement of their union is a pentad in the Witt design on 11 points associated with M_{11}.

Each S_3-subgroup in the class $\mathcal{S}_2$ fixes two points of an 11-set. Thus if two involutions are adjacent in $\Gamma(G, X, \mathcal{S}_2)$ then they have two fixed points in common, that is, their sets of fixed points are adjacent in $J(11, 3)$. Each involution of G is contained in twelve S_3-subgroups of $\mathcal{S}_2$ and so is adjacent to 24 involutions in $\Gamma(G, X, \mathcal{S}_2)$. This is the valency of $J(11, 3)$ and so $\Gamma(G, X, \mathcal{S}_2) \cong J(11, 3)$.

EXAMPLE 2.4. Let $G = \mathrm{AGL}(1, 3^n)$ for some positive integer n. Then G is the group of all maps

$$\begin{array}{rccc} t_{a,b}: & \mathrm{GF}(3^n) & \to & \mathrm{GF}(3^n) \\ & x & \mapsto & ax + b \end{array}$$

for any $a, b \in \mathrm{GF}(3^n)$ with $a \neq 0$. Let $X = \{t_{-1,b} \mid b \in \mathrm{GF}(3^n)\}$, the unique conjugacy class of involutions of G. Let x be the involution $t_{-1,0}$. Note that any involution $t_{-1,b} = t_{1,-b} t_{-1,0}$ is of the form hx, where h is an element of order 3. Now G contains $(3^n - 1)/(3 - 1)$ subgroups of order three and each element of order 3 is inverted by any involution in X. Thus G has $3^n(3^n - 1)/6$ subgroups isomorphic to S_3. Moreover, G acts transitively by conjugation on the set of subgroups of order three, while, given $h, h_1, h_2 \in G$ of order 3, the S_3-subgroups $\langle h, h_1 x\rangle$ and $\langle h, h_2 x\rangle$ are conjugate under the element $h_1^{-1} h_2$. Thus G has a unique conjugacy class $\mathcal{S}$ of S_3-subgroups. Moreover, given two distinct involutions $x_1, x_2 \in X$, we have that $x_1 x_2$ has order three. Hence $\Gamma(G, X, \mathcal{S})$ is the complete graph K_{3^n} on 3^n vertices.

In fact the following theorem shows that the only complete graphs that occur as S_3-involution graphs are those on 3^n vertices.

THEOREM 2.5. *Let G be a finite group with X a conjugacy class of involutions and $\mathcal{S}$ a union of conjugacy classes of S_3-subgroups. If $\Gamma(G, X, \mathcal{S})$ is the complete graph on X, then $|X| = 3^n$ for some positive integer n. Moreover, for each positive integer n there exists a group G with an S_3-involution graph isomorphic to K_{3^n}.*

PROOF. Suppose $\Gamma(G, X, \mathcal{S})$ is a complete graph. Then for all $x, y \in X$, xy has order three. Thus $\langle X\rangle$ is a 3-transposition group and by [1, (8.6)], $\langle X\rangle = N \rtimes \langle x\rangle$ for $x \in X$ and N a 3-subgroup. Thus $|x^{\langle X\rangle}|$ is a power of 3. Moreover, by Sylow's theorem,

$X = x^{\langle X\rangle}$ and so the first part follows. Example 2.4 provides the required examples for the second part. □

Our next two examples arise from subgroups of $\mathrm{AGL}(1, 3^n)$.

EXAMPLE 2.6. Let n be an even positive integer and G be the index-two subgroup of $\mathrm{AGL}(1, 3^n)$ given by all maps

$$\begin{array}{rccc} t_{a,b}: & \mathrm{GF}(3^n) & \to & \mathrm{GF}(3^n) \\ & x & \mapsto & ax+b \end{array}$$

for any $a, b \in \mathrm{GF}(3^n)$ with a a nonzero square. This group is isomorphic to $C_3^n \rtimes C_{(3^n-1)/2}$ and is a point stabilizer in $\mathrm{PSL}(2, 3^n)$. Then G still has a unique conjugacy class $X = \{t_{-1,b} \mid b \in \mathrm{GF}(3^n)\}$ of involutions. However, G has two conjugacy classes of S_3-subgroups: those for which the elements of order three are of the form $t_{1,a}$ with a a nonzero square; and those for which the elements of order three are of the form $t_{1,a}$ with a a nonsquare. Let $\mathcal{S}_1$ be the first class. Then $\langle t_{-1,b_1}, t_{-1,b_2}\rangle \in \mathcal{S}_1$ if and only if $b_2 - b_1$ is a nonzero square. Since X is in one-to-one correspondence with the elements of $\mathrm{GF}(3^n)$, it follows that $\Gamma(G, X, \mathcal{S}_1)$ is isomorphic to the graph with vertices the elements of $\mathrm{GF}(3^n)$ such that two elements are adjacent if and only if their difference is a square. (Note that $3^n \equiv 1 \bmod 4$ and so this relation is symmetric.) Thus $\Gamma(G, X, \mathcal{S}_1)$ is the Paley graph of $\mathrm{GF}(3^n)$.

EXAMPLE 2.7. Let $G = C_3^2 \rtimes C_2$, where elements of order two in G invert each element of order three. Then G has a unique conjugacy class X of involutions and four conjugacy classes $\mathcal{S}_i$ $(i = 1, 2, 3, 4)$ of S_3-subgroups, where each element from the same conjugacy class shares a common C_3-subgroup. For each i, $\Gamma(G, X, \mathcal{S}_i) \cong 3K_3$, while if $\mathcal{S} = \bigcup_i \mathcal{S}_i$ then $\Gamma(G, X, \mathcal{S}) \cong K_9$.

2.1. Automorphisms Given an S_3-involution graph $\Gamma(G, X, \mathcal{S})$, the group G acts on the set of vertices by conjugation and preserves adjacency. The following lemma collects some information about the action of G.

LEMMA 2.8. *Let G be a group with a set X of involutions closed under conjugation and a set $\mathcal{S}$ of S_3-subgroups closed under conjugation. Let $\Gamma = \Gamma(G, X, \mathcal{S})$.*

(1) *The orbits of G on the set of vertices are the conjugacy classes in X.*
(2) *The kernel of the action of G on the set of vertices is $C_G(\langle X\rangle)$.*
(3) *The orbits of G on the set of arcs are $\{(x, y) \mid \langle x, y\rangle \in \mathcal{S}_i\}$ for each conjugacy class $\mathcal{S}_i \subseteq \mathcal{S}$.*

PROOF. Part (1) is trivial. Part (2) follows as an element lies in the kernel if and only if it centralizes each element of X and hence of $\langle X\rangle$. Part (3) follows from the fact that two arcs lie in the same G-orbit if and only if the S_3-subgroups generated by their vertices are conjugate. □

The next lemma gives a natural way to find extra automorphisms of S_3-involution graphs.

https://doi.org/10.1017/S1446788708000839 Published online by Cambridge University Press

LEMMA 2.9. *Let G be a group with conjugacy classes X_1, X_2 of involutions and $\mathcal{S}_1, \mathcal{S}_2$ of S_3-subgroups. If there exists $g \in \mathrm{Aut}(G)$ such that $X_1^g = X_2$ and $\mathcal{S}_1^g = \mathcal{S}_2$, then $\Gamma(G, X_1, \mathcal{S}_1) \cong \Gamma(G, X_2, \mathcal{S}_2)$.*

