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ON PURIFIABLE SUBSOCLES OF A PRIMARY ABELIAN GROUP

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Introduction. In this paper we shall investigate an interesting connection between the structure of G/S and G, where S is a purifiable subsocle of G. The results are interesting in the light of a counterexample by Dieudonné [**3**, p. 142] who exhibits a primary abelian group G, where G/S is a direct sum of cyclic groups, but G is not a direct sum of cyclic groups. Surprisingly, the assumption of the purifiability of S allows G to inherit the structure of G/S. In particular, we show that if G/S is a direct sum of cyclic groups and S supports a pure subgroup H, then G is a direct sum of cyclic groups and H is a direct summand of G which is of course a direct sum of cyclic groups. It is also shown that if G/S is a direct sum of torsion-complete groups and S supports a pure subgroup H, then G is a direct sum of torsion-complete groups and H is a direct summand of G, and is also a direct sum of torsion-complete groups. Using some homological machinery, we show that if G/S is totally projective and S supports a p^α-pure subgroup H where α is an appropriately chosen ordinal, then G is totally projective and H is a direct summand of G, and is also totally projective. Consequently, if G/S is a direct sum of countable groups and S supports a p^α-pure subgroup H, where α is an appropriate ordinal, then G is a direct sum of countable groups and H is a direct summand of G, and is also a direct sum of countable groups.

All groups will be assumed to be additively written primary abelian groups for some prime p. We shall follow the notation and terminology of Fuchs [**3**]. All references to topological concepts will be relative to the p-adic topology on a primary group G which has the base $\{p^nG\}$ at 0. Let $\mathrm{ht}(x)$ denote the generalized p-height of x, that is the least ordinal α such that $x \notin p^{\alpha+1}G$, where $p^{\alpha+1}G = p(p^\alpha G)$ and $p^\alpha G = \bigcap_{\beta<\alpha} p^\beta G$ if α is a limit ordinal.

Definition 1. The subgroup H is *p^α-pure* in G if and only if the exact sequence $H \rightarrowtail G \twoheadrightarrow G/H$ is in $p^\alpha \mathrm{Ext}(G/H, H)$, where α is an ordinal.

Note that p^ω-purity is the same as the classical concept of purity for p-primary abelian groups. See [**14**].

Definition 2. The *subsocle S supports the subgroup H* if and only if $H[p] = S$.

Theorem 1 below will serve as a pattern and will motivate this paper. It is interesting in that its proof involves an application of the Kulikov criterion.

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Definition 3. The *subsocle S satisfies the Kulikov criterion* in the group G if and only if S can be expressed as the union of an ascending sequence of subgroups of bounded height.

Recall that Kulikov has shown that a p-group G is a direct sum of cyclic groups if and only if its socle $G[p]$ satisfies the Kulikov criterion.

THEOREM 1. *If G/S is a direct sum of cyclic groups and S is a subsocle which supports a pure subgroup H, then G is a direct sum of cyclic groups and H is a summand of G.*

Proof. Notice that $H/S \simeq pH$ is a subgroup of G/S, and so pH is a direct sum of cyclic groups. Consequently, H is a direct sum of cyclic groups. See [**2**] for results relating p^nG and G. To complete the proof, it is sufficient to show that G/H is a direct sum of cyclic groups. We show that $(G/H)[p]$ satisfies the Kulikov criterion. Consider the map $\pi: G/S \rightarrow G/H$. Using the purity of H, notice that $G[p]/S$ maps under π onto the socle of G/H. Since G/S is a direct sum of cyclic groups, any subsocle of G/S satisfies the Kulikov criterion in G/S. Consequently, $G[p]/S$ satisfies the Kulikov criterion in G/S. Using the purity of H, it can be shown that $\pi(G[p]/S) = (G/H)[p]$ satisfies the Kulikov criterion in G/H. Consequently, G/H is a direct sum of cyclic groups.

