ON THE STABILITY OF BARRELLED TOPOLOGIES II

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1. Introduction. If E is a Hausdorff barrelled space, which does not already have its finest locally convex topology, then the continuous dual E' may be enlarged within the algebraic dual E^*. Robertson and Yeomans [**10**] have recently investigated whether E can retain the barrelled property under such enlargements. Whereas finite-dimensional enlargements of the dual preserve barrelledness, they have shown that this is not always so for countable-dimensional enlargements $E'+M$. In fact, if E contains an infinite-dimensional bounded set, there always exists a countable-dimensional M for which the Mackey topology $\tau(E, E'+M)$ is not barrelled [**10**, Theorem 2].

Here, in Theorem 1 we give a construction which shows that for a large class of barrelled spaces E, there exists a countable-dimensional M with $E'\cap M=\{0\}$, for which $\tau(E, E'+M)$ does remain barrelled. Combining this with Theorem 2 of [**10**], we then see that if E contains a sufficiently large bounded set, there exist two countable-dimensional subspaces M_1, M_2 of E^* such that $E'\cap M_1=\{0\}=E'\cap M_2$ and $\tau(E, E'+M_1)$ is barrelled, while $\tau(E, E'+M_2)$ is not. Our Theorem 1 applies in several important special cases, for example Fréchet spaces, barrelled normed spaces and spaces of continuous functions.

The method of construction of M imposes a condition on the bounded sets of $E'+M$, namely that for each such set there is a finite-dimensional enlargement of E' which contains it. In the last section we show this condition is necessary whenever $\tau(E, E'+M)$ is barrelled, and therefore fundamental to any approach to finding such an M.

Generally we adopt the topological vector space notation of [**9**]. We use $\mathbf{K}$ to denote the scalars $\mathbf{R}$ or $\mathbf{C}$, and c for the cardinality of the continuum. The term *countable* is reserved for the infinite case only.

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2. Existence and construction of M.

THEOREM 1. *Let E be a Hausdorff barrelled space with a bounded subset which spans a subspace of dimension $\geq c$. Then there is a countable-dimensional subspace M of E^* such that $E'\cap M=\{0\}$ and $\tau(E, E'+M)$ is barrelled.*

Proof. It is clear from the hypothesis that E contains a bounded, absolutely convex subset A which spans a subspace G of dimension c. The gauge of A then defines on G a norm topology $\tau(G, G')$.

Since G and $\mathbf{K}^{\mathbf{N}}$ both have dimension c, there is an algebraic isomorphism $t: G\to\mathbf{K}^{\mathbf{N}}$, with injective transpose $t': \mathbf{K}^{(\mathbf{N})}\to G^*$. In $F=t'(\mathbf{K}^{(\mathbf{N})})$, only finite-dimensional sets are bounded. For if (x_r') is a linearly independent sequence in F, then $(t'^{-1}(x_r'))$ is a linearly independent sequence in $\mathbf{K}^{(\mathbf{N})}$ and hence cannot be $\sigma(\mathbf{K}^{(\mathbf{N})}, \mathbf{K}^{\mathbf{N}})$-bounded. But for any $k \in \mathbf{K}^{\mathbf{N}}$, $\exists x \in G$ such that $t(x) = k$, and

$$\langle k, t'^{-1}(x_r')\rangle = \langle t(x), t'^{-1}(x_r')\rangle = \langle x, x_r'\rangle \quad (r \in \mathbf{N}).$$

Consequently (x_r') cannot be $\sigma(G^*, G)$-bounded.

Now, let B be the closed unit ball of G'. Then $F \cap B$ is $\sigma(G^*, G)$-bounded, hence finite-dimensional, so $F \cap G'$ is also finite-dimensional. If N is an algebraic supplement of $F \cap G'$ in F, then $G' \cap N = \{0\}$ and dim $N = \aleph_0$.

Let H be any algebraic supplement of G in E, and extend each element of N to E by giving it the value 0 on H. If M is the set of all these extensions, M is a countable dimensional subspace of E^*, and since the norm topology of G is finer than the induced topology, we also have $E' \cap M = \{0\}$.