PROOF. The element g provides the isomorphism. □

COROLLARY 2.10. *If $g \in \mathrm{Aut}(G)$ fixes X_1 and $\mathcal{S}_1$ setwise, then g induces an automorphism of $\Gamma(G, X_1, \mathcal{S}_1)$.*

By [8, Theorem 9.1.2], if $n \geq 5$ then $\mathrm{Aut}(J(n, 2)) = S_n$ and so the graphs in Example 2.1 give examples where the full automorphism group of $\Gamma(G, X, \mathcal{S})$ is G. The line graph of the Petersen graph is the S_3-involution graph of A_5 and has full automorphism group S_5. This provides examples of automorphism provided by Corollary 2.10. We saw in Example 2.3 that M_{11} has an S_3-involution graph isomorphic to $J(11, 3)$. Thus its full automorphism group is S_{11}, which is much bigger than G. Similarly, M_{12} has an S_3-involution graph isomorphic to $J(12, 4)$ whose full automorphism group is S_{12}.

2.2. Connectivity We begin with the following lemma.

LEMMA 2.11. *Let $\Gamma = (G, X, \mathcal{S})$ be an S_3-involution graph. If $X = \dot{\bigcup} X_i$ with each X_i a G-conjugacy class, then Γ is the vertex disjoint union of the graphs $\Gamma(G, X_i, \mathcal{S})$.*

PROOF. Suppose that $x, y \in X$ lie in the same connected component of Γ. Then there exists a path $x = x_0, x_1, \ldots, x_d = y$ in Γ. For each $i = 0, \ldots, d-1$, we have $\langle x_i, x_{i+1}\rangle \cong S_3$ and so x_i is conjugate to x_{i+1}. Hence x is conjugate to y and the result follows. □

Note that if X is a single conjugacy class then $\Gamma(G, X, \mathcal{S})$ is not necessarily connected. For example, let $G = S_3 \operatorname{wr} S_2$ and let

$$X = \{(x, 1), (1, y) \mid x, y \in S_3, o(x) = o(y) = 2\},$$

a conjugacy class of involutions. If $\mathcal{S} = \{S_3 \times 1, 1 \times S_3\}$ then $\Gamma(G, X, \mathcal{S})$ consists of two disjoint triangles. (Note that in this example $\langle X\rangle \neq G$ and X is not an $\langle X\rangle$-conjugacy class.)

We have the following lemma.

LEMMA 2.12. *Let $N \lhd G$ and let $X \subset N$. Then $\Gamma(G, X, \mathcal{S}) = \Gamma(N, X, \mathcal{S})$. In particular, $\Gamma(G, X, \mathcal{S}) = \Gamma(\langle X\rangle, X, \mathcal{S})$.*

PROOF. Since the involutions of elements of $\mathcal{S}$ lie in X and hence N, it follows that for $S \in \mathcal{S}$ we have $S \leqslant N$. □

Lemma 2.12 leads to the following necessary condition for connectedness.

LEMMA 2.13. *If $\Gamma(G, X, \mathcal{S})$ is connected, then X is both a G-conjugacy class and an $\langle X\rangle$-conjugacy class.*

https://doi.org/10.1017/S1446788708000839 Published online by Cambridge University Press

PROOF. By Lemma 2.12, $\Gamma(G, X, \mathcal{S}) = \Gamma(\langle X\rangle, X, \mathcal{S})$. By Lemma 2.11, if $\Gamma(G, X, \mathcal{S})$ is connected, then X is a G-conjugacy class; while if $\Gamma(\langle X\rangle, X, \mathcal{S})$ is connected, then X is an $\langle X\rangle$-conjugacy class. □

Example 2.7 is an example where X is both a G-conjugacy class and an $\langle X\rangle$-conjugacy class but $\Gamma(G, X, \mathcal{S}_i)$ is still disconnected.

To obtain a necessary and sufficient condition for an S_3-involution graph $\Gamma(G, X, \mathcal{S})$ to be connected with X a single conjugacy class, we need to introduce a general method for constructing vertex-transitive graphs (see for example [20]).

Let G be a group with subgroup H and D a union of double cosets of H in G which is closed under inverses. We can define a graph $\mathcal{G}(G, H, D)$ with vertices the right cosets of H in G such that Hg_1 is adjacent to Hg_2 if and only if $g_1g_2^{-1} \in D$. Then G acts vertex-transitively on $\mathcal{G}(G, H, D)$ by right multiplication preserving adjacency. Moreover, $\mathcal{G}(G, H, D)$ is connected if and only if D generates G (see [20, Theorem 7]). Conversely, given a G-vertex-transitive graph Γ with arbitrary vertex v, $\Gamma \cong \mathcal{G}(G, H, D)$, where $H = G_v$ and D is the set of all elements of G that map v to a vertex adjacent to v (see [20, Theorem 1]).

LEMMA 2.14. *Let X be a conjugacy class of involutions in G and let $\mathcal{S} = \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_t$ be a union of t conjugacy classes of S_3-subgroups of G such that involutions in elements of $\mathcal{S}$ are contained in X. Let $x \in X$ and, for each i, let $y_i \in X$ such that $\langle x, y_i\rangle \in \mathcal{S}_i$. Then $\Gamma(G, X, \mathcal{S})$ is connected if and only if $\langle C_G(x), y_1, \ldots, y_t\rangle = G$.*

PROOF. Let $\Gamma = \Gamma(G, X, \mathcal{S})$ and let $H = C_G(x)$. For each i, $\{x, y_i, y_ixy_i\}$ forms a triangle in Γ and y_i maps x to the adjacent vertex y_ixy_i. Moreover, the double coset Hy_iH is the set of all elements of G mapping x to a neighbour u of x such that $\langle x, u\rangle \in \mathcal{S}_i$. Thus letting $D = Hy_1H \cup \cdots \cup Hy_tH$, [20, Theorem 1] implies that $\Gamma \cong \mathcal{G}(G, H, D)$. Hence by [20, Theorem 7], Γ is connected if and only if D generates G. Since $\langle D\rangle = \langle C_G(x), y_1, \ldots, y_t\rangle$ the result follows. □

2.3. Triangles Let $S \in \mathcal{S}$ and $T(S) = \{x, y, z\}$ be the set of three involutions in S. Then $T(S)$ is a triangle in $\Gamma(G, X, \mathcal{S})$. In particular, note that all S_3-involution graphs have girth three.

Given a graph Γ and partition $\mathcal{P}$ of the edge-set of Γ, we say that $(\Gamma, \mathcal{P})$ is a *G-arc-symmetrical decomposition* if G preserves $\mathcal{P}$, $G^{\mathcal{P}}$ is transitive, G acts transitively on the set of arcs of Γ and, for $P \in \mathcal{P}$, G_P is transitive on the set of arcs of P. Arc-symmetrical decompositions were introduced in [15].