It is possible to extend the above result, as Theorem 2 and its corollaries will indicate. First, we consider a definition.

Definition 4. The *subsocle S is purifiable* in G if and only if there is a pure subgroup H where $H[p] = S$.

THEOREM 2. *Let G be a p-primary group and S a subsocle which supports a pure subgroup H. If $G[p]/S$ is purifiable in G/S, then H is a direct summand of G.*

We need the following three lemmas to prove the above theorem.

LEMMA 3. *Let G be a p-primary group and S a subsocle of G. If S supports a pure subgroup H, then*

(i) $(G/S)[p] = (H/S)[p] \oplus G[p]/S$,

(ii) *$\pi: G/S \rightarrow G/H$ is height-preserving on $G[p]/S$,*

(iii) *If $h + S \in (H/S)[p]$ and $k + S \in G[p]/S$, then*

$$\mathrm{ht}(h + k + S) = \min\{\mathrm{ht}(h + S), \mathrm{ht}(k + S)\}.$$

Proof. (i) To see that $(H/S)[p] \cap G[p]/S = 0$, it is sufficient to notice that $H \cap G[p] = S$. Suppose that $x + S \in (G/S)[p]$. Map $x + S$ onto $x + H$. By [**9**, p. 15, Lemma 1], there is a $y \in G[p]$ such that $x + H = y + H$ and $x - y = h \in H$. Hence $x + S = (h + S) + (y + S)$ and so

$$(G/S)[p] = (H/S)[p] \oplus G[p]/S.$$

(ii) Suppose that $x + S \in G[p]/S$ and $x + H = p^n z + H$. Using the purity of H, we can assume that $p^n z \in G[p]$ and $p^n z \notin S$. Consequently,

$$x - p^n z \in H[p] = S$$

and so $x + S = p^n z + S$. Therefore, $\pi: G/S \to G/H$ is height-preserving on $G[p]/S$.

(iii) follows from (ii).

LEMMA 4. *Let G be a p-primary group with pure subgroups H and K, where $G[p] = H[p] \oplus K[p]$. If $\mathrm{ht}(h + k) = \min\{\mathrm{ht}(h), \mathrm{ht}(k)\}$ for all $h \in H[p]$ and $k \in K[p]$, then $G = H \oplus K$.*

Proof. By [**9**, p. 20, Lemma 7], $H \oplus K$ is a pure subgroup of G. Since $(H \oplus K)[p] = G[p]$, we have $G = H \oplus K$ by [**9**, p. 24, Lemma 12].

Finally, we need the following lemma of Hill and Megibben [**6**].

LEMMA 5. *Let G be a p-primary group containing subgroups H and K, where H is neat in G. Then $(H + K)[p] = H[p] + K[p]$ if and only if $H \cap K$ is neat in K.*

Proof of Theorem 2. By hypothesis, $G[p]/S$ supports a pure subgroup K/S. By Lemmas 3 and 4, $G/S = H/S \oplus K/S$ and so $G = H + K$. Since H is pure in G and $(H + K)[p] = H[p] + K[p]$, then by Lemma 5, $H \cap K$ is neat in K. Now $H \cap K = S$, and consequently S must be pure in K. Thus, S is a summand of K and so $G = H + K = H + (S \oplus K') = H \oplus K'$.

COROLLARY 6. *Let G satisfy the hypothesis of Theorem 2; then $G = H \oplus (G/H)$ and $G/S = H/S \oplus G/H \simeq pH \oplus (G/H)$.*

COROLLARY 7. *If G/S is a direct sum of cyclic groups and S supports H pure in G, then G is a direct sum of cyclic groups and H is a summand of G.*

Proof. Notice that every subsocle of a direct sum of cyclic groups is purifiable.

Definition 5. The group G is *pure-complete* if and only if every subsocle of G is purifiable.

Definition 6. The reduced p-group G is *quasi-closed* if and only if the closure of any pure subgroup is a pure subgroup.