Let X be any $\sigma(E' + M, E)$-bounded set and let Y be the set of restrictions to G of elements of X. Then Y is contained in $G' + N$ and Y is $\sigma(G' + N, G)$-bounded. Let p and q be the natural projections of $G' + N$ onto G' and N respectively. For each $y' \in Y$, define

$$z' = \begin{cases} \|p(y')\|^{-1} y', & \text{if } \|p(y')\| \geq 1, \\ y', & \text{otherwise,} \end{cases}$$

and let Z be the set of all these z'. Then Z is $\sigma(G' + N, G)$-bounded and $p(Z)$ is $\sigma(G', G)$-bounded. Consequently $q(Z)$ must be a $\sigma(G' + N, G)$-bounded subset of N and hence finite-dimensional. Since $q(Z)$ and $q(Y)$ span the same subspace of N, we also have that $q(Y)$ is finite-dimensional. From the construction of M out of N it is then clear that the natural projection of X into M is finite-dimensional. Thus there is a finite-dimensional subspace M_1 of M such that $X \subseteq E' + M_1$.

Since $\tau(E, E' + M_1)$ is barrelled [**10**, Theorem 1], X is $\tau(E, E' + M_1)$-equicontinuous. It is therefore $\tau(E, E' + M)$-equicontinuous, and so E is barrelled for $\tau(E, E' + M)$.

REMARK. In the proof of Theorem 1, barrelledness is not used in the actual construction, so that we may extract the following result which is perhaps of interest in its own right.

THEOREM 1a. *Let E be a Hausdorff locally convex space with a bounded subset which spans a subspace of dimension $\geq c$. Then there is a countable-dimensional subspace M of E^* such that $E' \cap M = \{0\}$ and for each $\sigma(E' + M, E)$-bounded set B there is a finite-dimensional subspace N of M such that $B \subseteq E' + N$.*

We now proceed to identify a number of special cases to which Theorem 1 applies.

COROLLARY 1. *The conclusion of Theorem 1 holds if E is a Hausdorff barrelled space with a fundamental sequence of bounded sets, and of dimension $\geq c$. In particular E may be a barrelled normed space of dimension $\geq c$.*

Proof. Suppose that the Hausdorff locally convex space E has a fundamental sequence of bounded sets (B_n). Irrespective of the continuum hypothesis, a countable union of sets of cardinality $< c$ also has cardinality $< c$ (e.g. apply König's Theorem [**5**, Ch. 2, 6.7]).

Thus if E has dimension $\geq c$, at least one of the B_n must span a subspace of dimension $\geq c$.

A normed space of course has a fundamental sequence of bounded neighbourhoods, but note also that Theorem 1 applies directly.

COROLLARY 2. *The conclusion of Theorem 1 holds if E is a barrelled metrisable space with dimension* $>c$.

Proof. Let (U_n) be a base of neighbourhoods of the origin in E. Let Λ be the set of sequences of positive numbers and for each $\boldsymbol{\lambda}=(\lambda_n)\in\Lambda$ let $B(\boldsymbol{\lambda})=\bigcap_{n=1}^{\infty}\lambda_n U_n$. Now the $B(\boldsymbol{\lambda})$ $(\boldsymbol{\lambda}\in\Lambda)$ form a fundamental family of bounded sets in E and $|\{B(\boldsymbol{\lambda}):\boldsymbol{\lambda}\in\Lambda\}|\leq c$. Thus if each $B(\boldsymbol{\lambda})$ spans a subspace of dimension $<c$ we have $\dim E\leq c.c=c$. Consequently $\exists\boldsymbol{\lambda}\in\Lambda$ such that $B(\boldsymbol{\lambda})$ spans a subspace of E with dimension $\geq c$.

REMARK. Suppose we assume the continuum hypothesis. Then in Corollary 1 we can replace the condition $\dim E\geq c$ by $\dim E$ infinite, provided we exclude from the first part the spaces which are countable direct sums of copies of $\mathbf{K}$. These are the only countable-dimensional Hausdorff barrelled spaces [**1**, p3, Exercise 4]. They are non-metrisable but they do have fundamental sequences of bounded sets. However they already have their finest locally convex topology.

Furthermore, the only case of infinite-dimensional metrisable barrelled spaces not covered by Theorem 1 or Corollary 2 would be that in which the space has dimension c, while its bounded sets span at most countable-dimensional subspaces. We have not been able to resolve this case. However we note that such a space would be an inductive limit of countable-dimensional normed spaces [**9**, p. 82], which as above are never barrelled. Consequently it is tempting to conjecture that this case does not occur.

In the remaining applications of Theorem 1, suitable completeness conditions remove this type of problem.

COROLLARY 3. *The conclusion of Theorem 1 holds when E is any Hausdorff barrelled space in which there is a sequentially complete infinite-dimensional absolutely convex bounded set. In particular E may be any infinite-dimensional Fréchet space.*

Proof. Let B be a sequentially complete, infinite-dimensional absolutely convex bounded subset of a Hausdorff locally convex space. The span of B is an infinite-dimensional Banach space with the gauge of B as norm [**2**, Proposition III. 1.9], and so has dimension $\geq c$ [**7**, Theorem I-1].