LEMMA 2.15. *Let G be a group with conjugacy class X of involutions and conjugacy class $\mathcal{S}$ of S_3-subgroups containing elements of X. Let $\Gamma = \Gamma(G, X, \mathcal{S})$ and $\mathcal{P} = \{T(S) \mid S \in \mathcal{S}\}$. Then $(\Gamma, \mathcal{P})$ is a G-arc-symmetrical decomposition.*

PROOF. Each edge of Γ lies in a unique triangle $T(S)$, $S \in \mathcal{S}$, and so $\mathcal{P}$ is a partition of $E\Gamma$ preserved by G. Since $\mathcal{S}$ is a G-conjugacy class, $G^{\mathcal{P}}$ is transitive. Moreover, given $P \in \mathcal{P}$, $G_P^P \cong S_3$, which is transitive on the six arcs of P. Hence $(\Gamma, \mathcal{P})$ is a G-arc-symmetrical decomposition. □

https://doi.org/10.1017/S1446788708000839 Published online by Cambridge University Press

An S_3-involution graph may or may not contain triangles other than those of the form $T(S)$ for $S \in \mathcal{S}$. The following theorem determines what subgroups arise in G if $\Gamma(G, X, \mathcal{S})$ has extra triangles.

THEOREM 2.16. *Let G be a finite group with conjugacy class X of involutions and union of conjugacy classes $\mathcal{S}$ of S_3-subgroups. If $\{x, y, z\}$ is a triangle in $\Gamma(G, X, \mathcal{S})$, then $\langle x, y, z\rangle \cong C_n^2 \rtimes S_3$ or $(C_{3n} \times C_n) \rtimes S_3$ for some $n \geq 1$.*

PROOF. Let $x, y \in S$ be involutions and hence $\{x, y\}$ is an edge of $\Gamma = \Gamma(G, X, \mathcal{S})$. Suppose that $z \in X$ such that $\{x, y, z\}$ is a triangle of Γ not obtained by a subgroup in $\mathcal{S}$ and let $R = \langle x, y, z\rangle$. Note that $\langle x, y\rangle < R$ and R satisfies the relations $x^2 = y^2 = z^2 = (xy)^3 = (xz)^3 = (yz)^3 = 1$, and hence is a finite quotient of the affine Coxeter group $\tilde{A}_2$, which is isomorphic to $\mathbb{Z}^2 \rtimes S_3$. We model this Coxeter group by the group $H = L \rtimes W$ where $L = \{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid x_1 + x_2 + x_3 = 0\}$ and $W \cong S_3$ acts on L by naturally permuting the coordinates. Note that L is generated by $(1, -1, 0)$ and $(0, -1, 1)$. The normal subgroups of H are determined in [21, Theorem 0.2] and are either subgroups of L or kernels of homomorphisms from H to a Coxeter group induced by a homomorphism from the Coxeter graph of H. As the Coxeter graph for H is a triangle, the only possible images of such homomorphisms have Coxeter graph a single edge or a single vertex and so the quotients obtained are C_2 or S_3. Since R contains the proper subgroup $\langle x, y\rangle \cong S_3$, it follows that R is the quotient of H by a proper subgroup of L. By [21, Proposition 7.2], the normal subgroups contained in L are integral multiples of one of the lattices $\Lambda_1 = L + \mathbb{Z}\omega$ and $\Lambda_3 = 3L + \mathbb{Z}\omega$, where $\omega = (1, 1, -2) \in L$. Since $\omega \in L$ we have $\Lambda_1 = L$ and so nontrivial integral multiples of Λ_1 give the quotients $C_n^2 \rtimes S_3$ for some integer $n \geq 2$. Now $L/\Lambda_3 = \langle (1, -1, 0) + \Lambda_3\rangle \cong C_3$. Let $W = \langle \sigma, \tau\rangle$ with $\sigma^3 = \tau^2 = 1$ such that $(1, -1, 0)^\sigma = (0, 1, -1)$ and $(1, -1, 0)^\tau = (-1, 1, 0)$. Then

$$((1, -1, 0) + \Lambda_3)^\sigma = (1, -1, 0) + \Lambda_3 \quad \text{and}$$
$$((1, -1, 0) + \Lambda_3)^\tau = (2, -2, 0) + \Lambda_3.$$

Hence $H/\Lambda_3 \cong C_3^2 \rtimes C_2$. For $n \geq 2$, we have

$$L/(n\Lambda_3) = \langle (1, -1, 0) + n\Lambda_3, (1, 1, -2) + n\Lambda_3\rangle \cong C_{3n} \times C_n$$

and hence $H/(n\Lambda_3) \cong (C_{3n} \times C_n) \rtimes S_3$. □

We can now prove Theorem 1.3.

PROOF OF THEOREM 1.3. By Theorem 2.16, if G does not contain any subgroups of the form $C_n^2 \rtimes S_3$ for $n \geq 2$, or $(C_{3n} \times C_n) \rtimes S_3$ for some $n \geq 1$, then the only triangles in $\Gamma(G, X, \mathcal{S})$ are those arising from subgroups in $\mathcal{S}$. Note that if p is a prime dividing n then $C_n^2 \rtimes S_3$ contains $C_p^2 \rtimes S_3$. Also $R = (C_{3n} \times C_n) \rtimes S_3$ contains a subgroup $C_3^2 \rtimes C_2$ seen as follows. Using the notation from the proof of Theorem 1.3, if $R = H/(n\Lambda_3)$ then R contains the subgroup $\langle (n, -n, 0) + n\Lambda_3\rangle \cong C_3$, which is normalized by W and centralized by σ. Hence $H/(n\Lambda_3)$ contains a subgroup isomorphic to $C_3^2 \rtimes C_2$. □

https://doi.org/10.1017/S1446788708000839 Published online by Cambridge University Press

TABLE 1. PSL(2, q) information.

q mod 12	$\|X\|$	$C_G(x)$, $x \in X$	# of classes of S_3	$\|\mathcal{S}\|$
4, 8	$q^2 - 1$	C_2^r	1	$\|G\|/6$
1	$q(q+1)/2$	D_{q-1}	2	$\|G\|/12$
3	$q(q-1)/2$	D_{q+1}	0	
5	$q(q+1)/2$	D_{q-1}	1	$\|G\|/6$
7	$q(q-1)/2$	D_{q+1}	1	$\|G\|/6$
11	$q(q-1)/2$	D_{q+1}	2	$\|G\|/12$
9	$q(q+1)/2$	D_{q-1}	2	$\|G\|/6$

The converse of Theorem 1.3 is not true. For example, we saw in Example 2.7 that if $G = C_3^2 \rtimes C_2$ then G has four conjugacy classes of S_3-subgroups and, for each of the classes $\mathcal{S}_i$, $\Gamma(G, X, \mathcal{S}_i) \cong 3K_3$ with the three triangles arising from the three subgroups in $\mathcal{S}_i$. The problem is that if, in the proof of Theorem 1.3, $R = C_3^2 \rtimes C_2$ then not all S_3-subgroups of R are R-conjugate and so are not necessarily contained in $\mathcal{S}$. The situation is similar, if $R = C_n^2 \rtimes S_3$ with 3 dividing n. On the other hand, if $(3, n) = 1$ then all S_3-subgroups of $R = C_n^2 \rtimes S_3$ are conjugate in R and hence contained in $\mathcal{S}$, so in this case we definitely obtain extra triangles.