COROLLARY 8. *If G/S is quasi-closed and S supports a pure subgroup H, then G is quasi-closed and H is a summand of G which is quasi-closed.*

Proof. Quasi-closed groups are pure complete and summands of quasi-closed groups are quasi-closed. Also, pG quasi-closed implies that G is quasi-closed. See [**6**] for additional properties of quasi-closed groups.

COROLLARY 9. *If G/S is pure complete and S supports a pure subgroup H, then G is pure complete and H is a summand of G.*

Proof. $G[p]/S$ supports a pure subgroup K/S. By Lemmas 3 and 4, $G/S = H/S \oplus K/S$. Note that if $G = A \oplus B$ and G is pure complete, then $G/B[p] \simeq A \oplus pB$ is pure complete. Consequently, $(G/S)/(K/S)[p]$ is pure complete. But $(G/S)/(K/S)[p] = (G/S)/(G[p]/S) \simeq G/G[p] \simeq pG$. Now G is pure complete if and only if p^nG is pure complete for some integer n. Consequently, G is pure complete.

COROLLARY 10. *If G/S is pure complete, S supports a pure subgroup H, and G/S has an unbounded direct sum of cyclic groups summand, then G has an unbounded direct sum of cyclic groups summand and H is a summand of G.*

Proof. O'Neill has proved in [**15**] that if $G = H \oplus K$ and G has an unbounded direct sum of cyclic groups summand, then either H or K has such a summand. If pH has an unbounded direct sum of cyclic groups summand, then H has such a summand.

Definition 7. A group G is *essentially indecomposable* if and only if whenever $G = H \oplus K$, either H or K is bounded.

COROLLARY 11. *If G/S is pure complete, essentially indecomposable, and S supports a pure subgroup H, then G is essentially indecomposable and H is a summand of G.*

Proof. Apply Corollary 6.

COROLLARY 12. *If G/S is a direct sum of torsion-complete groups and S supports a pure subgroup H, then G is a direct sum of torsion-complete groups and H is a summand of G which is a direct sum of torsion-complete groups.*

Proof. We use the following result which follows from a theorem by Hill [**4**]. If G is a direct sum of torsion-complete groups and $G[p] = S \oplus T$, where $\text{ht}(s + t) = \min\{\text{ht}(s), \text{ht}(t)\}$ for all $s \in S$ and $t \in T$, then S and T support summands of G which are direct sums of torsion-complete groups. By Hill's result, $G[p]/S$ supports a summand K/S in G/S which is a direct sum of torsion-complete groups.

Hill [**4**] and Warfield [**18**] have shown that a summand of a direct sum of torsion-complete groups is a direct sum of torsion-complete groups. Note that if pH is a direct sum of torsion-complete groups, then H is such a direct sum. Consequently, applying Corollary 6 we see that G is a direct sum of torsion-complete groups and H is a summand of G.

Definition 8. The group G is *semi-complete* if and only if G is the direct sum of a torsion-complete group and a direct sum of cyclic groups.

As an immediate consequence of Corollary 12, if G/S is semi-complete and S supports a pure subgroup H, then G is semi-complete and H is a summand of G. The condition that S supports a pure subgroup H is essential. Dieudonné [**3**, p. 142] has constructed an example where G/S is a direct sum of cyclic groups, but G is not such a direct sum. It is also easy to see that $G[p]/S$ is not always a purifiable subsocle of G/S. Consider the pure resolution $K \rightarrowtail G \twoheadrightarrow H$, where H is a p-group which is not a direct sum of cyclic groups and G is a direct sum of cyclic groups. Let $S = K[p]$. If $G[p]/S$ were purifiable in G/S, then by Theorem 2, K would be a summand of G. But this contradicts the fact that H is not a direct sum of cyclic groups.

Using the concept of large subgroup introduced by Pierce [**16**], we can relate the G/S problem to the class of thick groups and the class of thin groups.