If (x_n) is any sequence in a metrisable locally convex space, then we can choose $\lambda_n>0$ $(n\in\mathbf{N})$ such that $\lambda_n x_n\to 0$ as $n\to\infty$. If the space is infinite-dimensional we may choose the (x_n) to be linearly independent, and if it is complete the closed, absolutely convex hull of $\{\lambda_n x_n : n\in\mathbf{N}\}$ will then satisfy the requirements of the first part.

COROLLARY 4. *Let X be an infinite Hausdorff completely regular topological space and let C(X) be the space of real-valued continuous functions on X with the topology of compact convergence. If C(X) is barrelled, the conclusion of Theorem 1 holds for* $E=C(X)$.

Proof. The space F of bounded real-valued continuous functions on X is an infinite-dimensional Banach space under the topology of uniform convergence on X and so has dimension $\geq c$. The closed unit ball of F is also a bounded subset of $C(X)$.

REMARK. Nachbin and Shirota have characterized those spaces $C(X)$ which are barrelled [**8**, Theorem 1; **11**, Theorem 1].

COROLLARY 5. *If E is a Hausdorff ultrabornological space (i.e. inductive limit of Banach spaces) and if E does not have its finest locally convex topology, the conclusion of Theorem 1 holds for E.*

Proof. Let E be a Hausdorff ultrabornological space, and let $\mathscr{A}$ be the collection of all absolutely convex bounded subsets A of E such that E_A, the linear span of A, is a Banach space with the gauge of A as norm. Then E is the inductive limit of the spaces E_A $(A \in \mathscr{A})$ under the natural embeddings [**6**, p. 148]. This inductive limit topology is the finest locally convex topology on E if and only if each E_A is finite-dimensional. Thus, if E does not have its finest locally convex topology, some E_A has dimension $\geq c$.

3. Bounded sets in $E'+M$. The key to the success of the construction used in the proof of Theorem 1 is the fact that each $\sigma(E'+M, E)$-bounded set is actually contained in a corresponding finite-dimensional enlargement of E'. We now show that this is necessarily the case, whenever we have a countable-dimensional enlargement which preserves barrelledness.

LEMMA. *Let E be a Hausdorff barrelled space and let (f_n) be a sequence in E^*. For each $n \in \mathbf{N}$ denote by M_n the linear span of $\{f_1, f_2, \dots f_n\}$ and put $M = \bigcup_{n=1}^{\infty} M_n$. If X is a (non-empty) absolutely convex $\sigma(E'+M, E)$-compact set, $\exists n \in \mathbf{N}$ such that $X \subseteq E'+M_n$.*

Proof. Let H be the linear span of X. Then H is a Banach space under the gauge of X as norm. For each $n \in \mathbf{N}$, put $X_n = X \cap (E'+M_n)$. Clearly $X = \bigcup_{n=1}^{\infty} X_n$ and $X_n \subseteq X_{n+1}$ $(n \in \mathbf{N})$. We know from [**10**] that $\tau(E, E'+M_n)$ is barrelled so that $E'+M_n$ is $\sigma(E'+M_n, E)$-quasicomplete. Consequently X_n is $\sigma(E'+M_n, E)$-compact, and therefore $\sigma(E'+M, E)$-closed. Since the norm topology on H is finer than the restriction of $\sigma(E'+M, E)$ to H, each X_n is also closed in the Banach space H. It follows from Theorem 1 of [**3**] that $\exists n \in \mathbf{N}$ and $\lambda > 0$ such that $X \subseteq \lambda X_n$. Thus $X \subseteq E'+M_n$.

THEOREM 2. *If E is barrelled, and $E'+M$ a countable-dimensional enlargement of the dual, then $\tau(E, E'+M)$ is barrelled if and only if each $\sigma(E'+M, E)$-bounded set is contained in some finite-dimensional enlargement of E'.*

Proof. If $\tau(E, E'+M)$ is barrelled, the $\sigma(E'+M, E)$-closed, absolutely convex hull of each $\sigma(E'+M, E)$-bounded set is $\sigma(E'+M, E)$-compact. The necessity of the condition now follows from the Lemma.

The proof of the sufficiency of the condition is the same as the last part of the proof of Theorem 1.

REMARK. From the proof of Théorème 3 of [**4**] it follows that if in Theorem 2, the topology $\tau(E, E'+M)$ is barrelled, then for each $\sigma(E'+M, E)$-bounded set X there is a $\sigma(E', E)$-bounded set X_1 and a finite-dimensional bounded subset X_2 of M such that $X \subseteq X_1 + X_2$.

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