3. PSL(2, q) graphs

In this section we investigate the S_3-involution graphs for PSL(2, q). Note that PSL(2, 2) $\cong S_3$ while PSL(2, 3) $\cong A_4$, which does not contain any S_3-subgroups.

Table 1 collates information about the involutions and S_3-subgroups of PSL(2, q). This mostly follows from a theorem of Dickson [12, pp. 285–286] (and see [17, Theorem 2.1]). When there are two conjugacy classes of S_3-subgroups, the two classes are fused in PGL(2, q). Lemma 3.1 determines the conjugacy classes of S_3-subgroups when q is a power of 3.

LEMMA 3.1. *Let $G = \mathrm{PSL}(2, 3^r)$. Then the number and sizes of conjugacy classes of S_3-subgroups is given by the last two columns of Table 1. In particular, G contains S_3-subgroups if and only if r is even.*

PROOF. Let $q = 3^r$ and consider an S_3-subgroup S, which contains a C_3-subgroup C. Since the projective line contains $q + 1$ points and a C_3 cannot fix four points, it follows that C must be contained in a unique point stabilizer $P \cong C_3^r \rtimes C_{(q-1)/2}$, where P is the index-two subgroup of AGL(1, 3^r) given in Example 2.6. Since S normalizes C, it fixes the unique fixed point of C and hence $S \leqslant P$. Thus $(q-1)/2$ is even and so r is even. We saw in Example 2.6 that P has two conjugacy classes of S_3-subgroups and hence so does G. □

First we show that the S_3-involution graphs for PSL(2, q) are always connected.

https://doi.org/10.1017/S1446788708000839 Published online by Cambridge University Press

TABLE 2. Valencies of $\Gamma(\mathrm{PSL}(2, q), X, \mathcal{S})$.

q mod 12	Valency
4, 8	q
1	$(q-1)/2$
5	$q-1$
7	$q+1$
11	$(q+1)/2$
9	$q-1$

LEMMA 3.2. *Let $G = \mathrm{PSL}(2, q)$, X the set of involutions in G and $\mathcal{S}$ a conjugacy class of S_3-subgroups. Then $\Gamma(G, X, \mathcal{S})$ is connected.*

PROOF. By Lemma 2.14, we need to prove that in each case $\langle C_G(x), y\rangle = G$ for some $x, y \in X$ such that $\langle x, y\rangle \in \mathcal{S}$.

Suppose first that $q \equiv 1 \bmod 4$. Then $C_G(x) \cong D_{q-1}$, which is maximal in G for $q \geq 13$. Since $y \notin C_G(x)$, we are finished if $q \geq 13$. If $G = \mathrm{PSL}(2, 5)$, then $C_G(x) \cong C_2^2$ is contained only in maximal subgroups isomorphic to A_4. However, C_2^2 contains all involutions of the A_4-subgroup, and so $y \notin A_4$. Hence $\langle C_G(x), y\rangle = G$. If $G = \mathrm{PSL}(2, 9)$, then $C_G(x) \cong D_8$ is contained only in maximal subgroups isomorphic to S_4. Looking at the permutation representation on four points, we see that a central involution in a D_8 cannot be in an S_3-subgroup of S_4. Therefore $y \notin S_4$ and so $\langle C_G(x), y\rangle = G$.

Next suppose that $q \equiv 3 \bmod 4$. Then $C_G(x) \cong D_{q+1}$, which is maximal in G for $q \geq 11$, and so $y \notin C_G(x)$ and we are finished. If $G = \mathrm{PSL}(2, 7)$, then $C_G(x) \cong D_8$ is contained only in maximal subgroups isomorphic to S_4 and we can conclude as above.

Finally, suppose $q = 2^r$. Then $C_G(x) \cong C_2^r$, which lies in a unique maximal subgroup $H \cong C_2^r \rtimes C_{q-1}$. If $y \in H$, then y lies in the unique Sylow 2-subgroup of H, and so commutes with x. Therefore $y \notin H$ and hence $\langle C_G(x), y\rangle = G$. □

THEOREM 3.3. *Let $G = \mathrm{PSL}(2, q)$, X the set of involutions in G and $\mathcal{S}$ a conjugacy class of S_3-subgroups. Then the valency of $\Gamma(G, X, \mathcal{S})$ is given by Table 2 according to the value of q mod 12. Moreover if there are two conjugacy classes of S_3-subgroups in G, the corresponding graphs are isomorphic.*

PROOF. Knowing the number $|X|$ of involutions, and hence of vertices, and the number of S_3-subgroups in a conjugacy class $\mathcal{S}$, that is, a third of the number of edges, it is immediate to deduce the valency of $\Gamma(G, X, \mathcal{S})$. If there are two conjugacy classes of S_3-subgroups, the two classes are fused in $\mathrm{PGL}(2, q)$ and hence by Lemma 2.9 the corresponding graphs are isomorphic. □

3.1. Cliques First we analyse triangles.

LEMMA 3.4. *Let* $G = \mathrm{PSL}(2, q)$ *with* q *even or* $q \equiv \pm 3 \bmod 8$. *Then the only triangles in* $\Gamma(G, X, \mathcal{S})$ *are those arising from the elements of* $\mathcal{S}$.

PROOF. If q is a power of 3 and $q \equiv \pm 3 \bmod 8$ then q is an odd power of 3 and so G contains no S_3-subgroups. For the values of q given in the statement, we can see from [12, pp. 285–286] that G does not contain any subgroups of the form $C_3^2 \rtimes C_2$ or $C_p^2 \rtimes S_3$. Hence by Theorem 1.3, the only triangles in $\Gamma(G, X, \mathcal{S})$ are those arising from the elements of $\mathcal{S}$. $\square$

COROLLARY 3.5. *Let* $G = \mathrm{PSL}(2, q)$ *with* q *even or* $q \equiv \pm 3 \bmod 8$. *Then each edge of* $\Gamma(G, X, \mathcal{S})$ *lies in a unique triangle and the size of the largest clique is three.*

For the values of q where Γ contains triangles other than the natural ones, it is obvious to ask what is the size of the largest clique.

THEOREM 3.6. *Let* $G = \mathrm{PSL}(2, 9^e)$, X *the unique conjugacy class of involutions in* G *and* $\mathcal{S}$ *a conjugacy class of* S_3*-subgroups of* G. *Then the size of the largest clique in* $\Gamma(G, X, \mathcal{S})$ *is* 3^e.

PROOF. Let $\{x, y\}$ be an edge of $\Gamma = \Gamma(G, X, \mathcal{S})$. Let $\{x, y, z\}$ be a triangle such that $\langle x, y, z\rangle \cong S_3$ and suppose that $\{x, y, u\}$ is another triangle. By Theorem 2.16 and the subgroup structure of G, $\langle x, y, u\rangle \cong C_2^2 \rtimes S_3 \cong S_4$ or $\langle x, y, u\rangle \cong C_3^2 \rtimes C_2$. Indeed, either $\langle x, y, u\rangle$ is isomorphic to $C_n^2 \rtimes S_3$, and the only such possibility in G is for $n = 2$, or it contains an abelian subgroup $C_{3n} \times C_n$, and the only such possibility in G is for $n = 1$.