Definition 9. The subgroup L is a *large subgroup* of G if and only if L is fully invariant and $L + B = G$ for every basic subgroup B of G.

Definition 10. The group G is *thick* if and only if for every map $f\colon G \to \sum Z(p^n)$, the kernel contains a large subgroup of G.

LEMMA 13. *If L is a large subgroup of G and S is a subsocle of G, then $(L + S)/S$ contains a large subgroup of G/S.*

Proof. Pierce [**16**] has shown that a subgroup H contains a large subgroup if and only if for each integer k there is an integer n_k where $(p^{n_k}G)[p^k] \subseteq H$. Let k and n_k be the appropriate integers for L in G. For $(L + S)/S$ in G/S, let $N_k = n_{k+1}$ for each integer k. It is easy to see that $(p^{N_k}(G/S))[p^k] \subseteq (L + S)/S$. Consequently, $(L + S)/S$ contains a large subgroup of G/S.

THEOREM 14. *G is thick if and only if G/S is thick.*

Proof. Let $f\colon G/S \to \sum Z(p^n)$ be a map with kernel K/S. Consider the composite map

$$G \xrightarrow{\pi} G/S \xrightarrow{f} \sum Z(p^n).$$

G thick implies that $K \supseteq L$, where L is large in G. The subgroup K/S contains $(L + S)/S$ which contains a large subgroup of G/S. Consequently, G/S is thick. The converse follows from Lemma 13 and the following relation:

$$G[p]/S \rightarrowtail G/S \twoheadrightarrow G/G[p] \simeq pG.$$

Definition 11. The group G is *thin* if and only if for every map $f\colon \bar{B} \to G$, where $\bar{B}$ is the torsion completion of $\sum Z(p^n)$, the kernel of f contains a large subgroup of $\bar{B}$.

LEMMA 15. *The group G/S is thin if and only if G is thin.*

Proof. Richman [**17**] proved that extensions of thin groups by thin groups are thin groups. Applying this to the exact sequence $S \rightarrowtail G \twoheadrightarrow G/S$ proves the lemma one way. The converse is proved by considering the exact sequence

$$G[p]/S \rightarrowtail G/S \twoheadrightarrow G/G[p] \simeq pG.$$

Using basic homological techniques, we can gain a further insight into the relationship of the structure of G/S to the structure of G.

Definition 12. The group G is *cotorsion* if and only if G is a reduced group and any extension of G by a torsion-free group splits.

Definition 13. The group G is a *p-adic module* if and only if G is a module over the ring R_p which is the set of all rational numbers of the form a/b, where b is prime to p.

LEMMA 16. *Let G be a p-adic module. If G/S is cotorsion, then G is cotorsion.*

Proof. It is sufficient to show that $\mathrm{Hom}(Q, G) = 0 = \mathrm{Ext}(Q, G)$, where Q is the set of rational numbers. Consider the exact sequence

$$0 \to \mathrm{Hom}(Q, S) \to \mathrm{Hom}(Q, G) \to \mathrm{Hom}(Q, G/S) \to \mathrm{Ext}(Q, S) \to \mathrm{Ext}(Q, G) \to \mathrm{Ext}(Q, G/S) \to 0.$$

Since S and G/S are cotorsion, the lemma follows.

Definition 14. The group G is *algebraically compact* if and only if G is a direct summand of every group which contains G as a pure subgroup.

Definition 15. The subgroup $\mathrm{Pext}(A, B)$ of $\mathrm{Ext}(A, B)$ consists of all pure extensions of B by A. In fact, $\mathrm{Pext}(A, B)$ is the elements of infinite height of $\mathrm{Ext}(A, B)$. See [**3**].

Note. It is well known that a reduced group G is algebraically compact if and only if G is cotorsion and $\mathrm{Pext}(Q/Z, G) = 0$.