Since G has two conjugacy classes of S_4-subgroups [17] and two conjugacy classes of S_3-subgroups, it follows that all S_4-subgroups of G containing $\langle x, y\rangle$ are conjugate. There are $|G|/6$ conjugates of $\langle x, y\rangle$ and $|G|/24$ conjugates in each class of S_4-subgroups. Thus $\langle x, y\rangle$ lies in a unique S_4-subgroup. The edge $\{x, y\}$ lies in the two triangles $\{x, y, z\}$ and $\{x, y, u_1\}$ where $\langle x, y, u_1\rangle \cong S_4$. Under this isomorphism we can make the identifications $x = (1, 2)$, $y = (1, 3)$, $z = (2, 3)$ and $u_1 = (1, 4)$. Hence z is not adjacent to u_1 and so we do not obtain a clique of size four in this way.

Now $\langle x, y\rangle$ is contained in a unique parabolic subgroup $P \cong C_3^{2e} \rtimes C_{(9^e-1)/2}$ and if $\langle x, y, u\rangle \cong C_3^2 \rtimes C_2$ then $\langle x, y, u\rangle \leqslant P$. Now P is isomorphic to the index-two subgroup of $\mathrm{AGL}(1, 3^{2e})$ as in Example 2.6. Moreover, the two conjugacy classes of S_3-subgroups of P remain separate conjugacy classes in G. Thus the restriction of Γ to the involutions of P is the Paley graph of $\mathrm{GF}(9^e)$. By [7] the largest clique in the Paley graph of $\mathrm{GF}(9^e)$ has size 3^e. Moreover, [5] proved that such cliques are affine images of subfields $\mathrm{GF}(3^e)$. Thus a clique in P of size 3^e containing $x = t_{-1,a}$ and $y = t_{-1,b}$ corresponds to $(b - a)\mathrm{GF}(3^e) + a$ containing a, b and $-b - a$ (with notation for elements as in Example 2.6). Therefore $z = x^y = t_{-1,-a-b}$ also lies in this clique. Since z is not adjacent to u_1 we cannot make the clique larger by adding u_1. Thus the largest clique size of Γ is 3^e. $\square$

https://doi.org/10.1017/S1446788708000839 Published online by Cambridge University Press

THEOREM 3.7. *Let $G = \mathrm{PSL}(2, q)$ with $q \equiv \pm 1 \bmod 8$ not a power of 3. Then the size of the largest clique in $\Gamma(G, X, \mathcal{S})$ is four if $q = 25^e$ and three otherwise.*

PROOF. Assume that $\Gamma(G, X, \mathcal{S})$ contains a clique $\{x, y, z, t\}$ of size four. Let $S = \langle x, y\rangle \cong S_3$. Then S contains only one more involution, so we may assume without loss of generality that $\{x, y, t\}$ is a triangle not generating an S_3. By Theorem 2.16 and the subgroup structure of G it follows that $\langle x, y, t\rangle \cong S_4$. Note that G has two classes of S_4-subgroups each of length $|G|/24$ (see [12]) and has $|G|/6$ S_3-subgroups (in one or two conjugacy classes). An easy counting argument shows that S is contained in two S_4-subgroups B_1 and B_2. Obviously, all S_3-subgroups in a given S_4 are conjugate. Without loss of generality, we may assume that $\langle x, y, t\rangle = B_1$ and that the only other involution of B_1 adjacent to x and y is x^y. Moreover, t is not adjacent to x^y and so we must have that $\langle x, y, z\rangle = B_2$. Since z, t and x^y are the only elements of X adjacent to both x and y, we conclude that $\{x, y, z, t\}$ is a maximal clique. By symmetry, any three involutions in this clique generate an S_4-subgroup.

Let us look at the relations in $\langle x, y, z, t\rangle$. Because of the clique structure,

$$1 = x^2 = y^2 = z^2 = t^2 = (xy)^3 = (xz)^3 = (yz)^3 = (xt)^3 = (yt)^3 = (zt)^3.$$

Since x, y, z are three involutions generating an S_4 with pairs generating various S_3, we also have $1 = (xyxz)^2$ and the relations obtained from this one by permuting the three letters. Of course we have similar relations for any 3-subset of $\{x, y, z, t\}$. Now let $x' = x$, $y' = x^y$, $z' = y^z$ and $t' = z^t$. It is easily seen that $\langle x, y, z, t\rangle = \langle x', y', z', t'\rangle$. It is also easily proved from the relations described above that

$$1 = x'^2 = y'^2 = z'^2 = t'^2 = (x'y')^3 = (y'z')^3 = (z't')^3 = (z'x')^2 = (t'x')^2 = (y't')^2.$$

These relations yield a Coxeter group of type A_4, and so we have $\langle x, y, z, t\rangle = \langle x', y', z', t'\rangle \cong S_5$. Now $\mathrm{PSL}(2, q)$ contains a subgroup isomorphic to S_5 if and only if $q = 25^e$ (note that $\mathrm{PGL}(2, 5) \cong S_5$) and if G contains an S_5 then it does indeed have a clique of size four. Thus the size of the largest clique in $\Gamma(G, X, \mathcal{S})$ is four if $q = 25^e$ and three otherwise. □

Theorem 1.4 follows from Corollary 3.5 and Theorems 3.6 and 3.7.

3.2. Duality The *dual graph* of an S_3-involution graph $\Gamma(G, X, \mathcal{S})$ is the graph whose vertices are the S_3-triangles of $\Gamma(G, X, \mathcal{S})$ (that is, which correspond to elements of $\mathcal{S}$), with two triangles being adjacent if they share a vertex. It was seen in [11] that the graph $\Gamma(\mathrm{PSL}(2, 11), X, \mathcal{S})$, with X a conjugacy class of involutions and $\mathcal{S}$ a conjugacy class of S_3-subgroups, is isomorphic to its dual graph, with the duality between X and $\mathcal{S}$ induced by elements of $\mathrm{PGL}(2, 11) \setminus \mathrm{PSL}(2, 11)$. We now show that the only other value of q for which this happens is $q = 13$. Note that by Corollary 3.5 in both cases the only triangles in $\Gamma(G, X, \mathcal{S})$ are S_3-triangles.

THEOREM 3.8. *Let $G = \mathrm{PSL}(2, q)$, X the unique conjugacy class of involutions and $\mathcal{S}$ a conjugacy class of S_3-subgroups. Then $\Gamma(G, X, \mathcal{S})$ is isomorphic to its dual graph if and only if $q = 11$ or 13.*

PROOF. For $\Gamma(G, X, \mathcal{S})$ to have a duality between X and $\mathcal{S}$, the number of vertices must equal the number of S_3-subgroups in $\mathcal{S}$. It follows from Table 1 that $q = 11$ or 13. It remains to prove that in these two cases we do indeed have a duality.