LEMMA 17. *Let G be a p-adic module without elements of infinite height. If G/S is algebraically compact, then G is algebraically compact.*

Proof. We must show that $\mathrm{Hom}(Q,G) = 0 = \mathrm{Ext}(Q,G)$ and $\mathrm{Pext}(Q/Z,G) = 0$. Since G is necessarily cotorsion (by Lemma 16), the first two conditions follow. It is easy to see that $G \simeq \mathrm{Ext}(Q/Z, G)$ and consequently $\mathrm{Pext}(Q/Z, G) = 0$ since G has no elements of infinite height. Thus, G is algebraically compact.

LEMMA 18. *Let G be a p-primary group without elements of infinite height and S a closed subsocle of G. G is torsion-complete if and only if G/S is torsion-complete.*

Proof. A p-primary group G is torsion-complete if and only if

$$\mathrm{Pext}(Z(p^\infty), G) = 0.$$

Consider the exact sequence

$$\mathrm{Ext}(Z(p^\infty), S) \rightarrowtail \mathrm{Ext}(Z(p^\infty), G) \twoheadrightarrow \mathrm{Ext}(Z(p^\infty), G/S).$$

Now $\mathrm{Ext}(Z(p^\infty), S) \simeq S$ and the torsion subgroup of $\mathrm{Ext}(Z(p^\infty), G)$ is isomorphic to G. Now $G^1 = 0$ and $\mathrm{Pext}(Z(p^\infty), G/S) = 0$ imply that

$$\mathrm{Pext}(Z(p^\infty), G) = 0.$$

That is, G/S torsion-complete implies that G is torsion-complete.

Conversely, $\mathrm{Pext}(Z(p^\infty), G) = 0$ implies $\mathrm{Pext}(Z(p^\infty), G/S) = 0$; otherwise, since $\mathrm{Ext}(Z(p^\infty), G)/S \simeq \mathrm{Ext}(Z(p^\infty), G/S)$, we could construct a p-divisible subgroup of $\mathrm{Ext}(Z(p^\infty), G)$, but $\mathrm{Ext}(Z(p^\infty), G)$ is p-reduced.

Note that it is necessary that S be closed. Consider the standard $\bar{B}$ and let S be the socle of a basic subgroup of $\bar{B}$; then clearly $\bar{B}/S$ is not torsion-complete.

We can generalize the concept of a direct sum of cyclic groups by considering the class of projective and totally projective groups. First we list some fundamental results of Nunke [**13**].

Definition 16. The group G is *p^α-projective* if and only if $p^\alpha\text{Ext}(G, C) = 0$ for all groups C.

Definition 17. The functor $p^\alpha\text{Ext}$ is *hereditary* if and only if each p^α-pure subgroup of a p^α-projective group is p^α-projective.

THEOREM 19 [**14**, especially p. 163, Theorem 6.3]. *If H is p^α-pure in G, then the following sequences are exact, where C is any abelian group:*

$$0 \to \text{Hom}(C, H) \to \text{Hom}(C, G) \to \text{Hom}(C, G/H) \to p^\alpha\text{Ext}(C, H) \to p^\alpha\text{Ext}(C, G) \to p^\alpha\text{Ext}(C, G/H)$$

$$0 \to \text{Hom}(G/H, C) \to \text{Hom}(G, C) \to \text{Hom}(H, C) \to p^\alpha\text{Ext}(G/H, C) \to p^\alpha\text{Ext}(G, C) \to p^\alpha\text{Ext}(H, C).$$

If, in addition, $p^\alpha\text{Ext}$ *is hereditary, then the right-hand maps are epic.*

THEOREM 20 [**13**, p. 211, Theorem 4.4]. *Let $\beta \leqq \alpha < \beta + \omega$, where $\beta = 0$ or is a limit ordinal. Then* $p^\alpha\text{Ext}$ *is hereditary if and only if $\beta = 0$ or is the limit of a countable ascending sequence of ordinals.*

THEOREM 21 [**13**, p. 194, Proposition 2.5]. *If A is a p-group such that $A/p^\beta A$ is p^β-projective and $p^\beta A$ is p^γ-projective, then A is $p^{\beta+\gamma}$-projective.*

THEOREM 22 [**13**, p. 200, Proposition 3.1]. *If B is $p^{\alpha+1}$-pure in the p^α-projective p-group A, then B is a direct summand of A, hence B and A/B are p^α-projective.*

THEOREM 23 [**13**, p. 199, Theorem 2.12]. *A p-group is a direct sum of countable reduced groups if and only if it is totally projective and has length $\leqq \Omega$, where Ω is the first uncountable ordinal.*

Note that a p-group G is p^ω-projective if and only if G is a direct sum of cyclic groups. Also, $p^\alpha\text{Ext}$ is hereditary for countable ordinals.

THEOREM 24. *If G/S is p^α-projective, S supports H which is p^α-pure in G, and* $p^\alpha\text{Ext}$ *is hereditary, then G is p^α-projective.*

Proof. Consider the commutative diagram:

$$\text{(D}_1\text{)}\qquad \begin{array}{ccccc} E_1\colon H & \overset{i}{\rightarrowtail} & G & \overset{\pi}{\twoheadrightarrow} & \dfrac{G}{H} \\ f\downarrow & & g\downarrow & & \Vert \\ E_2\colon \dfrac{H}{S} & \overset{j}{\rightarrowtail} & \dfrac{G}{S} & \overset{P}{\twoheadrightarrow} & \dfrac{G}{H} \end{array}$$

Note that $E_2 \equiv fE_1$ are equivalent exact sequences and thus $E_1 \in p^\alpha\text{Ext}(G/H, H)$ implies $E_2 \in p^\alpha\text{Ext}(G/H, H/S)$ since $f(E + E') = fE + fE'$, where $E + E'$ is the Baer sum of two extensions. By Theorem 19, we obtain the exact sequences in the following diagram (D_2).

$$
(D_2)\quad
\begin{array}{ccccccccccccc}
0 \to \mathrm{Hom}\left(\frac{G}{H}, C\right) & \overset{P^*}{\to} & \mathrm{Hom}\left(\frac{G}{S}, C\right) & \overset{j^*}{\to} & \mathrm{Hom}\left(\frac{H}{S}, C\right) & \overset{\partial_1}{\to} & p^\alpha\mathrm{Ext}\left(\frac{G}{H}, C\right) & \overset{P^*}{\to} & p^\alpha\mathrm{Ext}\left(\frac{G}{S}, C\right) & \overset{j^*}{\to} & p^\alpha\mathrm{Ext}\left(\frac{H}{S}, C\right) \to 0 \\
\downarrow 1^* & & \downarrow g^* & & \downarrow f^* & & \downarrow 1^* & & \downarrow g^* & & \downarrow f^* \\
0 \to \mathrm{Hom}\left(\frac{G}{H}, C\right) & \overset{\pi^*}{\to} & \mathrm{Hom}(G, C) & \overset{i^*}{\to} & \mathrm{Hom}(H, C) & \overset{\partial_2}{\to} & p^\alpha\mathrm{Ext}\left(\frac{G}{H}, C\right) & \overset{\pi^*}{\to} & p^\alpha\mathrm{Ext}(G, C) & \overset{i^*}{\to} & p^\alpha\mathrm{Ext}(H, C) \to 0
\end{array}
$$

where ∂_1 and ∂_2 are the connecting homomorphisms and 1^* is the identity map. By the naturality of the maps, diagram (D_2) is commutative. G/S being p^α-projective implies that H/S is p^α-projective by considering diagram (D_2). $H/S \simeq pH$ being p^α-projective implies that H is p^α-projective by Theorem 21. By diagram chasing we see that G is p^α-projective.