When $q = 11$ or 13, the group G has two conjugacy classes $\mathcal{S}$ and $\mathcal{S}'$ of S_3-subgroups and $\Gamma(G, X, \mathcal{S}) \cong \Gamma(G, X, \mathcal{S}')$. By Theorem 3.3, $\Gamma(G, X, \mathcal{S})$ has valency six, which means that each involution is contained in three subgroups of $\mathcal{S}$ and in three subgroups of $\mathcal{S}'$. The centralizer in G of an involution x is isomorphic to D_{12} and each D_{12}-subgroup contains a unique S_3-subgroup in each conjugacy class. We will denote the unique subgroup of $\mathcal{S}$ (respectively $\mathcal{S}'$) in $C_G(x)$ by $s(x)$ (respectively $s'(x)$). Moreover, each subgroup $S \in \mathcal{S} \cup \mathcal{S}'$ is contained in a unique subgroup isomorphic to D_{12}, whose central involution will be denoted by $i(S)$. Notice that s and i are inverse bijections between involutions and elements of $\mathcal{S}$ and s' and i are inverse bijections between involutions and elements of $\mathcal{S}'$.

We claim that for $x \in X$, $T \in \mathcal{S}$, we have $x \in T$ if and only if x and $i(T)$ commute and $T \cap s'(x) = 1$.

First suppose that $x \in T$. Then $i(T)$ commutes with all elements in the D_{12} containing T and hence with x. We have $C_T(x) = \langle x \rangle$ and $x \notin s'(x) \subset C_G(x)$, and therefore $T \cap s'(x) = 1$.

Conversely, suppose that x and $i(T)$ commute, $T \cap s'(x) = 1$, and that $x \notin T$. If $x = i(T)$, then $T = s(x)$, and $s(x) \cap s'(x) \cong C_3$, a contradiction. Hence $x \neq i(T)$ and x is one of the six noncentral involutions in $C_G(i(T))$. Since $x \notin T = s(i(T))$, we have $x \in s'(i(T))$ and there exists a unique involution $t \in T$ commuting with x. Since $T \cap s'(x) = 1$ it follows that $t \in s(x)$. Now $s(x)$ and T are both in $\mathcal{S}$ and the normalizer of an element in $\mathcal{S}$ acts transitively on its three involutions. Thus there exists $g \in G$ mapping $s(x)$ to T and fixing t. Hence $g \in C_G(t)$ and g must map x onto $i(T)$. Moreover, x and $i(T)$ are two commuting involutions of $C_G(t)$ other than t, which implies that one is in $s(t)$ and the other in $s'(t)$. In other words, they are not conjugate in $C_G(t)$, and we get a contradiction. Hence the claim is proved.

Of course, by symmetry, we also have that a vertex $x \in T' \in \mathcal{S}'$ if and only if x and $i(T')$ commute and $T' \cap s(x) = 1$.

Consider the map $s' : X \to \mathcal{S}'$. By the claim, the vertices x and y are adjacent in $\Gamma(G, X, \mathcal{S})$ if and only if there exists $T \in \mathcal{S}$ such that $i(T)$ commutes with both x and y, and $T \cap s'(x) = T \cap s'(y) = 1$. Since $x = i(s'(x))$, $y = i(s'(y))$ and $s(i(T)) = T$, using the claim again implies that x and y are adjacent if and only if the involution $i(T)$ is contained in both $s'(x)$ and $s'(y)$. Therefore s' yields an isomorphism from $\Gamma(G, X, \mathcal{S})$ onto the dual graph of $\Gamma(G, X, \mathcal{S}')$. Since $\Gamma(G, X, \mathcal{S}) \cong \Gamma(G, X, \mathcal{S}')$, it follows that $\Gamma(G, X, \mathcal{S})$ is isomorphic to its dual graph. □

We note that the proof given in [11] for the duality in the case where $q = 11$ is different from the one here. It relies on a geometrical description of the graph.

3.3. Automorphism groups We now need more details on the different projective groups. The group $\mathrm{PGL}(2, q)$ is the group of all fractional linear transformations

$$t_{a,b,c,d} : z \mapsto \frac{az+b}{cz+d}, \quad ad - bc \neq 0,$$

of the projective line $L = \{\infty\} \cup \mathrm{GF}(q)$ with the conventions that $1/0 = \infty$ and $(a\infty + b)/(c\infty + d) = a/c$. Note that $t_{a,b,c,d} = t_{a',b',c',d'}$ if and only if

$$(a, b, c, d) = \lambda(a', b', c', d') \quad \text{for some } \lambda \neq 0.$$

The group $\mathrm{PSL}(2, q)$ is then the set of all $t_{a,b,c,d}$ such that $ad - bc$ is a square in $\mathrm{GF}(q)$. The Frobenius map $\phi : z \mapsto z^p$ also acts on L and $\phi^{-1} t_{a,b,c,d} \phi = t_{a^p,b^p,c^p,d^p}$. Then $\mathrm{P\Gamma L}(2, q) = \langle \mathrm{PGL}(2, q), \phi \rangle$ and $\mathrm{P\Sigma L}(2, q) = \langle \mathrm{PSL}(2, q), \phi \rangle$.

We split the determination of the full automorphism group into the cases where q is even and odd.

THEOREM 3.9. *Let $G = \mathrm{PSL}(2, q)$ with q odd, X the set of involutions in G, $\mathcal{S}$ a conjugacy class of S_3-subgroups, and let $\Gamma = \Gamma(G, X, \mathcal{S})$. If $q \equiv 1, 9, 11 \bmod 12$, then* $\mathrm{Aut}(\Gamma) = \mathrm{P\Sigma L}(2, q)$*. If $q \equiv 5, 7 \bmod 12$, then* $\mathrm{Aut}(\Gamma) = \mathrm{P\Gamma L}(2, q)$.

PROOF. Let $A = \mathrm{Aut}(\Gamma)$ and note that $G \leqslant A \leqslant \mathrm{Sym}(V\Gamma)$. Using MAGMA [6] the result can be verified for $q = 5, 7, 9$ and 11. Thus we may assume that $q \geq 13$. For $q \equiv \pm 1 \bmod 4$, we have that the stabilizer in G of a vertex is $D_{q\mp 1}$ and $|V\Gamma| = q(q \pm 1)/2$. Moreover, by Theorem 3.3, Γ has valency $q \mp 1$ or $(q \mp 1)/2$ and so $\mathrm{Alt}(V\Gamma)$ is not contained in A. Since $q \geq 13$, the subgroups D_{q-1} and D_{q+1} are maximal in G (see [12]) and hence both G and A act primitively on $V\Gamma$. Moreover, G and A share a common nontrivial orbital given by the edges of Γ. Using [19, Theorem 1], we conclude that either $\mathrm{soc}(A) = \mathrm{PSL}(2, q)$ or $q \equiv 1 \bmod 4$ and $\mathrm{soc}(A) = A_{q+1}$. In this last case, A acts on $V\Gamma$ as on 2-sets of a $(q+1)$-set. Therefore, the stabilizer in A of a vertex has orbit sizes 1, $2(q-1)$ and $\binom{q-1}{2}$ on vertices. Since Γ has valency $q - 1$ or $(q-1)/2$, we conclude that $\mathrm{soc}(A) \neq A_{q+1}$. Hence $\mathrm{soc}(A) = \mathrm{PSL}(2, q)$ and so $G \leqslant A \leqslant \mathrm{P\Gamma L}(2, q)$.

When $q \equiv 5, 7 \bmod 12$, Table 1 states that G has a unique conjugacy class of S_3-subgroups. Thus $\mathrm{P\Gamma L}(2, q)$ fixes X and $\mathcal{S}$ and so, by Corollary 2.10, $A = \mathrm{P\Gamma L}(2, q)$.