If $p^\alpha\mathrm{Ext}$ is not hereditary or if S does not support a p^α-pure subgroup, we obtain the following weaker result.

LEMMA 25. *If G/S is p^α-projective, then G is $p^{\alpha+1}$-projective, where $\alpha \geqq \omega$.*

Proof. Consider the exact sequence

$$G[p]/S \overset{i}{\rightarrowtail} G/S \overset{\pi}{\twoheadrightarrow} G/G[p] \simeq pG$$

which induces the exact sequence

$$0 \to \mathrm{Hom}\left(\frac{G}{G[p]}, C\right) \overset{\pi^*}{\to} \mathrm{Hom}\left(\frac{G}{S}, C\right) \overset{i^*}{\to} \mathrm{Hom}\left(\frac{G[p]}{S}, C\right) \overset{\partial}{\to} \mathrm{Ext}\left(\frac{G}{G[p]}, C\right) \overset{\pi^*}{\to} \mathrm{Ext}\left(\frac{G}{S}, C\right) \overset{i^*}{\to} \mathrm{Ext}\left(\frac{G[p]}{S}, C\right) \to 0.$$

Now

$$\frac{p^\alpha\mathrm{Ext}(G/G[p], C)}{p^\alpha\mathrm{Ext}(G/G[p], C) \cap \partial(\mathrm{Hom}(G[p]/S, C))} \simeq \pi^*(p^\alpha\mathrm{Ext}(G/G[p], C))$$

and

$$\pi^*\left(p^\alpha\mathrm{Ext}\left(\frac{G}{G[p]}, C\right)\right) \subseteq p^\alpha\mathrm{Ext}\left(\frac{G}{S}, C\right) = 0,$$

since G/S is p^α-projective. Thus

$$p^\alpha\mathrm{Ext}\left(\frac{G}{G[p]}, C\right) \subseteq \partial\left(\mathrm{Hom}\left(\frac{G[p]}{S}, C\right)\right) \simeq \sum_i Z_i(p)$$

since $\mathrm{Hom}(G[p]/S, C) \simeq \prod C[p]$ which is bounded of order p. Consequently, $p^{\alpha+1}\mathrm{Ext}(G/G[p], C) = 0$ or pG is $p^{\alpha+1}$-projective and by Theorem 21, G is then $p^{\alpha+1}$-projective.

Note that the above lemmas cannot in general be sharpened. Dieudonné has constructed an example of a p-primary group G without elements of infinite height where G/S is a direct sum of cyclic groups, but G is not a direct sum of cyclic groups. In homological terms, G/S is p^ω-projective, and consequently G is $p^{\omega+1}$-projective, but G is not p^ω-projective.

Definition 18. The group G is *totally projective* if and only if $G/p^\alpha G$ is p^α-projective for all ordinals α.

Definition 19. The *length* of a p-primary reduced group G is the least ordinal λ, where $p^\lambda G = 0$.

LEMMA 26. *If G/S is totally projective and S supports a $p^{\lambda+1}$-pure subgroup H, where λ is the length of G/S, then H is a direct summand of G, and G is totally projective.*

Proof. Now H/S is $p^{\lambda+1}$-pure in G/S which is p^λ-projective and so by Theorem 22, H/S is a summand of G/S. Consequently, G/H is totally projective and since $H/S \simeq pH$, H is totally projective. Consider the exact sequence $H \rightarrowtail G \twoheadrightarrow G/H$. Now H is p^λ-pure in G and G/H is p^λ-projective. Thus, the preceding exact sequence splits and H is a summand of G and G is totally projective.

COROLLARY 27. *If G/S is a direct sum of countable reduced p-groups and S supports a $p^{\lambda+1}$-pure subgroup H, where λ is the length of G/S, then H is a summand of G, and G is a direct sum of countable reduced p-groups.*

Proof. Use Theorem 23 and Lemma 26.

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