For $q \equiv 1, 9, 11 \bmod 12$, Table 1 states that G has two conjugacy classes $\mathcal{S}$ and $\mathcal{S}'$ of S_3-subgroups. These two classes are fused by any $g \in \mathrm{PGL}(2, q) \setminus \mathrm{PSL}(2, q)$ and so such elements do not induce automorphisms of Γ. Thus if $q = p$ then we have $A = G = \mathrm{P\Sigma L}(2, q)$. When $q = p^f$ with $f \geq 2$ and $p \neq 3$, by Table 1, the $\mathrm{PSL}(2, p)$-subgroup centralized by ϕ contains an S_3 and so ϕ fixes $\mathcal{S}$ and $\mathcal{S}'$. Thus again $A = \mathrm{P\Sigma L}(2, q)$. Finally when $p = 3$, by Lemma 3.1, f is even and so ϕ centralizes a $\mathrm{PGL}(2, 3)$-subgroup that contains an S_3. Hence ϕ fixes $\mathcal{S}$ and $\mathcal{S}'$ and so $A = \mathrm{P\Sigma L}(2, q)$. □

To deal with the q even case, we need the following lemma about the structure of the graph.

https://doi.org/10.1017/S1446788708000839 Published online by Cambridge University Press

LEMMA 3.10. *Let $G = \mathrm{PSL}(2, q)$ for $q = 2^e \geq 4$, X be the unique conjugacy class of involutions of G and $\mathcal{S}$ the unique class of S_3-subgroups. Let $\Gamma = \Gamma(G, X, \mathcal{S})$. Then X can be partitioned into $q + 1$ blocks $X_\alpha = X \cap G_\alpha$ of size $q - 1$ ($\alpha \in \mathrm{GF}(q) \cup \{\infty\}$), such that the subgraph induced by Γ on any two blocks is a matching between those blocks.*

PROOF. Let $\alpha, \beta \in \mathrm{GF}(q) \cup \{\infty\}$ with $\alpha \neq \beta$. Then $G_\alpha \cong \mathrm{AGL}(1, q) \cong C_2^e \rtimes C_{q-1}$ and $G_{\alpha,\beta} \cong C_{q-1}$. Thus each element of X lies in a unique point stabilizer of G. Moreover, since each G_α has a unique Sylow 2-subgroup and this subgroup is abelian, all involutions in X_α commute. Thus Γ is a multipartite graph with block set $\mathcal{P} = \{X_\alpha \mid \alpha \in \mathrm{GF}(q) \cup \{\infty\}\}$. Now $G_{X_\alpha} = G_\alpha$ and $G_{\alpha,\beta} = G_{X_\alpha, X_\beta} \cong C_{q-1}$ acts regularly on both X_α and X_β. Since G acts arc-transitively on Γ (Lemma 2.8), it follows that each vertex in X_α is adjacent to at most one vertex in any other block. Using the fact from Theorem 3.3 that Γ has valency q, we can conclude that the subgraph induced on $X_\alpha \cup X_\beta$ is a matching. □

THEOREM 3.11. *Let $G = \mathrm{PSL}(2, q)$ for $q \geq 4$ even, X the unique conjugacy class of involutions of G and $\mathcal{S}$ the unique class of S_3-subgroups. Let $\Gamma = \Gamma(G, X, \mathcal{S})$. Then* $\mathrm{Aut}(\Gamma) = \mathrm{P\Gamma L}(2, q)$.

PROOF. Let $A = \mathrm{Aut}(\Gamma)$. Since G has a unique conjugacy class of involutions and of S_3-subgroups it follows from Corollary 2.10 that $\mathrm{P\Gamma L}(2, q) \leqslant A$. By Lemma 3.10, Γ is a multipartite graph with block set $\mathcal{P} = \{X_\alpha \mid \alpha \in \mathrm{GF}(q) \cup \{\infty\}\}$, where $X_\alpha = X \cap G_\alpha$. Let $x \in X_\infty$. By Lemma 3.10, for each $\alpha \neq \infty$, $|\Gamma(x) \cap X_\alpha| = 1$. By Corollary 3.5, each edge of Γ lies in a unique triangle. Hence, A_x preserves a partition of $\mathcal{P}$ into blocks of size two given by the triangles containing x.

Let $y \in X_0$ be the unique element of X_0 adjacent to x and let $\{x, y, z\}$ be a triangle. Then $z \in X_\alpha$ for some $\alpha \in \mathrm{GF}(q) \backslash \{0\}$. Note that $G_{X_\infty, X_0, X_\alpha} = 1$. Now let $x' \in X_\infty$ with $x' \neq x$ and let $y' \neq y$ be the unique element of X_0 adjacent to x'. Since G is arc-transitive, there exists $g \in G$ such that $(x, y)^g = (x', y')$. Moreover, $\{x', y', z^g\}$ is the unique triangle of Γ containing $\{x', y'\}$. Since $G_{X_\infty, X_0, X_\alpha} = 1$ and g fixes X_∞ and X_0, it follows that $z^g \notin X_\infty \cup X_0 \cup X_\alpha$. Hence if $h \in A_{X_\infty, X_0, X_\alpha}$ then $h \in A_x$. Since x was arbitrary, if K is the kernel of the action of A on $\mathcal{P}$, it follows that K fixes X_∞ pointwise. Since $K \lhd A$ and A acts transitively on $\mathcal{P}$ we conclude that $K = 1$, that is, A acts faithfully on $\mathcal{P}$. Thus $\mathrm{P\Gamma L}(2, q) \leqslant A \leqslant S_{q+1}$. By [18] it follows that either $\mathrm{soc}(A) = \mathrm{PSL}(2, q)$ or $A_{q+1} \leqslant A$. If $q = 4$ then $\mathrm{PSL}(2, 4) = A_5$ and $\mathrm{P\Gamma L}(2, 4) = S_5$. Hence $\mathrm{Aut}(\Gamma) = \mathrm{P\Gamma L}(2, 4)$ in this case. Suppose now that $q \geq 8$ and $A_{q+1} \leqslant A$. Then there exists $h \in A$ that induces a 3-cycle on $\mathcal{P}$ and fixes X_∞, X_0 and X_α. As we have seen, this implies that $h \in A_x$ and so preserves a partition of $\mathcal{P}$ into blocks of size two, contradicting h inducing a 3-cycle on $\mathcal{P}$. Hence A_{q+1} is not contained in A and so $\mathrm{soc}(A) = \mathrm{PSL}(2, q)$. As $\mathrm{P\Gamma L}(2, q) \leqslant A$ it follows that $A = \mathrm{P\Gamma L}(2, q)$. □

Lemma 3.10 also enables us to determine the diameter of the S_3-involution graph of $\mathrm{PSL}(2, q)$ for q even.

THEOREM 3.12. *Let $G = \mathrm{PSL}(2, q)$ for $q \geq 4$ even, X the unique conjugacy class of involutions of G and $\mathcal{S}$ the unique class of S_3-subgroups. Let $\Gamma = \Gamma(G, X, \mathcal{S})$. Then Γ has diameter three.*

PROOF. For each $\alpha \in \mathrm{GF}(q) \cup \{\infty\}$ let $X_\alpha = X \cap G_\alpha$. By Lemma 3.10, Γ is multipartite with blocks $\{X_\alpha \mid \alpha \in \mathrm{GF}(q) \cup \{\infty\}\}$ such that the graph induced between any two blocks is a complete matching. Let $x \in X$. Without loss of generality we may assume $x \in X_\infty$. By Theorem 3.3, x has q neighbours and these are each in a different X_α, $\alpha \in \mathrm{GF}(q)$. By Corollary 3.5, each of these q vertices is adjacent to exactly one other neighbour of x. Hence each is adjacent to $q - 2$ vertices at distance 2 from x, and, by Lemma 3.10, none of these are in X_∞. Moreover, we claim that Γ contains no 4-cycle, and so there are exactly $q(q - 2)$ vertices at distance 2 from x. Therefore, the vertices at distance at most 2 from x cover exactly the involutions not in X_∞. Since the subgraph induced on two blocks of the partition is a matching, all vertices distinct from x in X_∞ are at distance 3 from x. Therefore any vertex is at distance at most 3 from x.

It remains to prove that there is no 4-cycle in Γ. Suppose that (x, y, z, t) is a 4-cycle. By Lemma 3.10, the four vertices are in four distinct blocks of the partition. Since G has one orbit on $V\Gamma$ (Lemma 2.8), we may assume that $x = t_{1,1,0,1} \in X_\infty$. A calculation shows that the unique involution in X_i ($i \in \mathrm{GF}(q)$) adjacent to x is $t_{i+1,i^2,1,i+1}$. Moreover, for $j \in \mathrm{GF}(q) \setminus \{i\}$, the image of the edge $\{t_{1,1,0,1}, t_{0,1,1,0}\}$ under

$$t_{i^2+ij,i^2+ij+j,i+j,i+j+1} \in G$$

is

$$\{t_{i+1,i^2,1,i+1}, t_{i^2+j^2+j,j^2,1,i^2+j^2+j}\}.$$

Thus the unique involution in X_j ($j \in \mathrm{GF}(q) \setminus \{i\}$) adjacent to $t_{i+1,i^2,1,i+1}$ is $t_{i^2+j^2+j,j^2,1,i^2+j^2+j}$. Hence

$$y = t_{i+1,i^2,1,i+1} \quad \text{and} \quad t = t_{i'+1,i'^2,1,i'+1}$$

for distinct $i, i' \in \mathrm{GF}(q)$. We know that $z \in G_j$ for some $j \in \mathrm{GF}(q) \setminus \{i, i'\}$. Since z is adjacent to both y and t, we have

$$z = t_{i^2+j^2+j,j^2,1,i^2+j^2+j} = t_{i'^2+j^2+j,j^2,1,i'^2+j^2+j}.$$

Hence $i^2 + j^2 + j = i'^2 + j^2 + j$, and so $i = i'$, which contradicts the fact that y and t are distinct. □

Acknowledgements

We thank Jean Doyen, Bernhard Mühlherr and Tim Pentilla for their helpful conversations. We are also grateful to the referee whose comments greatly improved the paper.

The first author is a Collaborateur Scientifique FRS–FNRS (Belgium). The second author holds an Australian Research Council Australian Research Fellowship. This paper was initiated during a visit to UWA by the first author, which was supported by the FNRS.

References

[1] M. Aschbacher, *3-Transposition Groups*, Cambridge Tracts in Mathematics, 124 (Cambridge University Press, Cambridge, 1997).

[2] C. Bates, D. Bundy, S. Hart and P. Rowley, 'Commuting involution graphs for sporadic simple groups', *J. Algebra* **316** (2007), 849–868.

[3] C. Bates, D. Bundy, S. Perkins and P. Rowley, 'Commuting involution graphs in special linear groups', *Comm. Algebra* **32** (2004), 4179–4196.

[4] ———, 'Commuting involution graphs for symmetric groups', *J. Algebra* **266** (2003), 133–153.

[5] A. Blokhuis, 'On subsets of GF(q^2) with square differences', *Indag. Math.* **46** (1984), 369–372.

[6] W. Bosma, J. Cannon and C. Playoust, 'The magma algebra system I: the user language', *J. Symbolic. Comput.* **24**(3/4) (1997), 235–265 (also see the MAGMA home page at http://www.maths.usyd.edu.au:8000/u/magma/).

[7] I. Broere, D. Döman and J. N. Ridley, 'The clique numbers and chromatic numbers of certain Paley graphs', *Quaest. Math.* **11** (1988), 91–93.

[8] A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance-regular Graphs*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 18 (Springer, Berlin, 1989).

[9] F. Buekenhout, *La géométrie des groupes de Fischer* (Université Libre de Bruxelles, 1972), unpublished notes.

[10] H. Cuypers and J. I. Hall, 'The 3-transposition groups with trivial centre', *J. Algebra* **178** (1995), 149–193.

[11] A. Devillers, M. Giudici, C. H. Li and C. E. Praeger, Some graphs related to the small Mathieu groups, submitted.

[12] L. E. Dickson, *Linear Groups with an Exposition of the Galois Field Theory* (Dover, New York, 1958).

[13] B. Fischer, 'Finite groups generated by 3-transpositions. I', *Invent. Math.* **13** (1971), 232–246.

[14] ———, 'Finite groups generated by 3-transpositions', Lecture Notes, University of Warwick, 1969.

[15] M. Giudici, C. H. Li and C. E. Praeger, 'Symmetrical covers, decompositions and factorisations of graphs', in: *Applications to Group Theory and Combinatorics* (eds. J. Koolen, J. H. Kwak and M. Xu) (Taylor and Francis, London, 2008), pp. 27–42.

[16] J. I. Hall, 'The general theory of 3-transposition groups', *Math. Proc. Cambridge Philos. Soc.* **114** (1993), 269–294.

[17] O. H. King, 'The subgroup structure of finite classical groups in terms of geometric configurations', in: *Surveys in Combinatorics 2005*, London Mathematical Society Lecture Note Series, 327 (Cambridge University Press, Cambridge, 2005), pp. 29–56.

[18] M. W. Liebeck, C. E. Praeger and J. Saxl, 'A classification of the maximal subgroups of the finite alternating and symmetric groups', *J. Algebra* **111** (1987), 365–383.

[19] ———, 'Primitive permutation groups with a common suborbit, and edge-transitive graphs', *Proc. London Math. Soc.* (3) **84** (2002), 405–438.

[20] P. Lorimer, 'Vertex-transitive graphs: symmetric graphs of prime valency', *J. Graph Theory* **8** (1984), 55–68.

[21] G. Maxwell, 'The normal subgroups of finite and affine Coxeter groups', *Proc. London Math. Soc.* (3) **76** (1998), 359–382.

ALICE DEVILLERS, Université Libre de Bruxelles, Département de Mathématiques, Géométrie – CP 216, Boulevard du Triomphe, B-1050 Bruxelles, Belgium
e-mail: adevil@ulb.ac.be

MICHAEL GIUDICI, School of Mathematics and Statistics, The University of Western Australia, 35 Stirling Highway, Crawley, WA 6009, Australia
e-mail: giudici@maths.uwa.edu.au